Физика

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Simple Two-Oscillators Model of C. Bjerknes's Vibrating Spheres Problem

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The simple two-oscillators model is suggested to illustrate the phase dependent interaction of vibrating spheres in hydrodynamics (the C. Bjerknes's problem). The integrability of this model is shown. Correspondence with the Fermi–Bose statistics in quantum mechanics is underlined.

Key words and phrases: Bjerknes, hydrodynamics, vibrating spheres.

1. Introduction. The C. Bjerknes's problem

The eminent Norwegian physicist C.A. Bjerknes discovered in 1875 the analogy between the motion of bodies in hydrodynamics and that of charged bodies in electrodynamics [1]. He considered two spheres plunged into incompressible fluid and separated by the large distance L, with their radii R_1, $R_2 \ll L$ oscillating in accordance with the law:

$$R_1 = R_0 + \epsilon \cos \omega t, \quad R_2 = R_0 + \epsilon \cos(\omega t + \varphi). \tag{1}$$

Introducing the velocity potential ϕ of the fluid by the substitution $\mathbf{v} = - \bigtriangledown \phi$, one gets for the single sphere the well-known expression

$$\phi_1 = \frac{R^2}{r'} \dot{R} \tag{2}$$

equivalent to the Coulomb's potential in electrodynamics, where r' stands for the distance from the center of the first sphere.

The influence of the potential (2) on the second sphere in the vicinity of the latter one can be found as follows:

$$\phi_1 = \frac{R_1^2}{L + r\cos\vartheta} \dot{R}_1 \approx \frac{R_1^2}{L} \dot{R}_1 \left(1 - \frac{r}{L}\cos\vartheta\right), \tag{3}$$

where r, ϑ denote the spherical coordinates with the origin at the center of the second sphere. In the dipole approximation the potential (3) induces the following potential of the second sphere:

$$\phi_2 \approx \frac{R_2^2}{r} \dot{R}_2 - \frac{R_1^2 R_2^3}{2L^2 r^2} \dot{R}_1 \cos\vartheta. \tag{4}$$

Using the resulting potential $\phi \approx \phi_1 + \phi_2$ one deduces from the Bernoulli's law the pressure $p \approx \rho \partial_t \phi$ in the vicinity of the second sphere and the z–component of the force acting on it:

$$F_{2z} = - \oint\limits_{r=R_2} p \cos\vartheta \, \mathrm{d}S = \rho \frac{4\pi R_2^2}{3L^2} \left[R_2 \frac{\mathrm{d}}{\mathrm{d}t}(R_1^2 \dot{R}_1) + \frac{\mathrm{d}}{R_2^2 \mathrm{d}t} \left(R_1^2 R_2^3 \dot{R} \right) \right], \tag{5}$$

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where ρ signifies the density of the fluid. After time averaging one gets from (1) and (5)

$$\langle F_{2z} \rangle = -2\pi\rho \frac{R_0^4}{L^2} \epsilon^2 \omega^2 \cos\varphi. \tag{6}$$

As follows from (6), one finds the attraction of the spheres for $\varphi = 0$ and their repulsion for $\varphi = \pi$. Using (6) C. Bjerknes tried to interpret the Coulomb's interaction of charges via hydrodynamical picture. However, the other interpretation of this result is possible if one takes into account the quantum mechanical behavior of identical particles. Namely, the repulsion is typical for Fermions and the attraction—for Bosons. Thus, the Bjerknes's effect (6) appears to generate, within the scope of the hydrodynamical model, the spin—statistics correlation in quantum mechanics.

2. The two–oscillators model of the C. Bjerknes's effect

Now we intend to illustrate the Bjerknes's effect within the frame of the simple two–oscillators mechanical model. Let us consider the dynamical behavior of two identical oscillators with the proper frequency ω, the equations of motion reading:

$$\ddot{q}_1 + \omega^2 q_1 = g\left(\omega^2 q_1 q_2 + \dot{q}_1 \dot{q}_2\right), \tag{7}$$

$$\ddot{q}_2 + \omega^2 q_2 = -g\left(\omega^2 q_1 q_2 + \dot{q}_1 \dot{q}_2\right). \tag{8}$$

The motivation for choosing the special interaction of the oscillators in (7) and (8) stems from the structure of the Bjerknes's force (6). Really, for the small coupling constant g in (7) and (8) one gets in the first approximation:

$$q_1 \approx A_1 \sin(\omega t + \varphi_1), \quad q_2 \approx A_2 \sin(\omega t + \varphi_2),$$

and the force in the r.h.s. of (7) and (8) reduces to that of Bjerknes, i.e. to (6).

The pleasant feature of the dynamical system (7) and (8) is its integrability. To show this fact we introduce the new variables by putting

$$2Q = q_1 + q_2, \quad q = q_1 - q_2. \tag{9}$$

Substituting (9) into (7) and (8), one gets the separated Q–oscillator:

$$\ddot{Q} + \omega^2 Q = 0, \quad Q = A\sin(\omega t + \varphi),$$

and the resulting q–dynamics:

$$\ddot{q} + \omega^2 q = 2g\omega^2 A^2 - \frac{g}{2}\left(\omega^2 q^2 + \dot{q}^2\right). \tag{10}$$

The equation (10) can be integrated if one introduces the new variable

$$I = \frac{1}{2}\left(\omega^2 q^2 + \dot{q}^2\right) - 2\omega^2 A^2 \tag{11}$$

satisfying the equation

$$\dot{I} + gI\dot{q} = 0. \tag{12}$$

The equation (12) admits the evident integral of motion:

$$I\exp(gq) = I_0 = \text{const}. \tag{13}$$

From (11) and (13) one deduces the quadrature

$$t = \pm \int_{q_0}^{q} \mathrm{d}q \left[\omega^2(4A^2 - q^2) + 2I_0 \exp(-gq)\right]^{-1/2},$$

that illustrates the Bjerknes's effect in the limit of small coupling constant g.

References

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Простая двухосцилляторная модель задачи К. Бьеркнеса о пульсирующих шарах

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Предложена простая двухосцилляторная модель, иллюстрирующая зависящее от фазы взаимодействие пульсирующих шаров в гидродинамике (задача Бьеркнеса). Доказывается интегрируемость модели. Проводится соответствие с Ферми–Бозе–статистиками в квантовой механике.