UDC 537.8, 514.742, 514.743

Maxwell's Equations in Arbitrary Coordinate System

D. S. Kulyabov*, A. V. Korolkova*, V. I. Korolkov†

* *Telecommunication Systems Department*
† *Radiophysics Department*
Peoples' Friendship University of Russia
6, Miklukho-Maklaya str., Moscow, 117198, Russia

The article is devoted to application of tensorial formalism for derivation of different types of Maxwell's equations. The Maxwell's equations are written in the covariant coordinate-free and the covariant coordinate forms. Also the relation between vectorial and tensorial formalisms and differential operators for arbitrary holonomic coordinate system in coordinate form is given. The results obtained by tensorial and vectorial formalisms are verified in cylindrical and spherical coordinate systems.

Key words and phrases: Maxwell's equations, tensorial formalism, covariant coordinate-free form, covariant coordinate form.

1. Introduction

Problems of waveguide mathematical modelling sometimes need curvilinear coordinate system to be applied. The choice of specific coordinate system is defined by the cross-section of the waveguide.

Usually the description of waveguide model is based on Maxwell's equations in Cartesian coordinate system. With the help of vector transformation property Maxwell's equations are rearranged for sertain coordinate system (spherical or cylindrical). But in some problems, e.g. simulation of a heavy-particle accelerator, the waveguide may has the form of a cone or a hyperboloid. Another example of a waveguide with a complex form is the Luneberg lens, which has the form of a part of a sphere or a cylinder attached to a planar waveguide. Therefore in the case of a waveguide with a complex form the Maxwell's equations should be written in a arbitrary curvilinear coordinate system.

It's well established to apply vectorial formalism to Maxwell's equations. But in this case Maxwell's equations in a curvilinear coordinate system are lengthy. In [1] some preliminary work on tensorial formalism resulting in a more compact form of Maxwell's equations is made. The tensorial formalism has a mathematical apparatus which allows to use covariant coordinate-free form of Maxwell's equations. In this case the transition to a certain coordinate system may be done on the final stage of research writing down the results. But tensorial formalism can't be directly applied to Maxwell's equations because the relation between vectorial and tensorial formalisms should be proven before.

Different forms of Maxwell's equations are used in problems of finding Hamiltonian of electromagnetic field applied in variational integrator (particularly, symplectic integrator) construction. The main task is the fulfillment of the condition of symplectic structure conservation during equations discretization. The several forms of Maxwell's equations are used in electromagnetic field Hamiltonian derivation:

– 3-vectors;
– momentum representation (complex form is used);
– momentum representation.

As a summary the main goals of the article may be formulated: to show connection between vectorial and tensorial formalisms (section 2); to apply tensorial formalism for different forms of Maxwell's equations (section 4); to verify the obtained results by representing Maxwell's equations in spherical and cylindrical coordinate systems (section 5).

Received 1st December, 2011.

2. Connection Between Vectorial and Tensorial Formalisms

Let's use the abstract indices formalism introduced in [2] in application to tensor algebra. In [2] α is the abstract index, $\underline{\alpha}$ — a tensor component index. The usage of a component index in some expression means that some arbitrary basis is introduced in this equation and indices obey the Einstein rule of summation (the sum is taken over every numeric index which occurs in one term of the expression twice — top and bottom). Abstract indices have an organizing value.

Let's consider an arbitrary n-dimensional vector and space $V^{\bullet}$ conjugated to $V^{\bullet}$ space $V_{\bullet}$.

In tensorial formalism the basis is given in coordinate form:

$$\delta^{i}_{\underline{i}} = \frac{\partial}{\partial x^{\underline{i}}} \in V^{\bullet}, \quad \delta^{\underline{i}}_{i} = \mathrm{d}x^{\underline{i}} \in V_{\bullet}, \quad \underline{i} = \overline{1, n}.$$

In vectorial formalism the basis is given by elements with the length $\mathrm{d}s^{\underline{i}'}$ upon the corresponding coordinate:

$$\delta^{i}_{\underline{i}'} = \frac{\partial}{\partial s^{\underline{i}'}}, \quad \delta^{\underline{i}'}_{i} = \mathrm{d}s^{\underline{i}'}, \quad \underline{i}' = \overline{1, n}.$$

In tensor form:

$$\mathrm{d}s^2 = g_{\underline{i}\underline{j}} \mathrm{d}x^{\underline{i}} \mathrm{d}x^{\underline{j}}, \quad \underline{i}, \underline{j} = \overline{1, n}, \tag{1}$$

where $g_{\underline{i}\underline{j}}$ — metric tensor.

In vector form:

$$\mathrm{d}s^2 = g_{\underline{i}'\underline{j}'} \mathrm{d}s^{\underline{i}'} \mathrm{d}s^{\underline{j}'}, \quad \underline{i}', \underline{j}' = \overline{1, n}. \tag{2}$$

In the case of orthogonal basis, (2) has the form:

$$\mathrm{d}s^2 = g_{\underline{i}'\underline{i}'} \mathrm{d}s^{\underline{i}'} \mathrm{d}s^{\underline{i}'}, \quad \underline{i}' = \overline{1, n}. \tag{3}$$

let's express the vector basis through the tensor one:

$$\mathrm{d}s^{\underline{i}'} = h^{\underline{i}'}_{\underline{i}} \mathrm{d}x^{\underline{i}}, \quad \frac{\partial}{\partial s^{\underline{i}'}} = h^{\underline{i}}_{\underline{i}'} \frac{\partial}{\partial x^{\underline{i}}},$$

where $h^{\underline{i}'}_{\underline{i}}$, $h^{\underline{i}}_{\underline{i}'}$, $\underline{i}$, $\underline{i}' = \overline{1, n}$, — matrix of Jacobi.

For orthogonal basis from (3)

$$g_{\underline{i}\underline{i}} \mathrm{d}x^{\underline{i}} \mathrm{d}x^{\underline{i}} = g_{\underline{i}'\underline{i}'} h^{\underline{i}'}_{\underline{i}} h^{\underline{i}'}_{\underline{i}} \mathrm{d}x^{\underline{i}} \mathrm{d}x^{\underline{i}}, \quad \underline{i}, \underline{i}' = \overline{1, n}.$$

Let's introduce the notation (for orthogonal coordinate system)

$$\left(h_{\underline{i}}\right)^2 := h^{\underline{i}'}_{\underline{i}} h^{\underline{i}'}_{\underline{i}} = \frac{g_{\underline{i}\underline{i}}}{g_{\underline{i}'\underline{i}'}}, \quad h_{\underline{i}} := h^{\underline{i}'}_{\underline{i}} = \sqrt{\frac{g_{\underline{i}\underline{i}}}{g_{\underline{i}'\underline{i}'}}}, \quad \underline{i}, \underline{i}' = \overline{1, n}.$$

Variables $h_{\underline{i}}$ are called Lame coefficients [3, Vol. 1, p. 34–35].

Let's express vector $f^i \in V^{\bullet}$ by its components $f^{\underline{i}}$ in tensor $\delta^{i}_{\underline{i}}$ and vector $\delta^{i}_{\underline{i}'}$ basises:

$$f^i = f^{\underline{i}} \delta^{i}_{\underline{i}} = f^{\underline{i}} \frac{\partial}{\partial x^{\underline{i}}},$$

$$f^i = f^{\underline{i}'} \delta^{i}_{\underline{i}'} = f^{\underline{i}'} \frac{\partial}{\partial s^{\underline{i}'}} = f^{\underline{i}'} \frac{1}{h^{\underline{i}'}_{\underline{i}}} \frac{\partial}{\partial x^{\underline{i}}},$$

and then

$$f^{\underline{i}'} = f^{\underline{i}} h^{\underline{i}'}_{\underline{i}}, \quad \underline{i}, \underline{i}' = \overline{1, n}.$$

In the similar way, for covectors:

$$f_i = f_{\underline{i}} \delta^{\underline{i}}_i = f_{\underline{i}} \mathrm{d}x^{\underline{i}},$$
$$f_i = f_{\underline{i}'} \delta^{\underline{i}'}_i = f_{\underline{i}'} \mathrm{d}s^{\underline{i}'} = f_{\underline{i}'} h^{\underline{i}'}_{\underline{i}} \mathrm{d}x^{\underline{i}},$$

and then

$$f_{\underline{i}'} = f_{\underline{i}} \frac{1}{h^{\underline{i}'}_{\underline{i}}}, \quad \underline{i}, \underline{i}' = \overline{1, n}.$$

So the connection between tensorial and vectorial formalisms is proved.

3. Tensorial Notation of Differential Operators in Components

Let's present the differential operators in the components (for connections associated with metric).

The expression for gradient:

$$(\operatorname{grad} \varphi)_i = (\operatorname{grad} \varphi)_{\underline{i}} \delta^{\underline{i}}_i, \quad (\operatorname{grad} \varphi)_{\underline{i}} = \nabla_{\underline{i}} \varphi = \partial_{\underline{i}} \varphi, \quad \underline{i} = \overline{1, n}. \tag{4}$$

The variable φ is a scalar.

The expression for an arbitrary vector divergence $\vec{f} \in V^{\bullet}$ is:

$$\operatorname{div} \vec{f} = \nabla_i f^i = f^i_{,i} - \Gamma^i_{ji} f^j = f^i_{,i} - f^i \frac{(\sqrt{|g|})_{,i}}{\sqrt{|g|}} = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} f^i \right), \tag{5}$$

or in components:

$$\operatorname{div} \vec{f} = \frac{1}{\sqrt{|g|}} \partial_{\underline{i}} \left(\sqrt{|g|} f^{\underline{i}} \right), \quad \underline{i} = \overline{1, n}. \tag{6}$$

Variable g is $\det \left(g_{\underline{i}\underline{i}} \right)$, $\underline{i} = \overline{1, n}$.

Because of the nonnegativity of radical expression and because of $\mathbb{M}^4$ $g < 0$ in Minkowsky space let's use the following notation $|g|$.

The expression for rotor is valid only in $\mathbb{E}^3$ space:

$$\left(\operatorname{rot} \vec{f} \right)^i = \left[\vec{\nabla}, \vec{f} \right] = \left(\operatorname{rot} \vec{f} \right)^{\underline{i}} \delta^i_{\underline{i}}, \quad \left(\operatorname{rot} \vec{f} \right)^{\underline{i}} = e^{\underline{i}\underline{j}\underline{k}} \nabla_{\underline{j}} f_{\underline{k}}, \quad \underline{i}, \underline{j}, \underline{k} = \overline{1, 3}, \tag{7}$$

where $e^{\underline{i}\underline{j}\underline{k}}$ is the alternating tensor expressed by Levi–Civita simbol $\varepsilon^{\underline{i}\underline{j}\underline{k}}$:

$$e_{\underline{i}\underline{j}\underline{k}} = \sqrt{|g|} \varepsilon_{\underline{i}\underline{j}\underline{k}}, \quad e^{\underline{i}\underline{j}\underline{k}} = \frac{1}{\sqrt{|g|}} \varepsilon^{\underline{i}\underline{j}\underline{k}}, \quad \underline{i}, \underline{j}, \underline{k} = \overline{1, 3}.$$

From (5) for divergence and (4) for gradient one can get Laplacian:

$$\Delta \varphi = \nabla_i \left(\nabla^i \varphi \right) = \nabla_i \left(g^{ij} (\operatorname{grad} \varphi)_j \right) = \nabla_i \left(g^{ij} \partial_j \varphi \right) = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j \varphi \right). \tag{8}$$

4. Maxwell's Equations Presentation

Let's consider Maxwell's equations in CGS-system [4]:

$$\begin{gathered}
\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}; \\
\vec{\nabla} \cdot \vec{D} = 4\pi\rho; \\
\vec{\nabla} \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c}\vec{j}; \\
\vec{\nabla} \cdot \vec{B} = 0.
\end{gathered} \tag{9}$$

Here $\vec{E}$ and $\vec{H}$ — electric and magnetic intensities, $\vec{j}$ is the current density, ρ is the charge density, c is the light velocity.

4.1. Maxwell's Equations Covariant Form by 3-vectors

Let's express the equation (9) in the covariant form

$$\begin{gathered}
e^{ijk}\nabla_j E_k = -\nabla_0 B^i; \\
\nabla_i D^i = 4\pi\rho; \\
e^{ijk}\nabla_j H_k = \nabla_0 D^i + \frac{4\pi}{c} j^i; \\
\nabla_i B^i = 0.
\end{gathered} \tag{10}$$

Let's rewrite the expression (9) in the tensorial formalism components with the help of (7) and (6):

$$\begin{gathered}
\frac{1}{\sqrt{|g^{(3)}|}}\left[\partial_{\underline{j}} E_{\underline{k}} - \partial_{\underline{k}} E_{\underline{j}}\right] = -\frac{1}{c}\partial_t B^{\underline{i}}, \quad \underline{i},\, \underline{j},\, \underline{k} = \overline{1,3}, \\
\frac{1}{\sqrt{|g^{(3)}|}}\partial_{\underline{i}}\left(\sqrt{|g^{(3)}|} D^{\underline{i}}\right) = 4\pi\rho, \quad \underline{i} = \overline{1,3}, \\
\frac{1}{\sqrt{|g^{(3)}|}}\left[\partial_{\underline{j}} H_{\underline{k}} - \partial_{\underline{k}} H_{\underline{j}}\right] = -\frac{1}{c}\partial_t D^{\underline{i}} + \frac{4\pi}{c} j^{\underline{i}}, \quad \underline{i},\, \underline{j},\, \underline{k} = \overline{1,3}, \\
\frac{1}{\sqrt{|g^{(3)}|}}\partial_{\underline{i}}\left(\sqrt{|g^{(3)}|} B^{\underline{i}}\right) = 0, \quad \underline{i} = \overline{1,3}.
\end{gathered} \tag{11}$$

4.2. Maxwell's Equations Covariant Form by 4-vectors

Let's rewrite (9) with the help of electromagnetic field tensors $F_{\alpha\beta}$ and $G_{\alpha\beta}$ [5], [6, p. 256, 263–264]:

$$\nabla_\alpha F^{\alpha\beta} = \frac{4\pi}{c} j^\beta, \tag{12}$$

$$\nabla_\alpha G_{\beta\gamma} + \nabla_\beta G_{\gamma\alpha} + \nabla_\gamma G_{\alpha\beta} = 0, \tag{13}$$

$$F_{\underline{\alpha\beta}} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B^3 & B^2 \\ -E_2 & B^3 & 0 & -B^1 \\ -E_3 & -B^2 & B^1 & 0 \end{pmatrix}, \quad G_{\underline{\alpha\beta}} = \begin{pmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & -H^3 & H^2 \\ -D_2 & H^3 & 0 & -H^1 \\ -D_3 & -H^2 & H^1 & 0 \end{pmatrix},$$

$E_{\underline{i}}$, $H^{\underline{i}}$, $\underline{i} = \overline{1,3}$, — components of electric and magnetic fields intensity vectors; $D_{\underline{i}}$, $B^{\underline{i}}$, $\underline{i} = \overline{1,3}$, — components of vectors of electric and magnetic induction.

The equation (13) may be rewritten in a simpler form

$$\nabla_\alpha {}^*G^{\alpha\beta} = 0, \tag{14}$$

where the tensor ${}^*G^{\alpha\beta}$ dual conjugated to $G^{\alpha\beta}$ is introduced

$${}^*G^{\alpha\beta} = \frac{1}{2} e^{\alpha\beta\gamma\delta} G_{\gamma\delta}, \tag{15}$$

where $e^{\alpha\beta\gamma\delta}$ is the alternating tensor.

The ordered pair (E_i, B^i) $(F_{\alpha\beta} \sim (E_i, B^i))$ may be assigned to $F_{\alpha\beta}$ by following

$$F_{0\underline{i}} = E_i, \quad F_{\underline{ij}} = -B^{\underline{k}}, \text{substitution } P(\underline{i}, \underline{j}, \underline{k}) \text{ — is even.} \tag{16}$$

So the following expressions may be written

$$\begin{aligned} F_{\alpha\beta} &\sim (E_i, B^i), \quad F^{\alpha\beta} \sim (-E^i, B_i), \\ G_{\alpha\beta} &\sim (D_i, H^i), \quad G^{\alpha\beta} \sim (-D^i, H_i), \\ {}^*G_{\alpha\beta} &\sim (H_i, -D^i), \quad {}^*G^{\alpha\beta} \sim (-H^i, -D_i). \end{aligned} \tag{17}$$

4.3. Complex Form of Maxwell's Equations

The complex form of Maxwell's equations was considered by various authors [7, p. 40–42], [8]

Similar to (17) let's introduce correspondence between an ordered pair and a complex 3-vector

$$\begin{aligned} F^i &\sim (E^i, B^i), \quad F^i = E^i + \mathrm{i}B^i; \\ G^i &\sim (D^i, H^i), \quad G^i = D^i + \mathrm{i}H^i. \end{aligned} \tag{18}$$

Let's express intensity and induction by means of complex vectors

$$\begin{aligned} E^i &= \frac{F^i + \bar{F}^i}{2}, \quad B^i = \frac{F^i - \bar{F}^i}{2\mathrm{i}}, \\ D^i &= \frac{G^i + \bar{G}^i}{2}, \quad H^i = \frac{G^i - \bar{G}^i}{2\mathrm{i}}. \end{aligned} \tag{19}$$

Two complementary vectors

$$K^i = \frac{G^i + F^i}{2}, \quad L^i = \frac{\bar{G}^i - \bar{F}^i}{2}. \tag{20}$$

The expression (10) assumes the form

$$\begin{aligned} &\nabla_i (K^i + L^i) = 4\pi\rho; \\ -\mathrm{i}\nabla_0 (K^i - L^i) &+ e^{ijk} \nabla_j (K_k - L_k) = \mathrm{i}\frac{4\pi}{c} j^i. \end{aligned} \tag{21}$$

4.3.1. Complex Form of Maxwell's Equations in Vacuum

From $D^i = E^i$, $H^i = B^i$ and (20) it follows

$$K^i = E^i + \mathrm{i}B^i = F^i, \quad L^i = 0. \tag{22}$$

Then the equations (21) will have the form

$$\begin{aligned} \nabla_i F^i &= 4\pi\rho; \\ -\mathrm{i}\nabla_0 F^i + e^{ijk}\nabla_j F_k &= \mathrm{i}\frac{4\pi}{c} j^i. \end{aligned} \tag{23}$$

4.3.2. Complex Representation of Maxwell's Equations in Homogeneous Isotropic Space

In homogeneous isotropic space the following relations $D^i = \varepsilon E^i$, $\mu H^i = B^i$ (where ε — dielectric permittivity and μ — magnetic permeability) are correct.

The resulting expressions may be simplified as follows. In (23) we need the formal substitutions $c \to c' = \dfrac{c}{\sqrt{\varepsilon\mu}}$ (the speed of light in vacuum is substituted by the speed of light in medium) and $j^\alpha \to \dfrac{j^\alpha}{\sqrt{\varepsilon}}$. The result:

$$\begin{aligned} F^i &= \sqrt{\varepsilon} E^i + \mathrm{i}\frac{1}{\sqrt{\mu}} B^i, \\ \nabla_i F^i &= \frac{4\pi}{\sqrt{\varepsilon}}\rho; \\ e^{ijk}\nabla_j F_k &= \mathrm{i}\frac{4\pi\sqrt{\mu}}{c} j^i + \mathrm{i}\frac{\sqrt{\varepsilon\mu}}{c}\frac{\partial F^i}{\partial t}. \end{aligned} \tag{24}$$

4.4. Momentum Representation of Maxwell's Equations

Let's expand the vectors of electric and magnetic fields intensity in a wavevector Fourier series k^j, j — abstract index:

$$\begin{aligned} E^i(t, x^j) &= \frac{1}{\sqrt{(2\pi)^3}} \int \mathrm{d}^3 k_j E^i(t, k_j) \mathrm{e}^{\mathrm{i} k_j x^j}, \\ H^i(t, x^j) &= \frac{1}{\sqrt{(2\pi)^3}} \int \mathrm{d}^3 k_j H^i(t, k_j) \mathrm{e}^{\mathrm{i} k_j x^j}, \\ B^i(t, x^j) &= \frac{1}{\sqrt{(2\pi)^3}} \int \mathrm{d}^3 k_j B^i(t, k_j) \mathrm{e}^{\mathrm{i} k_j x^j}, \\ D^i(t, x^j) &= \frac{1}{\sqrt{(2\pi)^3}} \int \mathrm{d}^3 k_j D^i(t, k_j) \mathrm{e}^{\mathrm{i} k_j x^j}, \\ \rho(t, x^j) &= \frac{1}{\sqrt{(2\pi)}} \int \mathrm{d}^3 k_j \rho(t, k_j) \mathrm{e}^{\mathrm{i} k_j x^j}, \\ j^i(t, x^j) &= \frac{1}{\sqrt{(2\pi)^3}} \int \mathrm{d}^3 k_j j^i(t, k_j) \mathrm{e}^{\mathrm{i} k_j x^j}. \end{aligned} \tag{25}$$

Let's note that the vector components $E^i(t, x^j)$ and $E^i(t, k_j)$, (similarly: $H^i(t, x^j)$ and $H^i(t, k_j)$, $D^i(t, x^j)$ and $D^i(t, k_j)$, $B^i(t, x^j)$ and $B^i(t, k_j)$, $j^i(t, x^j)$ and $j^i(t, k_j)$) are used in different basises:

$$\begin{aligned} E^i(t, x^j) &= E^{\underline{i}}(t, x^j)\delta^i_{\underline{i}}, \\ E^i(t, k_j) &= E^{\underline{\hat{i}}}(t, k_j)\delta^i_{\underline{\hat{i}}}, \end{aligned}$$

where the basis $\delta^i_{\underline{\hat{i}}}$ is given according to the vector k_i. For all k_i the independent basis is defined and that is why one can use expressions under integral sign putting down

Maxwell's equations from (25):

$$\begin{aligned} \mathrm{i}\frac{1}{\sqrt{|g^{(3)}|}}\varepsilon^{ijk}k_j E_k(t,k_j) &= -\frac{1}{c}\partial_t B^i(t,k_j),\\ \mathrm{i}\frac{1}{\sqrt{|g^{(3)}|}}\varepsilon^{ijk}k_j H_k(t,k_j) &= \frac{1}{c}\partial_t D^i(t,k_j) + \frac{4\pi}{c}j^i(t,k_j),\\ \mathrm{i}k_i D^i(t,k_j) &= 4\pi\rho(t,k_j),\\ \mathrm{i}k_i B^i(t,k_j) &= 0. \end{aligned} \tag{26}$$

Because of the complex form of the resulting equations the complex form of Maxwell's equations (18) is recommended to use

$$\begin{aligned} F^i(t,x^j) &= \frac{1}{\sqrt{(2\pi)^3}}\int \mathrm{d}^3k_j F^i(t,k_j)\mathrm{e}^{\mathrm{i}k_j x^j},\\ G^i(t,x^j) &= \frac{1}{\sqrt{(2\pi)^3}}\int \mathrm{d}^3k_j G^i(t,k_j)\mathrm{e}^{\mathrm{i}k_j x^j},\\ \rho(t,x^j) &= \frac{1}{\sqrt{(2\pi)}}\int \mathrm{d}^3k_j \rho(t,k_j)\mathrm{e}^{\mathrm{i}k_j x^j},\\ j^i(t,x^j) &= \frac{1}{\sqrt{(2\pi)^3}}\int \mathrm{d}^3k_j j^i(t,k_j)\mathrm{e}^{\mathrm{i}k_j x^j}. \end{aligned} \tag{27}$$

Remark. In terms of classical electrodynamics vectors E^j, H^j, B^j, D^j decompositions in wavevector k^j Fourier series correspond to these vectors decomposition in momentum Fourier series in quantum mechanics. That is why the representation (26) may be considered as momentum representation.

4.5. Spinor Form of Maxwell's Equations

The tensor of electromagnetic field $F_{\alpha\beta}$ and its components $F_{\underline{\alpha}\underline{\beta}}$, $\underline{\alpha},\underline{\beta}=\overline{0,3}$ may be considered in spinor form [2, p. 153] (and similarly for $G_{\alpha\beta}$):

$$\begin{gathered} F_{\alpha\beta} = F_{AA'BB'};\\ F_{\underline{\alpha}\underline{\beta}} = F_{\underline{A}\underline{A'}\underline{B}\underline{B'}} g_{\underline{\alpha}}{}^{\underline{A}\underline{A'}} g_{\underline{\beta}}{}^{\underline{B}\underline{B'}}, \quad \underline{A},\underline{A'},\underline{B},\underline{B'}=\overline{0,1}, \quad \underline{\alpha},\underline{\beta}=\overline{0,3}, \end{gathered} \tag{28}$$

where $g_{\underline{\alpha}}{}^{\underline{A}\underline{A'}}$, $\underline{\alpha}=\overline{0,3}$, — Infeld–Van der Waerden symbols defined in real spinor basis $\varepsilon_{\underline{A}\underline{B}}$ in the following way [2, p. 161]:

$$g_{\underline{\alpha}}{}^{\underline{A}\underline{A'}} := g_{\underline{\alpha}}{}^{\alpha}\varepsilon_{A}{}^{\underline{A}}\varepsilon_{A'}{}^{\underline{A'}}, \quad g_{\underline{A}\underline{A'}}{}^{\underline{\alpha}} := g^{\underline{\alpha}}{}_{\alpha}\varepsilon^{A}{}_{\underline{A}}\varepsilon^{A'}{}_{\underline{A'}}, \tag{29}$$

$$\varepsilon_{\underline{A}\underline{B}} = \varepsilon_{\underline{A'}\underline{B'}} = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}, \quad \varepsilon_{\underline{A}}{}^{A}\varepsilon_{A}{}^{\underline{B}} = \varepsilon_{\underline{A}}{}^{\underline{B}} = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}. \tag{30}$$

Let's write Maxwell's equations using the spinors.

The tensor $F_{\alpha\beta}$ is real and antisymmetric, it can be represented in the form

$$F_{\alpha\beta} = \varphi_{AB}\varepsilon_{A'B'} + \varepsilon_{AB}\bar{\varphi}_{A'B'}, \tag{31}$$

where φ_{AB} is a spinor of electromagnetic field:

$$\varphi_{AB} := \frac{1}{2}F_{ABC'}{}^{C'} = \frac{1}{2}F_{AA'BB'}\varepsilon^{A'B'} = \frac{1}{2}F_{\alpha\beta}\varepsilon^{A'B'}.$$

Similarly

$$G_{\alpha\beta} = \gamma_{AB}\varepsilon_{A'B'} + \varepsilon_{AB}\bar{\gamma}_{A'B'}, \tag{32}$$

$$^{*}G^{\alpha\beta} = -\mathrm{i}\gamma^{AB}\varepsilon^{A'B'} + \mathrm{i}\varepsilon^{AB}\bar{\gamma}^{A'B'}. \tag{33}$$

Replacing in (12) abstract indices α by AA' and β by BB', we can write:

$$\nabla_{AA'}F^{AA'BB'} = \frac{4\pi}{c}j^{BB'}.$$

Using (31) we will get

$$\nabla^{AB'}\varphi_A^B + \nabla^{BA'}\varphi_{A'}^{B'} = \frac{4\pi}{c}j^{BB'}. \tag{34}$$

Similarly, from (14) and (33) it follows

$$\nabla^{A'B}\gamma_B^A - \nabla^{AB'}\bar{\gamma}_{B'}^{A'} = 0. \tag{35}$$

In so doing the system of Maxwell's equations can be written as

$$\begin{aligned} \nabla^{AB'}\varphi_A^B + \nabla^{BA'}\varphi_{A'}^{B'} &= \frac{4\pi}{c}j^{BB'}, \\ \nabla^{A'B}\gamma_B^A - \nabla^{AB'}\bar{\gamma}_{B'}^{A'} &= 0. \end{aligned} \tag{36}$$

The spinor form of Maxwell's equations system in vacuum can be written in the form of one equation [2, p. 385]:

$$\nabla^{AB'}\varphi_A^B = \frac{2\pi}{c}j^{BB'}. \tag{37}$$

The components of electromagnetic field spinor:

$$\varphi_{\underline{A}\,\underline{B}} = \frac{1}{2}F_{\underline{\alpha}\underline{\beta}}\varepsilon^{\underline{A'}\,\underline{B'}}g^{\underline{\alpha}}{}_{\underline{A}\,\underline{A'}}g^{\underline{\beta}}{}_{\underline{B}\,\underline{B'}}, \quad \underline{A}, \underline{A'}, \underline{B}, \underline{B'} = \overline{0,1}, \quad \underline{\alpha}, \underline{\beta} = \overline{0,3}.$$

Using the equations (29), (30) and notation $F_i = E_i - \mathrm{i}B^i$, we will get [2, p. 386]:

$$\varphi_{00} = \frac{1}{2}\left(F_1 - \mathrm{i}F_2\right), \quad \varphi_{01} = \varphi_{10} = -\frac{1}{2}F_3, \quad \varphi_{11} = -\frac{1}{2}\left(F_1 + \mathrm{i}F_2\right).$$

5. Maxwell's Equations Presentation in Some Coordinate Systems

5.1. Maxwell's Equations in Cylindric Coordinate System

Due to the standard ISO 31-11 the coordinates (x^1, x^2, x^3) are denoted as (ρ, φ, z). In order to avoid some collisions with charge density symbol ρ the following notation (r, φ, z) will be used.

The law of coordinate transition from Cartesian coordinates to cylindric ones:

$$\begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi, \\ z = z. \end{cases} \tag{38}$$

The law of coordinate transition from cylindric coordinates to Cartesian ones:

$$\begin{cases} r = \sqrt{x^2 + y^2}, \\ \varphi = \operatorname{arctg}\left(\dfrac{y}{x}\right), \\ z = z. \end{cases} \tag{39}$$

The metric tensor:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{40}$$

$$\sqrt{g} = r. \tag{41}$$

Lame coefficients: $h_1 \equiv h_r = 1, \quad h_2 \equiv h_\varphi = r, \quad h_3 \equiv h_z = 1.$
Maxwell's equations in cylindric coordinates (r, φ, z):

$$\begin{gathered} \frac{1}{r}\left[\partial_{\underline{j}} E_{\underline{k}} - \partial_{\underline{k}} E_{\underline{j}}\right] = -\frac{1}{c}\partial_t B^{\underline{i}}, \quad \underline{i}, \underline{j}, \underline{k} = \overline{1,3}, \\ \frac{1}{r}\left[\partial_{\underline{j}} H_{\underline{k}} - \partial_{\underline{k}} H_{\underline{j}}\right] = -\frac{1}{c}\partial_t D^{\underline{i}} + \frac{4\pi}{c} j^{\underline{i}}, \quad \underline{i}, \underline{j}, \underline{k} = \overline{1,3}, \\ \frac{1}{r}\partial_{\underline{i}}\left(r D^{\underline{i}}\right) = 4\pi\rho, \quad \underline{i} = \overline{1,3}, \\ \frac{1}{r}\partial_{\underline{i}}\left(r B^{\underline{i}}\right) = 0, \quad \underline{i} = \overline{1,3}. \end{gathered} \tag{42}$$

The final result after some rearrangements:

$$\begin{gathered} \frac{1}{r}\left[\partial_\varphi E_3 - \partial_z E_2\right] = -\frac{1}{c}\partial_t B^1, \\ \frac{1}{r}\left[\partial_z E_1 - \partial_r E_3\right] = -\frac{1}{c}\partial_t B^2, \\ \frac{1}{r}\left[\partial_r E_2 - \partial_\varphi E_1\right] = -\frac{1}{c}\partial_t B^3, \\ \frac{1}{r}\left[\partial_\varphi H_3 - \partial_z H_2\right] = -\frac{1}{c}\partial_t D^1 + \frac{4\pi}{c} j^1, \\ \frac{1}{r}\left[\partial_z H_1 - \partial_r H_3\right] = -\frac{1}{c}\partial_t D^2 + \frac{4\pi}{c} j^2, \\ \frac{1}{r}\left[\partial_r H_2 - \partial_\varphi H_1\right] = -\frac{1}{c}\partial_t D^3 + \frac{4\pi}{c} j^3, \\ \frac{1}{r} D^1 + \frac{\partial D^1}{\partial r} + \frac{\partial D^2}{\partial \varphi} + \frac{\partial D^3}{\partial z} = 4\pi\rho, \\ \frac{1}{r} B^1 + \frac{\partial B^1}{\partial r} + \frac{\partial B^2}{\partial \varphi} + \frac{\partial B^3}{\partial z} = 0. \end{gathered} \tag{43}$$

5.2. Maxwell's equations in Spherical Coordinate System

Due to standard ISO 31-11 coordinates (x^1, x^2, x^3) are denoted as (r, ϑ, φ).
The law of coordinate transition from Cartesian coordinates to spherical ones:

$$\begin{cases} x = r \sin\vartheta \cos\varphi, \\ y = r \sin\vartheta \sin\varphi, \\ z = r \cos\vartheta. \end{cases} \tag{44}$$

The law of coordinate transition from spherical coordinates to Cartesian ones:

$$\begin{cases} r = \sqrt{x^2+y^2+z^2}, \\ \vartheta = \arccos\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right) = \operatorname{arctg}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right), \\ \varphi = \operatorname{arctg}\left(\dfrac{y}{x}\right). \end{cases} \tag{45}$$

The metric tensor:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\vartheta \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{r^2} & 0 \\ 0 & 0 & \dfrac{1}{r^2\sin^2\vartheta} \end{pmatrix} \tag{46}$$

$$\sqrt{g} = r^2 \sin\vartheta. \tag{47}$$

Lame coefficients: $h_1 \equiv h_r = 1, \quad h_2 \equiv h_\vartheta = r, \quad h_3 \equiv h_\varphi = r\sin\vartheta$.
Maxwell's equations in spherical coordinates (r, ϑ, φ):

$$\begin{gathered} \frac{1}{r^2\sin\varphi}\left[\partial_{\underline{j}}E_{\underline{k}} - \partial_{\underline{k}}E_{\underline{j}}\right] = -\frac{1}{c}\partial_t B^{\underline{i}}, \quad \underline{i},\, \underline{j},\, \underline{k} = \overline{1,3}, \\ \frac{1}{r^2\sin\varphi}\left[\partial_{\underline{j}}H_{\underline{k}} - \partial_{\underline{k}}H_{\underline{j}}\right] = -\frac{1}{c}\partial_t D^{\underline{i}} + \frac{4\pi}{c}j^{\underline{i}}, \quad \underline{i},\, \underline{j},\, \underline{k} = \overline{1,3}, \\ \frac{1}{r^2\sin\varphi}\partial_{\underline{i}}\left(rD^{\underline{i}}\right) = 4\pi\rho, \quad \underline{i} = \overline{1,3}, \\ \frac{1}{r^2\sin\varphi}\partial_{\underline{i}}\left(rB^{\underline{i}}\right) = 0, \quad \underline{i} = \overline{1,3}. \end{gathered} \tag{48}$$

$$\begin{gathered} \frac{1}{r^2\sin\vartheta}\left[\partial_\vartheta E_3 - \partial_\varphi E_2\right] = -\frac{1}{c}\partial_t B^1, \\ \frac{1}{r^2\sin\vartheta}\left[\partial_\varphi E_1 - \partial_r E_3\right] = -\frac{1}{c}\partial_t B^2, \\ \frac{1}{r^2\sin\vartheta}\left[\partial_r E_2 - \partial_\vartheta E_1\right] = -\frac{1}{c}\partial_t B^3, \\ \frac{1}{r^2\sin\vartheta}\left[\partial_\vartheta H_3 - \partial_\varphi H_2\right] = -\frac{1}{c}\partial_t D^1 + \frac{4\pi}{c}j^1, \\ \frac{1}{r^2\sin\vartheta}\left[\partial_\varphi H_1 - \partial_r H_3\right] = -\frac{1}{c}\partial_t D^2 + \frac{4\pi}{c}j^2, \\ \frac{1}{r^2\sin\vartheta}\left[\partial_r H_2 - \partial_\vartheta H_1\right] = -\frac{1}{c}\partial_t D^3 + \frac{4\pi}{c}j^3, \\ \frac{2}{r}D^1 + \partial_r D^1 + \operatorname{ctg}\vartheta D^2 + \partial_\vartheta D^2 + \partial_\varphi D^3 = 4\pi\rho, \\ \frac{2}{r}B^1 + \partial_r B^1 + \operatorname{ctg}\vartheta B^2 + \partial_\vartheta B^2 + \partial_\varphi B^3 = 0. \end{gathered} \tag{49}$$

6. Conclusion

The main results of the article are:

1. It is shown that the usage of tensorial formalism instead of vectorial one for Maxwell's equations may simplify mathematical expressions (particularly in non-Cartesian coordinate systems).
2. The connection between tensorial and vectorial formalisms is shown.

3. The covariant coordinate representation of differential operators for holonomic coordinate systems is given.
4. It is shown how to use tensor formalism for different forms of Maxwell's equations.
5. Maxwell's equations are presented in covariant coordinate-free and covariant coordinate forms.
6. It is shown that the results obtained by tensorial and vectorial formalisms are the same for cylindrical and spherical coordinate systems.

Using tensorial formalism instead of vectorial one can simplify the form of equations and intermediate results in non-Cartesian coordinate systems due to well developed formalism of tensor analysis. The transition to vectorial formalism can be done at a final stage if necessary.

References

1. *Kulyabov D. S., Nemchaninova N. A.* Maxwell's Equations in Curvilinear Coordinates (in russian) // Bulletin of Peoples' Friendship University of Russia. Series "Mathematics. Information Sciences. Physics". — 2011. — No 2. — Pp. 172–179.
2. *Penrose R., Rindler W.* Spinors and Space-Time. Two-Spinor Calculus and Relativistic Fields. — Camgridge University Press, 1987. — Vol. 1, 472 p.
3. *Морс Ф. М., Фешбах Г.* Методы теоретической физики. — М.: Издательство иностранной литературы, 1960. [*Mors F. M., Feshbakh G.* Metodih teoreticheskoyj fiziki. — M.: Izdateljstvo inostrannoyj literaturih, 1960.]
4. *Васильев А. Н.* Классическая электродинамика. Краткий курс лекций. — С.-П.: БХВ-Петербург, 2010. [*Vasiljev A. N.* Klassicheskaya ehlektrodinamika. Kratkiyj kurs lekciyj. — S.-P.: BKhV-Peterburg, 2010.]
5. *Minkowski H.* Die Grundlagen für die electromagnetischen Vorgönge in bewegten Körpern // Math. Ann. — 1910. — No 68. — Pp. 472–525.
6. *Терлецкий Я. П., Рыбаков Ю. П.* Электродинамика: Учебное пособие для студентов физ. спец. университетов. — 2-е, перераб. издание. — М.: Высш. шк., 1990. — 352 с. [Terleckii Ya. P., Rybakov Yu. P. Ehlektrodinamika: Uchebnoe posobie dlya studentov fiz. spec. universitetov. — 2-e, pererab. izdanie. — M.: Vihssh. shk., 1990. — 352 s.]
7. *Stratton J. A.* Electromagnetic Theory. — MGH, 1941.
8. *Silberstein L.* Electromagnetische Grundgleichungen in bivectorieller Behandlung // Annalen der Physik. — 1907. — Vol. 22. — Pp. 579–586.

УДК 537.8, 514.742, 514.743

Уравнения Максвелла в произвольной системе координат

Д. С. Кулябов*, А. В. Королькова*, В. И. Корольков†

* *Кафедра систем телекоммуникаций*

† *Кафедра радиофизики*

Российский университет дружбы народов

ул. Миклуха-Маклая, д. 6, Москва, 117198, Россия

В работе продемонстрировано применение тензорного формализма для получения разных форм записи уравнений Максвелла. Получены уравнения Максвелла в ковариантной бескоординатной и ковариантной координатной формах. Предварительно установлена связь между векторным и тензорным формализмами, выписано координатное представление дифференциальных операторов для произвольных голономных систем координат. Проведена верификация результатов, полученных с помощью тензорного и векторного формализмов, на примере цилиндрической и сферической систем координат.

Ключевые слова: уравнения Максвелла, тензорный формализм, ковариантная бескоординатная форма, ковариантная координатная форма.