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General Integral for a Class of Non-Steady Atmospheric Flights and Applications to Trajectory Analysis

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A complete analytical integration of the aircraft kinematic and dynamic equations of motion is presented. Different applications of defined integrals to trajectory analysis are considered. The dynamic equations are obtained under the assumptions, that acceleration due to aerodynamic lift, the difference between the accelerations due to propulsive thrust and aerodynamic drag are not changed, the aircraft body rate about the velocity axis is zero and the sideslip angle is zero. The general integral of these equations consists of six independent first integrals of motion and describes a class of non-steady flight trajectories in a maneuver plane. It will be shown that the dynamic equations can be derived and completely integrated in a closed-form for more general assumptions. The problem of computing thrust for a given trajectory has been considered. The trajectory is defined by constraint equation. Constraints stabilization equations, which have asymptotically stable trivial solution, are constructed. Explicitness can make the integrals applicable to modeling the trajectories of spacecraft, re-entry vehicles and missiles, and to the design of on-board targeting and guidance. An illustrative example is presented.

Key words and phrases: analytical integration, general integral, analytical solutions, aircraft nonlinear model, programmed constraints

1. Introduction

This paper presents a complete analytical integration of the aircraft kinematic and dynamic equations obtained under the following assumptions: (a) acceleration due to aerodynamic lift, and the difference between the accelerations due to propulsive thrust and aerodynamic drag are not changed; (b) the aircraft body rate about the velocity axis is zero; (c) the sideslip angle is zero. It will be shown that the general integral of these equations consists of six independent first integrals which lead to the closed-form analytical solutions. The studies of the existing literature show that the aircraft equations can be integrated in a closed-form for some specific cases of quasi-steady and non-steady flights, including the cases of climb and cruise with constant altitude, velocity or lift acceleration, negligible flight path angle or small angle of attack [1, 2]. In some cases of optimal quasi-steady cruise trajectories, the equations of motion have been implicitly integrated or reduced to quadratures [2]. It should be noted that the studies presented in this paper were initiated with the purpose of integration of the 3rd order differential equation,

$$\frac{\mathrm{d}\ddot{\mathbf{r}}}{\mathrm{d}t} = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{|\dot{\mathbf{r}}|^2} \times (\ddot{\mathbf{r}} - \mathbf{g}), \tag{1}$$

obtained for the nonlinear model of the aircraft tracking problem under the following assumptions: (1) acceleration due to aerodynamic lift, and the difference between the

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accelerations due to propulsive thrust and aerodynamic drag are not changed; (2) the aircraft body rate about the roll axis is zero; (3) the angle of attack and the sideslip angle are zero [3–5]. Note that the left hand side of Eq. (1) represents the jerk vector, and its expression does not explicitly depend on the accelerations due to thrust, drag and lift. Analysis show, however, that as the drag is a function of the square of the velocity, it would be very difficult to hold the lift, and the thrust-drag accelerations constant with zero angle of attack. In this paper, it will be shown that Eq. (1) can also be derived and completely integrated in a closed-form for a more general assumptions (a-c) with non-zero and variable angles of attack. It is demonstrated that the assumptions (a-b) can significantly extend the applicability of Eq. (1). Explicitness can make the integrals applicable to modeling the trajectories of spacecraft, re-entry vehicles and missiles, and to the design of on-board targeting and guidance [4].

2. Equations and Integrals for Non-steady Flight

Consider the F-frame formed by the triad of orthogonal unit vectors $\mathbf{e}_1^F$, $\mathbf{e}_2^F$, $\mathbf{e}_3^F$ and with the origin O at the aircraft center of mass (COM): the unit vector $\mathbf{e}_1^F$ is aligned with the velocity vector, $\mathbf{e}_3^F$ forms the angle ϕ with lift and $\mathbf{e}_2^F$ completes the right handed system (see Fig. 1). The angle ϕ is measured in the $O\mathbf{e}_2^F\mathbf{e}_3^F$-plane. It is assumed that the non-steady flight trajectory lies in a vertical plane xz containing $\mathbf{e}_1^F$ and $\mathbf{e}_3^F$. Then if $\mathbf{P}$ is the sum of external forces acting on the aircraft, that is thrust, weight, drag and lift, then [1]

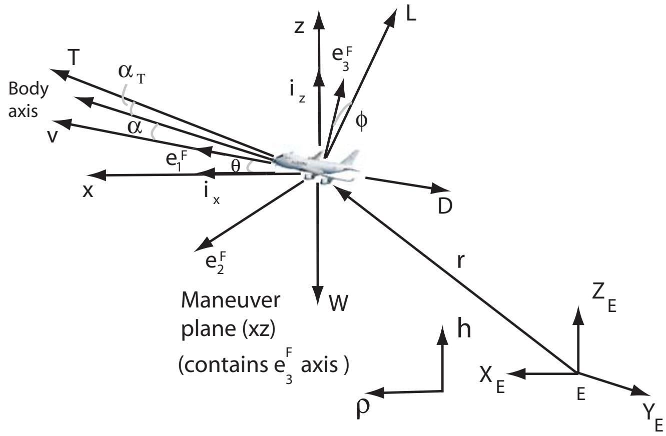

Figure 1. To the nonlinear aircraft model

$$\mathbf{P} = \mathbf{W} + \mathbf{T} + \mathbf{D} + \mathbf{L}$$

can be rewritten in the form:

$$\mathbf{P} = (T\cos\bar{\alpha} - D - g_0\sin\theta)\mathbf{e}_1^F + [(T\sin\bar{\alpha} + L)\cos\phi - g_0\cos\theta]\mathbf{e}_3^F,$$

where $\bar{\alpha} = \alpha + \alpha_T$ is the angle between the thrust vector and the velocity vector. If $\mathbf{a} = \dot{v}\,\mathbf{e}_1^F + v\dot{\theta}\,\mathbf{e}_3^F$, then with the assumptions given above, the Newton's second law yields the following equations valid in the maneuver xz-plane [3]:

$$\dot{v} = -g_0\sin\theta + c_1, \quad v\dot{\theta} = -g_0\cos\theta + c_2, \quad \dot{c}_1 = 0, \quad \dot{c}_2 = 0, \tag{2}$$

where

$$c_1 = \frac{g_0}{W}(T\cos\bar{\alpha} - D), \quad c_2 = \frac{g_0}{W}(T\sin\bar{\alpha} + L)\cos\phi \tag{3}$$

with ϕ = const. Complete analytical integration of Eqs. (2) and application of the resulting solutions to trajectory analysis is the main purpose of this paper. As will be shown below, the complete integration of Eqs. (2) reveals a general integral which consists of six independent first integrals with their constants. These constants will be denoted below by η_i, $(i = 1, \ldots, 6)$, and one can accept that $\eta_1 = c_1 =$ const and $\eta_2 = c_2 =$ const. Eqs. (2) are valid for a flight with the assumptions (a-c) in the maneuver plane. The first integrals of Eqs. (2) for η_1 and η_2 represent the relationships between the velocity magnitude, flight path angle, the propulsive and aerodynamic accelerations.

3. Integrals for Velocity Vector, Time and Position Vector

Integrals for magnitude of velocity vector

In this subsection, it will be shown that the first two equations of Eqs. (2) can be explicitly integrated in elementary and transcendental functions in terms of the angle θ. By considering θ as an independent variable instead of time, t, we have $\dot{v} = \mathrm{d}v/\mathrm{d}t = \mathrm{d}v/\mathrm{d}\theta\,\mathrm{d}\theta/\mathrm{d}t$. Then by eliminating $\mathrm{d}\theta/\mathrm{d}t$ from Eqs. (2), one can obtain

$$\frac{\mathrm{d}v}{\mathrm{d}\theta}v^{-1} = \frac{-g_0\sin\theta + c_1}{-g_0\cos\theta + c_2}, \tag{4}$$

which can be integrated in the form [6]:

$$\begin{aligned}
v(\theta) &= \eta_3(a + b\sin x)^{-1}\exp\left[\frac{2A}{d_1}\arctan\frac{a\tan\bar{x} + b}{d_1}\right], & [a^2 > b^2],\\
v(\theta) &= \eta_3(a + b\sin x)^{-1}\left[\frac{a\tan\bar{x} + d_3}{a\tan\bar{x} + d_4}\right]^{(A/d_2)}, & [a^2 < b^2],\\
v(\theta) &= \eta_3(a + b\sin x)^{-1}\ \exp\left[\frac{A}{a}\ \tan\left(\bar{x} - \frac{\pi}{4}\right)\right], & [a^2 = b^2],
\end{aligned} \tag{5}$$

where η_3 is the integration constant, $x = \theta + \frac{\pi}{2}$, $\bar{x} = x/2$ and $a + b\sin x \neq 0$, and the following constants are used:

$$A = c_1, \quad a = c_2, \quad b = -B = -g_0, \tag{6}$$

$$d_1 = \sqrt{a^2 - b^2}, \quad d_2 = \sqrt{b^2 - a^2}, \quad d_3 = b - \sqrt{b^2 - a^2}, \quad d_4 = b + \sqrt{b^2 - a^2}. \tag{7}$$

Note that in a particular case when $a + b\sin x = 0$, the system of equations in Eqs. (2) describes a motion with constant v and θ. This case is of a very limited theoretical and practical interest, and not considered in this paper.

Integrals for time

Once $v = v(\theta)$ is determined, the second equation of Eqs. (2) can be integrated as:

$$t = \int\limits_{x_0-\gamma}^{x-\gamma} \frac{v(x)\mathrm{d}x}{a + b\sin x} + \eta_4, \tag{8}$$

which can be reduced to the following final forms:

$$
\begin{aligned}
t &= \frac{\eta_3}{A^2+a^2-b^2}\exp\left[\frac{2A}{d_1}\arctan\frac{a\tan\bar{x}+b}{d_1}\right]\left(\frac{A+b\cos x}{a+b\sin x}+\frac{a}{A}\right)+\eta_4, && [a^2>b^2],\\
t &= \frac{\eta_3}{A^2+a^2-b^2}\exp\left[\frac{A}{d_2}\ln\frac{a\tan\bar{x}+d_3}{a\tan\bar{x}+d_4}\right]\left(\frac{A+b\cos x}{a+b\sin x}+\frac{a}{A}\right)+\eta_4, && [a^2<b^2],\\
t &= \frac{\eta_3}{A^2}\exp\left[\frac{A}{a}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\right]\left(\frac{A+b\cos x}{a(1+\sin x)}+\frac{a}{A}\right)+\eta_4, && [a^2=b^2],
\end{aligned}
\tag{9}
$$

where η_4 is the new integration constant, $\sin x \neq -1$. As Eq. (1) is a 3rd-order vector differential equation, which describes the motion in the maneuver plane, its complete integration would require to find *six independent first integrals with six scalar integration constants* of motion in the maneuver plane. So far, four independent first integrals and four new constants have been found above, that is η_1 and η_2 in Eq. (2), η_3 in Eqs. (5) and η_4 in Eqs. (9). Eqs. (5) and Eqs. (9) also represent the general solution of Eqs. (2) with constants η_3, η_4, c_1 and c_2.

Integrals for position vector components

It can be shown that the magnitude of the velocity vector and angle between the velocity vector and the local horizon are not enough to uniquely determine the position of the aircraft in the maneuver plane. If $\mathbf{v} = v\cos\theta\,\mathbf{i}_x - v\sin\theta\,\mathbf{i}_z$ and $\mathbf{r} = \rho\,\mathbf{i}_x - h\,\mathbf{i}_z$, where ρ and h are the aircraft horizontal and vertical coordinates (crossrange and downrange respectively), then the equation $\mathbf{v} = \dot{\mathbf{r}}$ written in terms of its components yields

$$
\dot\rho = v\cos\theta, \qquad \dot h = v\sin\theta. \tag{10}
$$

Noting that $\theta = x - \pi/2$, $\dot\rho = \mathrm{d}\rho/\mathrm{d}x\,\mathrm{d}x/\mathrm{d}t$ and $\dot h = \mathrm{d}h/\mathrm{d}x\,\mathrm{d}x/\mathrm{d}t$, one can rewrite Eqs. (10) as

$$
\frac{\mathrm{d}\rho}{\mathrm{d}x} = v\frac{\mathrm{d}t}{\mathrm{d}x}\sin x, \qquad \frac{\mathrm{d}h}{\mathrm{d}x} = -v\frac{\mathrm{d}t}{\mathrm{d}x}\cos x, \tag{11}
$$

Integration of Eqs. (11) yields the aircraft coordinates ρ and h:

$$
\begin{aligned}
\rho(x) &= P\exp\left[\frac{4A}{d_1}\arctan\frac{a\tan\bar{x}+b}{d_1}\right]+\eta_5, && [a^2>b^2],\\
\rho(x) &= P\exp\left[\frac{2A}{d_2}\ln\frac{a\tan\bar{x}+d_3}{a\tan\bar{x}+d_4}\right]+\eta_5, && [a^2<b^2]\\
\rho(x) &= P\exp\left[\frac{2A}{a}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\right]+\eta_5, && [a^2=b^2],
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
h(x) &= Q\exp\left[\frac{4A}{d_1}\arctan\frac{a\tan\bar{x}+b}{d_1}\right]+\eta_6, && [a^2>b^2],\\
h(x) &= Q\exp\left[\frac{2A}{d_2}\ln\frac{a\tan\bar{x}+d_3}{a\tan\bar{x}+d_4}\right]+\eta_6, && [a^2<b^2],\\
h(x) &= Q\exp\left[\frac{2A}{a}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\right]+\eta_6, && [a^2=b^2],
\end{aligned}
\tag{13}
$$

where η_5 and η_6 are the new integration constants, $\bar{x} = x/2$, and P and Q are known functions of x. Eqs. (12) and (13) represent the first integrals of Eqs. (10) (and Eqs. (2)), and allow us to determine the aircraft's horizontal and vertical cartesian coordinates (crossrange and downrange) in the maneuver plane. The first integrals presented in Eqs. (2), (5), (9), (12) and (13) with constants η_i, $i = 1, \ldots, 6$ represent the general integral of Eq. (2). Any point on the trajectory can be considered as a target point and the constants can be chosen to achieve this point. Consequently, the targeting problem can be solved at any desired point thereby providing a foundation for the development and design of the targeting and guidance schemes.

4. Expressions for Thrust and Angle of Attack

As mentioned above, the assumptions (a)–(c) can be justified and validated by analyzing the thrust, drag and list accelerations using the solutions for altitude and velocity, and by comparing the results for the angle of attack and the thrust to existing ranges of these quantities [2, 7, 8]. The assumptions (a) and (b) mean that

$$\frac{T\cos\bar{\alpha} - D}{m} = c_1 = \text{const}, \quad \frac{(T\sin\bar{\alpha} + L)\cos\phi}{m} = c_2 = \text{const}, \tag{14}$$

where [1]

$$\phi = \text{const}, \quad m = \frac{W}{g_0}, \quad D = \frac{1}{2}C_D\rho_a S v^2, \quad L = \frac{1}{2}C_L\rho_a S v^2,$$

and it is assumed that $\phi \neq \pi/2 + k\pi$, $k = 0, 1, 2 \ldots$, the drag and lift coefficients, C_D and C_L can be computed according to Ref. [2] and the air density, ρ_a is changed according to the exponential law. From Eq. (14) one can obtain

$$(mc_1 + D)\sin\bar{\alpha} - \left(\frac{mc_2}{\cos\phi} - L\right)\cos\bar{\alpha} = 0. \tag{15}$$

Eq. (15) is a transcendental equation and solvable for $\alpha = \alpha(h, v)$ only by numerical schemes. Once $\alpha = \alpha(h, v)$ is determined, then the thrust can be computed as

$$T = \sqrt{(mc_1 + D)^2 + \left(\frac{mc_2}{\cos\phi} - L\right)^2}. \tag{16}$$

5. Illustrative Example

Consider the example of a flight simulation using the analytical solutions presented above for $a^2 > b^2$. One can compute magnitude of velocity vector, time, altitude and downrange in terms of the angle θ which is assumed to satisfy the inequality: $\theta_0 \leq \theta \leq \theta_1$. The following values have been accepted: $t_0 = 0.0$, $v_0 = 250$, $g_0 = 9.8$, $b = -9.8$, $\rho_0 = 28500$, $h_0 = 7000$, $W = 5000$, $a = 10.0$, $A \in [0.1, 0.5]$, $\theta_0 = -10°$, $\Delta\theta = 65°$. The results of the simulations for this case are illustrated in figures 2–4. The plots of the angle of attack and the thrust vs magnitude of velocity are presented in figures 2 and 3. These figures show that when the velocity magnitude is decreased in the beginning of the simulation for $\sim$ 10 ft/s, α is increased for $\sim 0.5°$ and T is decreased for $\sim$ 45 lbs. Analysis has also shown that an increase in values of W leads to the higher values for α and T, and also T is increased together with altitude and Mach number, computed as $M = v/v_a$, where v_a is the speed of sound [2]. From Fig. 4 it can also be seen that each additional value 0.1 to A would yield an increase of the final altitude by only 100 m.

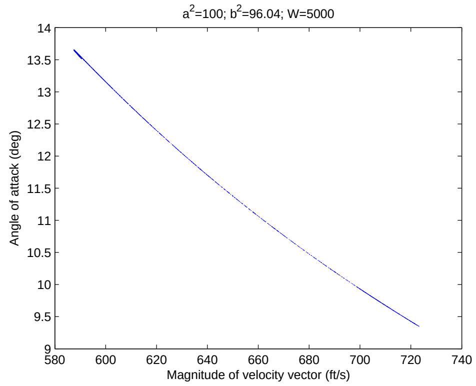

Figure 2. Angle of attack vs magnitude of velocity

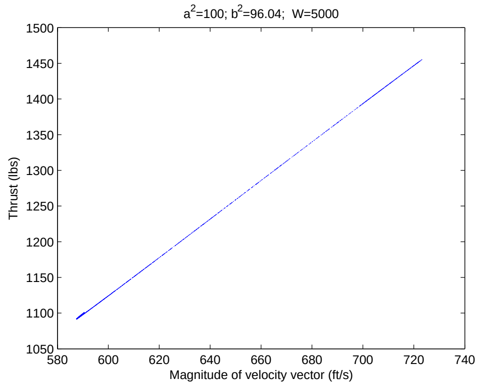

Figure 3. Propulsive thrust vs magnitude of velocity

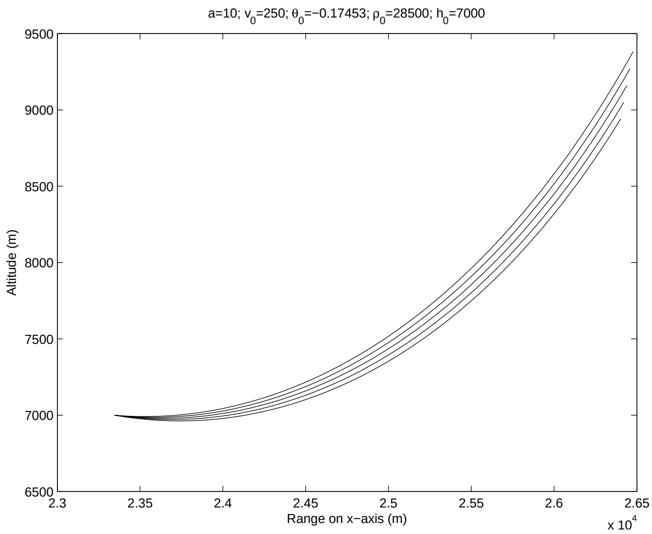

Figure 4. Downrange vs altitude

6. Computation of Thrust with Stabilization of Constraints

Let the aircraft trajectory be given in the form $h = f(\rho)$. Then the equations of a programmed constraint and its derivative are $y = h - f(\rho)$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = \nu_y$ with [9]

$$\nu_y = -\nu\big(\sin\theta + f'(\rho)\cos\theta\big), \quad f'(\rho) = \frac{\mathrm{d}f(\rho)}{\mathrm{d}\rho}, \tag{17}$$

The equations of the perturbed constraint can be given as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \nu_y, \quad \frac{\mathrm{d}\nu_y}{\mathrm{d}t} = K(y, \nu_y, \rho, h, \nu, \theta), \quad K(0, 0, \rho, h, \nu, \theta) = 0, \tag{18}$$

and in particular, $\frac{\mathrm{d}\nu_y}{\mathrm{d}t} = -cy - k\nu_y$. Eqs. (2), (17) and (18) yield the equation with respect to the thrust:

$$\frac{\mathrm{d}\nu_y}{\mathrm{d}t} = -\frac{\mathrm{d}\nu}{\mathrm{d}t}\big(\sin\theta + f'(\rho)\cos\theta\big) + \nu\big(f'(\rho)\sin\theta - \cos\theta\big)\frac{\mathrm{d}\theta}{\mathrm{d}t} - \nu f''(\rho)\cos\theta\frac{\mathrm{d}\rho}{\mathrm{d}t},$$

$$f''(\rho) = \frac{\mathrm{d}f'(\rho)}{\mathrm{d}\rho}.$$

After substitution of these expressions into Eqs. (18), which provides

$$\frac{\mathrm{d}\nu_y}{\mathrm{d}t} = -\frac{\mathrm{d}\nu}{\mathrm{d}t}\big(\sin\theta + f'(\rho)\cos\theta\big) + \nu\big(f'(\rho)\sin\theta - \cos\theta\big)\frac{\mathrm{d}\theta}{\mathrm{d}t} - \nu f''(\rho)\cos\theta\frac{\mathrm{d}\rho}{\mathrm{d}t},$$

one can obtain the equation for determination of T: $T = A/B$, where

$$\begin{aligned} A = {} & mK - \big(\sin\theta + f'(\rho)\cos\theta\big)(mg\sin\theta + D) - \\ & - \big(f'(\rho)\sin\theta - \cos\theta\big)(-mg\cos\theta + L\cos\varphi) + \\ & + m\nu^2 f''(\rho)\cos^2\theta, \\ B = {} & -\big(\sin\theta + f'(\rho)\cos\theta\big)\cos\alpha + \big(f'(\rho)\sin\theta - \cos\theta\big)\cos\varphi\sin\alpha. \end{aligned}$$

Assume now that $K = -cy - k\nu_y = c\big(f(\rho) - h\big) + k\nu\big(\sin\theta + f'(\rho)\cos\theta\big)$, with $c > 0$, $k > 0$. Then one can show that

$$\begin{aligned} A = {} & -mch + mk\nu\sin\theta - mg + D\sin\theta + L\cos\varphi\cos\theta + \\ & + mcf(\rho) + (mk\nu\cos\theta - D\cos\theta - L\cos\varphi\sin\theta)f'(\rho) + \\ & + m\nu^2 f''(\rho)\cos^2\theta, \\ B = {} & -\cos\alpha\sin\theta - \sin\alpha\cos\varphi\cos\theta + (\cos\alpha\cos\theta + \sin\alpha\cos\varphi\sin\theta)f'(\rho). \end{aligned}$$

7. Conclusions

The general integral of the aircraft's kinematic and dynamic equations of motion in the non-steady flight conditions has been obtained. These equations represent the 3rd order vector differential equation, the general integral of which consists of six independent first integrals with six corresponding constants. All integrals are expressed in elementary and transcendental functions in terms of the flight path angle. The applications may also include the flight trajectories in the transonic, low and high supersonic conditions. These results can find potential applications in the design of on-board targeting and guidance schemes.

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Общий интеграл для одного класса нестационарных атмосферных летательных аппаратов и приложения для анализа траекторий

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Представлено полное интегрирование уравнений кинематики и динамики движения самолёта. Рассмотрены различные применения полученных интегралов к анализу траекторий. Уравнения динамики получены в предположении, что разница между ускорением, вызванным аэродинамической подъёмной силой, и ускорением тяги не меняется, направление курса самолёта относительно продольной оси остаётся постоянным, угол атаки и угол скольжения равны нулю. Общее решение состоит из шести первых интегралов уравнений движения и описывает множество траекторий в вертикальной плоскости. Показано, что уравнения динамики могут быть получены и проинтегрированы в замкнутой форме при более общих предположениях. Рассматривается задача определения величины тяги, соответствующей данной траектории, заданной уравнением связи. Строится уравнение возмущений связи, имеющее асимптотически устойчивое тривиальное решение. Предлагаемый метод построения интегралов может быть использован в задачах построения траекторий космических аппаратов, ракет и спускаемых аппаратов, а также при проектировании бортовых систем целеуказания и наведения. Приводится иллюстрационный пример.

Ключевые слова: аналитическое интегрирование, общий интеграл, аналитические решения, нелинейная модель самолёта, программные связи

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