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Properties of Wigner Distribution Functions Applied to Quantum Mechanics

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An operational model of quantum measurements was presented befor. In order to obtain constructive theoretical results from this model there is a need to define previously not described properties of Wigner distribution functions. The report contains the proof of these properties. Multidimensional generalization and relationships with different conventions of the Fourier transform were described.

Key words and phrases: Wigner distribution functions, operational model of quantum measurement, quantum distribution function.

1. Introduction

Most of the works on quantum mechanics describes fundamental researches and update parameters of the many-particle systems. One of the important problems in this domain is the quantum measurement problem. There are significant works made by the experimenters like Braginskiy [1, 2] and others. Significant contribution to theoretical justification of the problem was made by Holevo [3], Helstrom [4], and by others. This report does not contain the discussion of a philosophical aspect of the problem.

The report is based on the operational model of a quantum-mechanical measurement of observables [5–7]. Practical realization of the model is possible according to the works of Kuryshkin [8] and Wodkiewicz [9]. According to this model, we can describe qunatum measurement procedure as an interaction of a quantum object presented by the Wigner distribution function [10] and the measurement instrument with the associated Wigner distribution function. Results of the quantum measurement is convolution of this two functions - a positively defined functions corresponding to the physical picture of the world. This brings us to the formulation of the Kuryshkin–Wodkiewicz [7] quantization rule. Formulation of this new quantization rule requires existence of some previously unmentioned certain mathematical properties of the Wigner distribution functions.

2. Properties of Wigner Distribution Functions Applied to Quantum Mechanics

We define the Wigner distribution functions (WDF) in the form they were first introduced in the original work of Wigner [10]. In the simplest case the wave function in the coordinate representation of the physical system in configuration space with only one degree of freedom can be described by the WDF

$$W_{\varphi}(q,p) = \frac{1}{\pi\hbar}\int\limits_{\mathbb{R}} \varphi^{*}(q+q')\,\varphi(q-q')\,e^{\frac{2ipq'}{\hbar}}dq', \tag{1}$$

where $\varphi^{*}(q)$ — complex conjugated function of $\varphi(q)$. In case of generalization of the formula (1) for the physical systems with n degrees of freedom we can define WDF as

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$$W_\varphi(q,p) = (\pi\hbar)^{-n} \int \ldots \int \varphi^*(q_1 + q_1' + \ldots + q_n + q_n') \times \\ \times \varphi(q_1 - q_1' + \ldots + q_n - q_n')\, e^{\frac{2i(p_1 q_1' + \ldots + p_n q_n')}{\hbar}} dq_1' \ldots dq_n'. \quad (2)$$

We can introduce the WDF not only in the coordinate representation, but in the momentum representation as well:

$$W_\varphi(q,p) = \frac{1}{\pi\hbar} \int \varphi^*(p+p')\, \varphi(p-p')\, e^{-\frac{2ip'q}{\hbar}} dp'. \quad (3)$$

It can be shown that the two definitions (1) and (3) are equivalent up to the constant multiplier depending on the form of the Fourier transform.

Proposition 1. If the Fourier transform is defined in the form

$$\tilde{\varphi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \varphi(q)\, e^{-\frac{iqp}{\hbar}} dq, \quad (4)$$

then

$$W_\varphi(q,p) = 2\pi\hbar \cdot W_{\tilde{\varphi}}(p,q). \quad (5)$$

Proof. We define the Wigner distribution function as follows:

$$W_\varphi(q,p) = \frac{1}{\pi\hbar} \int \varphi^*(q+q')\, \varphi(q-q')\, e^{\frac{2ipq'}{\hbar}} dq', \quad (6)$$

here

$$\varphi^*(q+q') = \frac{1}{\sqrt{2\pi\hbar}} \int \tilde{\varphi}^*(p')\, e^{-ip'(q+q')/\hbar} dp', \quad (7)$$

$$\varphi(q-q') = \frac{1}{\sqrt{2\pi\hbar}} \int \tilde{\varphi}(p'')\, e^{ip''(q-q')/\hbar} dp''. \quad (8)$$

Let's substitute equations (7) and (8) into the equation (6):

$$W_\varphi(q,p) = \frac{1}{\pi\hbar} \int \left(\frac{1}{\sqrt{2\pi\hbar}} \int \tilde{\varphi}^*(p')\, e^{-ip'(q+q')/\hbar} dp' \right) \times \\ \times \left(\frac{1}{\sqrt{2\pi\hbar}} \int \tilde{\varphi}(p'')\, e^{ip''(q-q')/\hbar} dp'' \right) e^{\frac{2ipq'}{\hbar}} dq'. \quad (9)$$

After changing the order of integration in the equation (9):

$$W_\varphi(q,p) = \frac{1}{\pi\hbar} \int \tilde{\varphi}^*(p') \int \frac{1}{2\pi\hbar} \int e^{-iq'(p''+p'-2p)/\hbar} dq' \times \\ \times \tilde{\varphi}(p'')\, e^{-i(-p''+p')q/\hbar} dp'' dp' \quad (10)$$

the integral $\frac{1}{2\pi\hbar} \int e^{-iq'(p''+p'-2p)/\hbar} dq'$ is equal to the delta function:

$$\frac{1}{2\pi\hbar} \int e^{-iq'(p''+p'-2p)/\hbar} dq' = \delta(p''+p'-2p).$$

Thus we can rewrite the equation (10) in the form

$$W_\varphi(q,p) = \frac{1}{\pi\hbar} \int \tilde{\varphi}^*(p') \int e^{-iq(-p''+p')/\hbar} \delta(p''+p'-2p)\, \tilde{\varphi}(p'')\, dp'' dp'.$$

Using the convolution property of the delta function:

$$\frac{1}{2\pi\hbar}\int e^{-iq(-p''+p')/\hbar}\tilde{\varphi}(p'')\,\delta(p''-(2p-p'))\,dp''=e^{-2iq(p'-p)/\hbar}\tilde{\varphi}(2p-p').$$

We can see that

$$W_\varphi(q,p)=\frac{1}{\pi\hbar}\int\tilde{\varphi}^*(p')\,(2\pi\hbar)\,e^{-2iq(p'-p)/\hbar}\tilde{\varphi}(2p-p')\,dp'. \tag{11}$$

After changing the variables $p'''=p'-p$ $(dp'\to dp''')$ in the equation (11):

$$W_\varphi(q,p)=2\pi\hbar\left(\frac{1}{\pi\hbar}\int\tilde{\varphi}^*(p+p''')\,\tilde{\varphi}(p-p''')\,e^{-2iqp'''/\hbar}dp'''\right)$$

finally, we can see that $W_\varphi(q,p)=2\pi\hbar\cdot W_{\tilde{\varphi}}(p,q)$. □

There are other popular forms of the Fourier transform and we can define this property for them as well:

Proposition 2. If the Fourier transform is defined in the form

$$\tilde{\varphi}(p)=\int\varphi(q)\,e^{-\frac{iqp}{\hbar}}dq, \tag{12}$$

then

$$W_\varphi(q,p)=W_{\tilde{\varphi}}(p,q). \tag{13}$$

Proposition 3. If the Fourier transform is defined in the form

$$\tilde{\varphi}(p)=\int\varphi(q)\,e^{-2\pi iqp}dq, \tag{14}$$

then

$$W_\varphi(q,p)=W_{\tilde{\varphi}}(p,q). \tag{15}$$

We can easily modify this proposition for the case of the physical systems with n degrees of freedom in configuration space $\mathbb{R}^n$:

Proposition 4. If the Fourier transform is defined in the form

$$\tilde{\varphi}(\vec{p})=\frac{1}{(2\pi\hbar)^{n/2}}\int\limits_{\mathbb{R}^n}\varphi(\vec{q})\,e^{-\frac{i\vec{q}\vec{p}}{\hbar}}d\vec{q}, \tag{16}$$

then

$$W_\varphi(q,p)=(2\pi\hbar)^n\cdot W_{\tilde{\varphi}}(p,q). \tag{17}$$

Proof. For the n-dimensional physical systems we can write down WDF in the form:

$$W_\varphi(\vec{q},\vec{p})=\frac{1}{(\pi\hbar)^n}\int\limits_{\mathbb{R}^n}\varphi^*\left(\vec{q}+\vec{q'}\right)\varphi\left(\vec{q}-\vec{q'}\right)e^{\frac{2i\vec{p}\vec{q'}}{\hbar}}d\vec{q'}, \tag{18}$$

where

$$\varphi^*\left(\vec{q}+\vec{q'}\right)=\frac{1}{(2\pi\hbar)^{n/2}}\int\limits_{\mathbb{R}^n}\tilde{\varphi}^*\left(\vec{p'}\right)e^{-i\vec{p'}\left(\vec{q}+\vec{q'}\right)/\hbar}d\vec{p'}, \tag{19}$$

$$\varphi\left(\vec{q}-\vec{q'}\right)=\frac{1}{(2\pi\hbar)^{n/2}}\int\limits_{\mathbb{R}^n}\tilde{\varphi}\left(\vec{p''}\right)e^{i\vec{p''}\left(\vec{q}-\vec{q'}\right)/\hbar}d\vec{p''}. \tag{20}$$

Let's substitute (19) and (20) into the equation (18), then

$$W_\varphi(\vec{q},\vec{p}) = \frac{1}{(\pi\hbar)^n}\int\limits_{\mathbb{R}^n}\left(\frac{1}{(2\pi\hbar)^{n/2}}\int\limits_{\mathbb{R}^n}\tilde{\varphi}^*\left(\vec{p'}\right)e^{-i\vec{p'}\left(\vec{q}+\vec{q'}\right)/\hbar}d\vec{p'}\right)\times$$
$$\times\left(\frac{1}{(2\pi\hbar)^{n/2}}\int\limits_{\mathbb{R}^n}\tilde{\varphi}\left(\vec{p''}\right)e^{i\vec{p''}\left(\vec{q}-\vec{q'}\right)/\hbar}d\vec{p''}\right)e^{\frac{2i\vec{p}\vec{q'}}{\hbar}}d\vec{q'}. \quad (21)$$

Changing the order of integration in the equation (21), we can see that

$$W_\varphi(\vec{q},\vec{p}) = \frac{1}{(\pi\hbar)^n}\int\limits_{\mathbb{R}^n}\tilde{\varphi}^*\left(\vec{p'}\right)\int\limits_{\mathbb{R}^n}\frac{1}{(2\pi\hbar)^{n/2}}\int\limits_{\mathbb{R}^n}e^{-i\vec{q'}\left(\vec{p''}+\vec{p'}-2\vec{p}\right)/\hbar}d\vec{q'}\times$$
$$\times\,\tilde{\varphi}\left(\vec{p''}\right)e^{-i(-\vec{p''}+\vec{p'})\vec{q}/\hbar}d\vec{p''}d\vec{p'}. \quad (22)$$

Here the integral $\frac{1}{(2\pi\hbar)^n}\int\limits_{\mathbb{R}^n}e^{-i\vec{q'}\left(\vec{p''}+\vec{p'}-2\vec{p}\right)/\hbar}d\vec{q'}$ can be replaced with the delta function:

$$\frac{1}{(2\pi\hbar)^n}\int\limits_{\mathbb{R}^n}e^{-i\vec{q'}\left(\vec{p''}+\vec{p'}-2\vec{p}\right)/\hbar}d\vec{q'} = \delta\left(\vec{p''}+\vec{p'}-2\vec{p}\right),$$

So the equation (22) will be

$$W_\varphi(\vec{q},\vec{p}) = \frac{1}{(\pi\hbar)^n}\int\limits_{\mathbb{R}^n}\tilde{\varphi}^*\left(\vec{p'}\right)\int\limits_{\mathbb{R}^n}e^{-i\vec{q}(-\vec{p''}+\vec{p'})/\hbar}\delta\left(\vec{p''}+\vec{p'}-2\vec{p}\right)\tilde{\varphi}\left(\vec{p''}\right)d\vec{p''}d\vec{p'}.$$

After using the convolution property of the delta function:

$$\frac{1}{(2\pi\hbar)^n}\int\limits_{\mathbb{R}^n}e^{-i\vec{q}(-\vec{p''}+\vec{p'})/\hbar}\tilde{\varphi}\left(\vec{p''}\right)\delta\left(\vec{p''}-\left(2\vec{p}-\vec{p'}\right)\right)d\vec{p''} = e^{-2i\vec{q}(\vec{p'}-\vec{p})/\hbar}\tilde{\varphi}\left(2\vec{p}-\vec{p'}\right).$$

This will lead us to the equation

$$W_\varphi(\vec{q},\vec{p}) = \frac{1}{(\pi\hbar)^n}\int\limits_{\mathbb{R}^n}\tilde{\varphi}^*\left(\vec{p'}\right)(2\pi\hbar)^n e^{-2i\vec{q}(\vec{p'}-\vec{p})/\hbar}\tilde{\varphi}\left(2\vec{p}-\vec{p'}\right)d\vec{p'}. \quad (23)$$

Changing of the variables $\vec{p'''} = \vec{p'}-\vec{p}$ $(d\vec{p'} \to d\vec{p'''})$ in the equation (23):

$$W_\varphi(\vec{q},\vec{p}) = (2\pi\hbar)^n\left(\frac{1}{(\pi\hbar)^n}\int\limits_{\mathbb{R}^n}\tilde{\varphi}^*\left(\vec{p}+\vec{p'''}\right)\tilde{\varphi}\left(\vec{p}-\vec{p'''}\right)e^{-2i\vec{q}\vec{p'''}/\hbar}d\vec{p'''}\right)$$

finally, we can see, that $W_\varphi(q,p) = (2\pi\hbar)^n\cdot W_{\tilde{\varphi}}(p,q)$. □

Considering physical systems with n degrees of freedom we can also define similar equalities for the other forms of the Fourier transform:

Proposition 5. If the Fourier transform is defined in the form

$$\tilde{\varphi}\left(\vec{p}\right)=\int\limits_{\mathbb{R}^n}\varphi\left(\vec{q}\right)e^{-\frac{i\vec{q}\vec{p}}{\hbar}}d\vec{q}, \tag{24}$$

then

$$W_{\varphi}\left(q,p\right)=W_{\tilde{\varphi}}\left(p,q\right). \tag{25}$$

Proposition 6. If the Fourier transform is defined in the form

$$\tilde{\varphi}\left(\vec{p}\right)=\int\limits_{\mathbb{R}^n}\varphi\left(\vec{q}\right)e^{-2\pi i\vec{q}\vec{p}}d\vec{q}, \tag{26}$$

then

$$W_{\varphi}\left(q,p\right)=\left(2\pi\hbar\right)^n\cdot W_{\tilde{\varphi}}\left(p,q\right). \tag{27}$$

There is a theorem introduced in [7] which states, that quantization rule of Kuryshkin-Wodkiewicz corresponds a continuous linear operator of the form $O_{\rho_2}\left(A\right)=O_w\left(A*W_{\rho_2}\right):$ $S\left(Q\right)\to S'\left(Q\right)$ to the distribution $A\in S'\left(T*Q\right)$. For this theorem to be true we need to define another property of the WDF.

Proposition 7. For the Wigner distribution function the statement is true

$$\mathrm{W}_{\varphi}\left(-q,p\right)=-W_{\varphi}\left(q,p\right). \tag{28}$$

Proof. Consider left part of the equality (28):

$$W_{\varphi}\left(-q,p\right)=\frac{1}{\pi\hbar}\int\varphi^*\left(-q+q'\right)\varphi\left(-q-q'\right)e^{\frac{2ipq'}{\hbar}}dq'$$

after changing the variable $q'\to-q''$:

$$W_{\varphi}\left(-q,p\right)=-\frac{1}{\pi\hbar}\int\varphi^*\left(-q-q''\right)\varphi\left(-q+q''\right)e^{-\frac{2ipq''}{\hbar}}dq''. \tag{29}$$

Here the Fourier transform is used as follow:

$$\varphi^*\left(-q-q''\right)=\frac{1}{\sqrt{2\pi\hbar}}\int\tilde{\varphi}^*\left(p'\right)e^{-ip'\left(-q-q''\right)/\hbar}dp', \tag{30}$$

$$\varphi\left(-q+q''\right)=\frac{1}{\sqrt{2\pi\hbar}}\int\tilde{\varphi}\left(p''\right)e^{ip''\left(-q+q''\right)/\hbar}dp''. \tag{31}$$

Let's substitute equations (30) and (31) into the equation (29), then

$$\begin{aligned}W_{\varphi}\left(q,p\right)=-\frac{1}{\pi\hbar}\int&\left(\frac{1}{\sqrt{2\pi\hbar}}\int\tilde{\varphi}^*\left(p'\right)e^{-ip'\left(-q-q''\right)/\hbar}dp'\right)\times\\&\times\left(\frac{1}{\sqrt{2\pi\hbar}}\int\tilde{\varphi}\left(p''\right)e^{ip''\left(-q+q''\right)/\hbar}dp''\right)e^{-\frac{2ipq'}{\hbar}}dq'=\\=-\frac{1}{\pi\hbar}\int\tilde{\varphi}^*\left(p'\right)&\int\frac{1}{2\pi\hbar}\int e^{-iq''\left(-p''-p'+2p\right)/\hbar}dq''\tilde{\varphi}\left(p''\right)e^{-i\left(p''-p'\right)q/\hbar}dp''dp'=\\=-\frac{1}{\pi\hbar}&\int\tilde{\varphi}^*\left(p'\right)\int e^{-iq\left(p''-p'\right)/\hbar}\delta\left(p''+p'-2p\right)\tilde{\varphi}\left(p''\right)dp''dp'=\\&=-\frac{1}{\pi\hbar}\int\tilde{\varphi}^*\left(p'\right)\left(2\pi\hbar\right)e^{2iq\left(p'-p\right)/\hbar}\tilde{\varphi}\left(2p-p'\right)dp'.\end{aligned} \tag{32}$$

If we change the variables $p''' = p' - p$ $(dp' \to dp''')$ in the equation (32), then:

$$W_\varphi(-q,p) = -2\pi\hbar\left(\frac{1}{\pi\hbar}\int\tilde{\varphi}^*(p+p''')\,\tilde{\varphi}(p-p''')\,e^{2iqp'''/\hbar}dp'''\right).$$

Finally, we can see that $W_\varphi(-q,p) = -2\pi\hbar\cdot W_{\tilde{\varphi}}(p,q) = -W_\varphi(q,p)$. □

We used Fourier transform in the form (4). It is easy to proof that the equality (28) will be the same no matter what form of the Fourier transform is used. Using the same scheme of proof we can proof more important properties of the WDF:

Proposition 8. For the Wigner distribution function the statement is true

$$W_\varphi(q,-p) = -W_\varphi(q,p). \tag{33}$$

Proposition 9. For the Wigner distribution function the statement is true

$$W_\varphi(-q,-p) = W_\varphi(q,p). \tag{34}$$

Generalization of the WDF properties (28), (33) and (34) for the systems with n degrees of freedom can be done easily: $W(-q,p) = (-1)^n W(q,p)$, $W(q,-p) = (-1)^n W(q,p)$, $W(-q,-p) = W(q,p)$.

3. Conclusion

For the first time the Wigner distribution functions were proposed in physics in 1932, and up to now they are used to study quantum corrections to classical statistical mechanics. According to the Weyl quantization procedure [11], the WDF determines quantum expectation values of the observables of an isolated quantum object. However this function fails to satisfy the property of the real distribution functions due to possible negative probability values. Though these probabilities are supposed to be responsible for the “non-classical” states, it is obvious, that they are not in common with the real process of quantum measurements.

Kuryshkin and Wodkiewicz proposed an alternative approach, which uses a positive-definite convolution of two WDF to describe the behavior of the composite system “object+measurement instrument”. This corresponds to the operational model of quantum measurements which associates the continuous linear pseudo-differential operators in the rigged Hilbert space to a classical observable from the class of tempered distributions. The which were made first calculations [12] for the hydrogen, lithium and sodium atoms based on this model showed good correspondence between the observed experimental data from NIST and modeled values.

4. Appendix

In the course of the discussion of the Wigner distribution function’s properties we applied several theorems from the mathematical physics. They describe connections between the Fourier transform and the inverse Fourier transform of different types. This section contain the proofs of these theorems according to [13]. Assume that φ is a function from space of tempered distributions: $\varphi\in\Im(\mathbb{R})$: $\sup\limits_x\left|x^p\varphi^{(k)}(x)\right| < \infty$ $\forall\, p,\, k$. Then the theorem is true:

Theorem 1. *If we define the Fourier transform so, that:*

$$\hat{\varphi}(y) = \frac{1}{\sqrt{2\pi\hbar}}\int\limits_{-\infty}^{+\infty}\varphi(x)\,e^{-iyx/\hbar}dx, \tag{35}$$

then $\forall\, x \in \mathbb{R}$ *is an inverse Fourier transform:*

$$\varphi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-\infty}^{+\infty} \hat{\varphi}(y)\, e^{iyx/\hbar} dy. \tag{36}$$

Proof. To prove the existence of the inverse Fourier transform we can use a modified method of L. Fejér (1904) used to describe Fourier series. Let's look at the right side of the equation (36):

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-n}^{n} \hat{\varphi}(y)\, e^{iyx/\hbar} \left(1 - \frac{|y|}{n}\right) dy =$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-n}^{n} \left[\frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-\infty}^{+\infty} \varphi(x')\, e^{-iyx'/\hbar} dx' \right] e^{iyx/\hbar} \left(1 - \frac{|y|}{n}\right) dy. \tag{37}$$

We can see that absolute value of the element of integration is integrable on the specified interval and that the result of integration is finite. Thus we can change the order of integration in (37):

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-\infty}^{+\infty} \int\limits_{-n}^{n} e^{iy(x-x')/\hbar} \left(1 - \frac{|y|}{n}\right) dy \frac{1}{\sqrt{2\pi\hbar}} \varphi(x')\, dx'. \tag{38}$$

Equation (38) can be modified using the formula

$$\frac{1}{2\pi\hbar} \int\limits_{-n}^{n} e^{iy(x-x')/\hbar} \left(1 - \frac{|y|}{n}\right) dy = \frac{1 - \cos\left(n(x-x')\right)}{n\pi (x-x')^2}. \tag{39}$$

The substitution (39) into (38) will result in

$$\frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-\infty}^{+\infty} 2\pi\hbar \lim_{n\to\infty} \frac{1 - \cos\left(n(x-x')\right)}{n\pi (x-x')^2} \frac{1}{\sqrt{2\pi\hbar}} \varphi(x')\, dx'. \tag{40}$$

We know that $\dfrac{1 - \cos nx}{n\pi x^2} \to \delta(x)$ if $n \to \infty$, and that's why we can modify (40):

$$\frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-\infty}^{+\infty} 2\pi\hbar\delta(x - x') \frac{1}{\sqrt{2\pi\hbar}} \varphi(x')\, dx' = \int\limits_{-\infty}^{+\infty} \delta(x - x')\, \varphi(x')\, dx' = \varphi(x).$$

Other forms of the Fourier transform can be obtained by simple substitution of the argument. For example, if we use the substitution $\hat{\varphi}(y) \to \sqrt{2\pi\hbar} \cdot \hat{\varphi}(y)$, the following theorem will be true:

Theorem 2. *If we define the Fourier transform so, that:*

$$\hat{\varphi}(y) = \int\limits_{-\infty}^{+\infty} \varphi(x)\, e^{-iyx/\hbar} dx, \tag{41}$$

then $\forall\, x \in \mathbb{R}$ *is an inverse Fourier transform:*

$$\varphi\left(x\right) = \frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-\infty}^{+\infty} \hat{\varphi}\left(y\right) e^{iyx/\hbar} dy. \tag{42}$$

If we modify the Fourier transform using the substitution $\hat{\varphi}\left(y\right) \to \sqrt{2\pi\hbar} \cdot \hat{\varphi}\left(2\pi y\right)$, then

Theorem 3. *If we define the Fourier transform so, that:*

$$\hat{\varphi}\left(y\right) = \int\limits_{-\infty}^{+\infty} \varphi\left(x\right) e^{-2\pi i y x} dx, \tag{43}$$

then $\forall\, x \in \mathbb{R}$ *is an inverse Fourier transform:*

$$\varphi\left(x\right) = \int\limits_{-\infty}^{+\infty} \hat{\varphi}\left(y\right) e^{2\pi i y x} dy. \tag{44}$$

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Свойства квантовой функции распределения Вигнера в применении к квантовой механике

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В процессе построения операциональной модели квантовых измерений возникла необходимость установить ряд ранее не описанных свойств квантовой функции распределения Вигнера. Данная работа посвящена доказательству этих свойств, так как они необходимы для получения ряда конструктивных теоретических результатов. Сделано обобщение на многомерный случай и показана зависимость от выбора формы записи преобразования Фурье.

Ключевые слова: квантовая функция распределения Вигнера, операциональная модель квантовых измерений, квантовая функция распределения.