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The 8th Congress of the International Society for Analysis, its Applications, and Computation (ISAAC) was organized by Peoples' Friendship University of Russia, Division of Mathematics of the Russian Academy of Sciences, V. A. Steklov Institute of Mathematics of Russian Academy of Sciences and V. M. Lomonosov Moscow State University, and took place at Peoples' Friendship University of Russia, Moscow, through 22-27 August 2011.

The programme of the Congress included most of the topics of contemporary mathematical analysis, in particular, real, functional, complex analysis, operator theory, theory of ordinary differential equations, theory of partial differential equations, nonlinear analysis, optimization theory, variational analysis, approximation theory, applications of analysis (inverse problems, functional and difference equations, mathematics in medicine, stochastic analysis), teaching analysis at universities and schools, history of analysis.

Vol. 3 contains papers by the plenary speakers Z. Kruszewski, L. Pepe, and by the participants of Session VI (Teaching analysis at universities and schools) and of Session VII (History of analysis).

This volume would be first of all of interest for mathematicians interested in teaching analysis at different levels and history of analysis, but also for the general community of analysts because traditionally research in mathematics and teaching mathematics are done in parallel.

Prepared by the Organizing Committee of the 8th Congress of the ISAAC.

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THE PROBLEMS OF HIGHER EDUCATION IN POLAND IN THE CONTEXT OF TEACHING MATHEMATICS IN HIGH SCHOOLS

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AMS Mathematics Subject Classification: 97A40

Abstract. The contemporary man needs to be prepared for changes in the surrounding world. These expectations must be satisfied by an increasingly better education system.An extremely important, if not decisive, stage of education is elementary school. The errors committed at this stage may be irreparable. It is elementary school which shapes young people's attitude to education and to their curiosity of the world, their courage to pose questions, their willingness to search for the truth and the heart of the matter, their interest in reading or mathematics. High schools and high school final examinations are an intermediate stage. University education is the final stage. One of very common weaknesses of the majority of Polish high schools is very poor mathematical training that their graduates receive.Polish universities search for candidates who possess scientific abilities and talents. This phenomenon has been enhanced by the demand for education in the fields of study which contributes to an increase in the national economic growth of Poland. The supply of properly prepared university applicants presents a challenge to higher education in Poland. Important role play the examinations system. In the talk the main problems concerning the different aspects of mathematical education in Poland will be discussed in the context of teaching mathematics in high schools.

1 Introduction

Contemporary people need to be prepared for the changes in the surrounding world. This influences institutions responsible for preparing them for these changes. These expectations must be satisfied by an increasingly better education system.

Politicians in different countries, including Poland, wish to respond to the term "better education". Education in united Europe is not subject to unification, although there have been certain attempts to standardize this system. Each country may freely shape its education and examination systems. Yet, it needs to be taken into account that young Polish people are searching for jobs and competing for university places against their peers from other countries on the European market.

The structure of the Polish education system should increase their chances in this competition.

Formally, the description of education may be limited to the number of years during which a student receives institutionalized education. Thus it may be stated that the more years we spend at school, the better educated we are.

An extremely important, if not decisive, stage of education is elementary school. The errors committed at this stage may be beyond repair during further education stages. It is elementary school which shapes young people's attitude to education, their curiosity of the world, their courage to pose questions, their willingness to search for the truth and the heart of the matter, their interest in reading or mathematics.

High school and high school final examinations are another important stage. At this stage, all young people complete their general education. They learn about various areas of life, which are not discussed at further education stages, and frequently the knowledge acquired at high school will be conclusive for them. For example, if someone studies philosophy, then their knowledge of chemistry of physics will remain at the level acquired at high school. It is possible that a history student will never expand his or her knowledge of mathematics. Therefore high school is such an important stage of education, as it prepares its graduates for taking actions for the benefit of the society and themselves.

The final stage of formal education is a university course. A good higher school, in order to achieve its principal objective, i.e. preparing students for independent, active lives, often needs to correct the errors committed at previous stages of education.

Academic teachers become acquainted with the work quality of tens of high schools, and they are able to compile a regional map of the teaching quality in secondary education[1] [2].

The situation is better in the case of Mathematics students, although 25% of the class obtained the results below the world average. Students, future mathematics teachers for primary school, are not able to make independent decisions, to define mathematical concepts, to link different elements of knowledge, or to draw conclusions. The Institute for Educational Research provided a very notable example. The students were asked to solve a problem, i.e. to calculate the perimeter of a given figure 1. This task could have been solved by a junior high school student. However, more than half of examined students did not manage to do it.

[1]Z. Kruszewski *"The function and role of higher schools in the contemporary economic life"* [in:] Selected issues of mechanical engineering, Wyd. Politechniki Warszawskiej, Piock 2006.

[2]A. Grabarek "Our mathematic skills lag behind in Europe", DGP, no. 125 (3011), 30.06.2011.

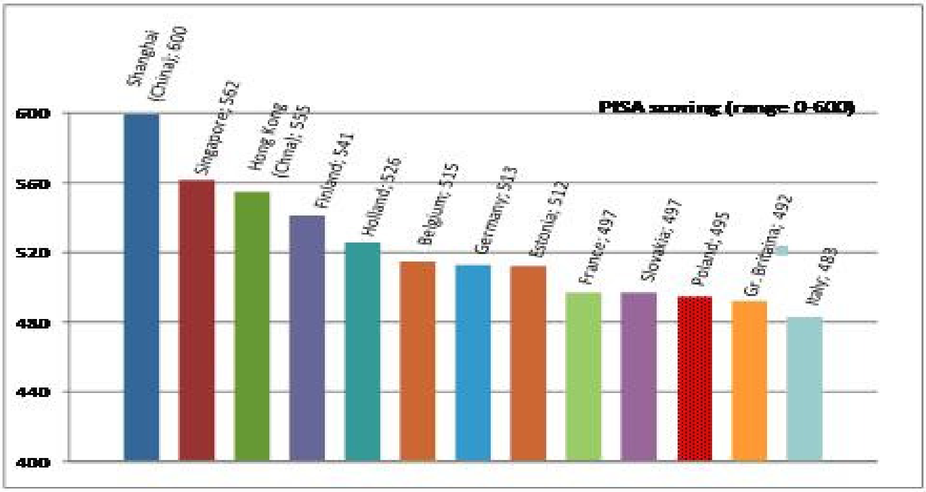

Figure 1. Mathematical skills of students according to PISA (2009)

Experts with the Institute for Educational Research stated that the low level of students' skills primarily results from low university entrance requirements, and are a consequence of the lack of mathematical knowledge. This situation is supposed to improve through reintroducing mathematics, after 25 years of absence, as an obligatory high school final exam [3].

The supply of properly prepared candidates for university courses presents a challenge to secondary education in Poland.

2 The system of teaching mathematics in Poland

The Polish system of education was reformed in 1999. The reform was aimed at "improving the quality of the formal system of education and adjusting it to the requirements of a knowledge-based economy, providing the entire population with access to the system of continuing education, as well as creating closer connections between universities, academic community and business, whose presence in other countries enhances the process of technology transfer, thus increasing the competitiveness of the Polish economy"[4].

The education system in Poland, reformed in 1999, consists of two systems: the system of secondary education and the system of higher education (Fig. 2).

[3] Ibid.

[4] *Prospects for knowledge-based economy in Poland - World Bank report results*, KBN, Warsaw, 2004.

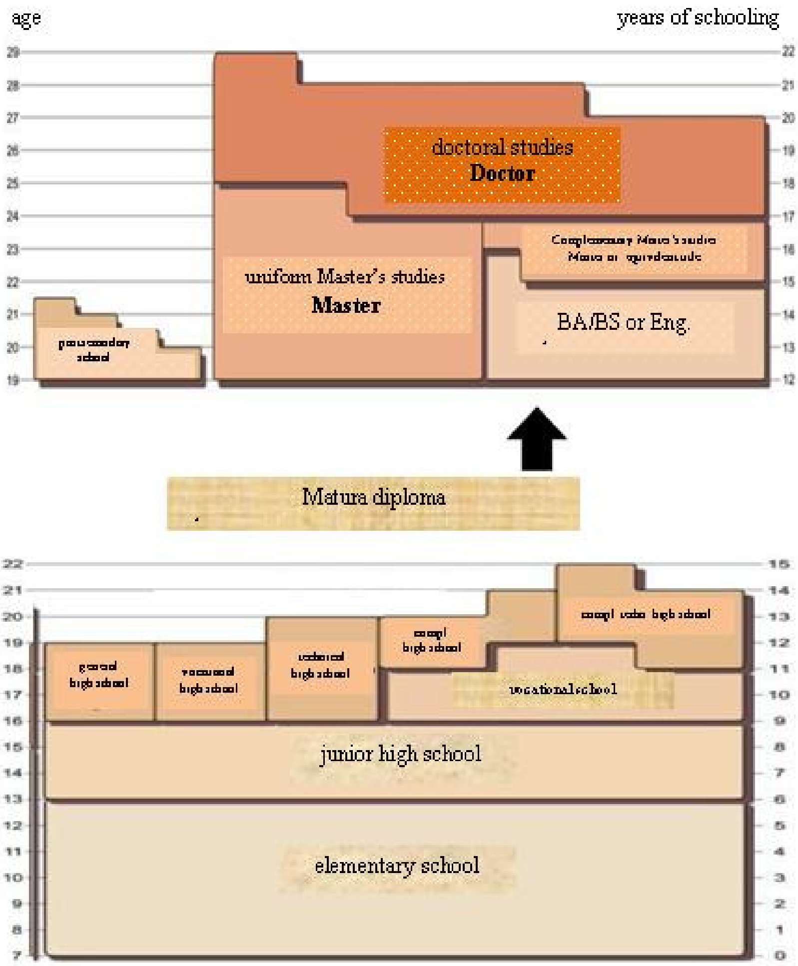

Figure 2. The education system in Poland as of 1 September 2011

The education system encompasses elementary school (6 years) and junior high school (3 years), which are obligatory, pursuant to the Constitution of the Republic of Poland, until the age of 18. Students attend classes five days a week. In Poland, the annual number of classes amounts to 635 hours, and is lower than the EU average by c. 350 hours a year, i.e. 2 hours a day.

Early childhood education constitutes the first three years of elementary school. Training during the remaining years of schooling is divided into separate courses. During early childhood education mathematics is included in general classes. After the third year of schooling, i.e. grades IV - VI and junior high school, the classes in mathematics are conducted as a separate course, amounting to 12 hours a week, during a three-year period of schooling.

Post-secondary schools include:

— three-year high schools, the completion of which allows students to obtain a school-leaving examination certificate (in Poland referred to as 'Matura');
— three-year vocational high schools, providing general vocational training, the completion of which allows students to obtain a school-leaving examination certificate (Matura diploma);
— four-year technical high schools, the completion of which allows students to obtain a school-leaving examination certificate;
— two- or three-year vocational schools, the completion of which allows students to obtain diplomas certifying their vocational qualifications after passing examinations, and to continue education in two-year complementary high schools or three-year complementary technical high schools;
— two-year complementary high schools, addressed to graduates of basic vocational schools, the completion of which allows students to obtain a school-leaving examination certificate after passing final examinations;
— three-year complementary technical high schools, addressed to graduates of vocational schools, the completion of which allows students to obtain diplomas certifying their vocational qualifications after passing an examination, and also to obtain high school-leaving examination certificates after passing finals;
— post-secondary schools, where schooling lasts not longer than 2.5 years, the completion of which allows people who graduated from high schools to obtain diplomas certifying their vocational qualifications after passing finals;
— three-year special schools, preparing students with special needs with mild or severe disabilities, and for students with multiple disability, the completion of which allows students to obtain certificates confirming their vocational training.

The number of mathematics, depending on the type of school, varies from 2 to 15 hours a week.

In August 2007, a significant change in the mathematics curriculum for general high schools was made. Mathematics in high schools is taught in a differentiated manner, some students choose basic classes, whilst others prefer extended classes. The curriculum for extended classes is far richer than that for the basic level. Mathematics at basic level is taught 9 hours a week for three years. Mathematics instruction at extended level is conducted in the amount of 12 to 15 hours a week for three years. The following general requirements of mathematics instruction have been distinguished:

— using and obtaining information,
— using and interpreting of representations,
— mathematical modeling,
— using and developing strategies,
— understanding and interpreting.

Among the most important skills expected from students, which are developed throughout the entire period of education, such as mathematical thinking, and not only performing calculations or memorizing formulas. Also scientific thinking is required in physics, biology and social sciences.

The process of elementary education finishes with a test, which is prepared and assessed by an external institution. One part tests humanistic knowledge, the other part tests mathematics and natural sciences. The results of this test are used to evaluate schools, not students.

After junior high school, there is another test in liberal arts and mathematics, prepared and assessed by an external institution. The results of this test are used to evaluate both schools and students. The test results influence ranking lists of students, and their possibilities of choosing proper high schools. At this stage, students decide about their further education. Presently, about 85% of students choose general high schools.

Vocational schools graduates complete them with graduation diplomas, and with skills at a specific occupation. Students of technical high schools receive technician diplomas, and they can take high school final examinations and obtain school-leaving diplomas, similarly to high school students.

A high school-leaving diploma is a necessary condition for university candidates.

3 Obligatory mathematics at final high school examinations.

Mathematics is a useful tool, necessary for using a vast part of the achievements of our civilization. It is a difficult tool, requiring many years of regular study, and, what is essential, it needs to be learned at a proper age. If certain mathematical skills are not acquired or developed by students at a proper age, it is very unlikely that they will make up for this at a later time. Many young people are reluctant to learn mathematics. This unwillingness results from the fact that they are not interested in overcoming even slight difficulties in learning mathematics, or they do not even attempt to do so. Generally, young people behave rationally. As mathematics was optional at Matura, they avoided this hardship and did not choose mathematics. What is more, many elementary school graduates planned not to choose mathematics as one of their final examinations, and thus they did not have motivation to learn it. Social consequences of this have recently become visible in Poland.

In this situation, it has been decided to reintroduce mathematics as an obligatory examination subject. In 2010, after 25 years, mathematics became obligatory at final high school examinations. Obligatory mathematics should force students to learn basic skills (operations on fractions, simple algebraic transformations, essentials of geometry), about the lack of which academic teachers often complain. It is also known that "mathematical thinking" is expected by teachers of other faculties than technology, mathematics, physics, i.e. philosophy, law, marketing, sociology and the like.

There are many mathematically talented students in each class, or people with higher aspirations who wish to acquire more knowledge than the minimum required. These are future students of mathematics, physics, computer science or other technical faculties.

In elementary school, junior high school and in the first grade of high school that is for 10 years of schooling, the curriculum is the same for all students. Whereas during the next two years, the curriculum is significantly changed for students who wish to know more. Students can learn more extensively the subjects which they find useful in their future careers, including mathematics. It is assumed that students who would like to study the aforementioned faculties will choose extended mathematics. And as it has already been mentioned, the curriculum for the extended level is much richer in content than the basic one; students learn and master more complex mathematical issues.

Universities accept students according to ranking lists, compiled on the basis of Matura results. Thus high school diplomas decide whether candidates are accepted into a given faculty, or not. In order to pass Matura, it is necessary to receive credits in five obligatory examinations, that is the Polish language and a foreign language in an oral form, as well as the Polish language, a foreign language, mathematics and an additional, optional subject in a written form. The passing result is 30% of correct answers. The results can be improved a year later.

Matura, in its present form, is supposed to show the real face of school. It also verifies teachers, and provides data for a ranking of high schools. Matura plays another important role; it is a way to curb corruption attempts at schools. Its basic assumption is objectivity. Examination tasks are identical for all students, and they are prepared by the Central Examination Commission. The tasks are assessed by external examination commissions, supervised by the Central Commission. The comparability of the results of high school-leaving examinations meets the standards of most EU countries.

In May 2010, Matura examination was taken by 366,623 people, and in 2011 by 355,166 people. This year's graduates constituted 96% and 97% of the total number of candidates in 2010 and 2011 respectively. The remaining group were previous years' graduates. The examination was also taken by those graduates who had already obtained their Matura diplomas in order to improve their results.

Table 1 presents the number of this year's graduates who took all the obligatory examinations, and the percentage of people who were successful in particular years.

Tab. 1 This year's graduates who took Matura.

This year's graduates	Year	
	2010	2011
Number of candidates who took all the obligatory examinations	351852	343824
Success rate	81%	75.5%

This year's Matura results have been the poorest since the introduction of the new test forms, i.e. since 2005, when the exam was not passed by people who failed one exam with 30% of correct answers. The success rate in subsequent years amounted to: 2005 - 86.5%; 2006 - 79%; 2007 - 89%; 2009 - 78%. Whereas in Tab. 2 the success rate in different school types is shown.

Tab. 2 Success rate in different school types

Type of school	2010		2011	
	The number of candidates	Success rate	The number of candidates	Success rate
General high schools	229516	91%	213969	86%
Vocational high schools	19357	64%	11403	50%
Vocational high schools	105391	70%	107439	63%
Complementary high schools	9367	37%	8553	23%
Complementary technical high schools	2992	34%	2463	19%

General high school students obtained the best Matura results in 2011 - 86% passed (previous year's result was 91%), the scores of technical high schools dropped to 63% (previous year 70%). As in previous years, the Matura results in vocational high schools were really low (only 50% of their students passed compared to 64% last year). The Matura examination came out poorly also in schools attended by adults, i.e. complementary high schools and complementary technical high schools.

In these schools, final examinations were passed by only 23% (last year 37%) and 19% (last year 34%) respectively.

Obligatory mathematics came out more poorly than last year as well. Only 79% of the candidates passed it at the basic level, whereas last year this number amounted to 87%. The average results of Matura are also lower by about 10 percentage points than last year. The most difficult were geometry, the probability calculus and open tasks, which were failed by as many as 72,000 students. A year ago, i.e. in 2010, students laughed at the examination in mathematics as the tasks had been prepared in a way preventing candidates from becoming afraid of this new obligatory subject. This opinion was confirmed by the Vice-Director of the Central Examination Commission, who said that "the first obligatory examination created a social belief that Matura in mathematics is so easy that junior high school students could pass it. Maybe too many young people believed it"[5]. Another opinion was as follows - "After Matura students claimed that the tasks were not easy, and mathematicians said that some of them were untypical"[6].

The central panel of Experts in Mathematics, operating at the Central Examination Commission, has elaborated assessment criteria for particular tasks[7]. A holistic system, which involves a comprehensive approach to solving problems, has been accepted. The results primarily depend on how close candidates are to solving the tasks.

Open tasks included in the Matura examination sheet are divided into two groups: short answer tasks (2 points each), and extended answer tasks (from 4 to 6 points each). In short answer tasks candidates receive 1 point for solutions which are not complete, or which contained mistakes; yet a certain minimum is determined, which needs to be included in order to receive 1 point. In extended answer tasks the most important phase has been singled out; it is called "overcoming fundamental difficulty of the task". It has been assumed that at least half of the points should be given for overcoming fundamental difficulty of the task, which a candidate could receive for a correct solution to this task. Therefore in the case of 4-point tasks, 2 or 3 points are given for overcoming fundamental difficulties (depending on the task). In the case of 5-point tasks, generally 3 points are given for overcoming fundamental difficulties. In the case of 6-point tasks generally 3 or 4 points are given.

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[5]R. Czeladko R., *"Every fourth high-school graduate failed"* Rzeczpospolita 1.07.2011.

[6]Ibid.

[7]*"On the methods of evaluating open tasks during the mock final exam in mathematics conducted on 3 November 2009".* Materials from the Central Examination Commission Central Panel of Mathematics Experts. www.cke-efs.pl of 20.06.2011.

Three categories of phases have been distinguished in the solutions to short answer tasks, which are presented in Tab. 3.

Tab. 3 Categories of phases in the assessment of 2-point tasks

Phase	Solution accuracy	Number of points
1.	solution, in which fundamental difficulties have not been overcome	0
2.	fundamental difficulties of the task have been overcome, but there have been mistakes	1
3.	correct solution to the task	2

Table 4 shows an example of giving points for particular phases of the solutions to extended answer tasks, e.g. 6-point tasks.

Tab. 4 Categories of phases in the assessment of 6-point tasks

Phase	Solution accuracy	Number of points
1.	solution, in which there is no significant progress	0
2.	solution, in which there is slight progress, but necessary in order to solve the tasks completely	1
3.	there has been a significant progress in solving the task, but fundamental difficulties have not been overcome	2
4.	fundamental difficulties have been overcome, the solution to the task has not been finalized, and there have been mistakes or inaccuracies while overcoming fundamental difficulties of the task	3
5.	fundamental difficulties have been correctly overcome, and the candidate stopped solving the task, or continued solving the task incorrectly	4
6.	fundamental difficulties have been overcome, the candidate has solved the task completely, but the solution contains errors (calculation mistakes, the result has been lost, the correct solutions has not been selected and the like)	5
7.	correct solution to the task	6

The exam conducted in 2010 has been used as an example presenting the difficulty of the examination in mathematics. The examination lasted 170 minutes.

The examination sheet for basic level consisted of 34 tasks, including 25 closed tasks, 6 short answer tasks and 3 extended answer tasks.

Closed tasks checked, first of all, the knowledge and understanding of basic mathematical concepts, definitions and theorems, as well as the ability to use this knowledge in practice. Open tasks checked the ability to analyze and interpret mathematical problems, as well as to formulate mathematical descriptions of given situations.

The basic level examination sheet tested most of the issues determined by the curriculum. The maximum number of points that candidates could receive amounted to 50.

In order to assess the difficulty of tasks, the so-called 'easiness index' (extent to which a tasks is completed) has been used - it is a quotient of the number of points received by all candidates and the maximum number of points for the examination tasks. This index also describes the efficiency of candidates. The easiness index allows to group tasks according to their difficulty.

Tab. 5 presents the number of tasks in mathematics in 2010 (basic level), occurring in specific groups of the task easiness index. The 2010 basic examination sheet consisted of 34 tasks.

Tab. 5 Number of tasks in particular groups

Task easiness index	Task interpretation	Number of tasks
0.00-0.19	very difficult task	2
0.20-0.49	difficult task	4
0.50-0.69	moderately difficult task	11
0.70-0.89	easy task	12
0.90-1.00	very easy task	5

50% of the tasks of the 2010 Matura in mathematics were categorized as easy and very easy ones. That is why candidates considered the examination in mathematics to be very easy. This exam had not been elaborated to give a reliable picture of knowledge and skills of candidates, but its purpose was to adjust the difficulty level of tasks to the percentage of unsuccessful candidates that the Ministry of Education desired. In fact it appeared that Matura does not have much in common with a reliable assessment of students' competences in relation to the curriculum, and to the necessity of acquiring certain knowlege and skills.

4 Higher education that meets social expectations.

It seems that the future of the world has never been so vague before. Our everyday life is characterized by unpredictability, significant changeability and dynamism of chaotic changes in the environment. The world is globalized to a large extent. Economies and financial resources of countries are strongly interdependent. On the other hand, the lives of individual people are subject to dynamic changes, which are unpredictable to a significant extent and surrounded by chaos in the areas of work, living conditions, or personal life[8] .

Thus arises a question: what kind of people will deal with these dynamic changes better? It may be stated that the current system of education in Poland has not been very successful in preparing people for the openness, diversity and vagueness of the world, as well as for the constant change of both their destiny and their roles.

School systems in many countries focus on educating people towards the national economic growth, aimed at taking advantage of the globalized market. Actions taken by the Ministry of Science and Higher Education in Poland also concentrate on the shortcomings related to sciences, technology and engineering. These are primarily connected with applied sciences, which can quickly help to develop a strategy bringing economic benefits. Due to this financial motivation most politicians concerned about the country's economic condition regard sciences and technology as essential for their country's prosperity.

Certain international institutions, such as the International Monetary Fund, or the World Bank, consider this economic development model to be appropriate. This model is being implemented in many European countries, including Poland.

Education aimed at the economic growth only poses certain threats. Such education can lead to ignoring such elements of upbringing as critical thinking, history teaching, philosophy, culture, fine arts, and mathematics as the queen of sciences. In this situation it becomes obvious that education should be aimed at sciences and technical faculties. Young people should be encouraged to develop their knowledge of these fields. The Ministry of Science and Higher Education established, in response to these issues, so-called "ordered faculties", including mathematics, computer science, mechanics and mechanical engineering, automation and robotics, environmental engineering, civil engineering, technical physics, biotechnology, mechatronics, industrial design, physics and chemistry. Students of these faculties receive very high scholarships, and higher schools receive additional funds from the state budget. These actions, similarly to other technical courses, cause the number

[8] T. Bauman *"University versus social and cultural transformations"*, Wyd. U.G., Gdacsk, 2001.

of students of these faculties to rise. This translates into a greater interest of young people in mathematics, physics and chemistry.

Yet, it needs to be remembered that education is supposed to develop students' minds, imagination and practical knowledge.

Universities are to prepare graduates for their future individual lives. Work is an important element of our lives. People who graduate in mathematics-related faculties find work more easily.

It may be assumed the the contemporary perception of professional careers of university graduates refers to chaos theory, and in this context it approaches the present reality seen from the angle of a tumultuous environment, where coincidence and intuition play vital roles. Yet, monitoring one's own activity, professional competence, predicting hardhsips and dangers generated by the employment market constitute an important component of a career understood in such a way[9] .

It is assumed that people who possess mathematical skills are better at taking decisions and preparing materials in relation to unstable objectives. They can constantly reconstruct their individual plans and make quick decisions connected with self-management skills, which are helpful in unpredictable and chaotic situations. Such people display openness to new experiences, communicativeness, risk-taking, involvement in their own businesses, they can also deal with fear and uncertainty. They also demonstrate responsibility and reflection during the constant process of learning and self-study.

5 Conclusion

The last Matura in 2011 gave rise to a heated discussion. Great many authorities questioned the knowledge level of graduates, as well as the minimum requirement of 30% needed to pass an examination. Good high school-leaving examinations are such which test whether young people are able to link facts and draw conclusions, they should not involve giving dates, as it often happens during history examinations. Modern schools must teach how to take advantage of information, which is often excessive and overwhelming, in a sensible way. Simultaneously, contemporary pedagogy confirms that children memorize facts better when they find solutions to problems on their own.

All these observations lead to the conclusion that Polish students deal well with tasks requiring a memorized arsenal of algorithms and school interpretations, yet they are not equally successful in situations where reflection, linking facts and creativity are needed. And these are the skills which do not only help students to

[9]B.M. Lichtenstein, M. Mendenhall, *Non - linearity and response - ability: Emergest order in 21st century careers*, "Human Relations", 2002, vol. 39, no. 1.

be successful in the labor market, but also to take rational decisions concerning their health and finances, or to participate in public life more efficiently.

In the recent study, the entrepreneurs who are members of BUSNESSEUROPE[10] , the most influential organization of employers in Europe, mention the shortage of employees qualified within the scope of sciences, technology, engineering and mathematics as one of major obstacles to economic development in the nearest future. Admittedly, during the last decade the total number of students and graduates of such faculties increased in the EU countries from 630,400 in 1999 to 916,100 in 2008; however, in comparison to the total number of university graduates it dropped from 24.8% in 1999 to 22.7% in 2005.

It is expected that in the coming years a great number of representatives of all the professions connected with technology will retire within the entire European Union, therefore the demand for their successors is definitely on the increase.

In Europe, and in Poland as well, there have been actions aimed at increasing social awareness of the significance of skills in sciences, technology and engineering.

Owing to the involvement of teachers in elementary and high schools, and also to the supply of materials for teaching sciences, the social interest in these fields of education has been gradually growing. Within these areas, the introduction of the aforementioned obligatory examination at Matura can bring positive results in Poland in about 5 or 6 years. However, various entities, including government administration bodies and educational institutions, should be even more engaged, and involved in taking appropriate actions.

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MASCHERONI AND *GAMMA*

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Key words: Lorenzo Mascheroni, XVIII Century, Euler-Mascheroni Constant

AMS Mathematics Subject Classification: 01A50

Abstract. The aim is to present a brief history of Euler-Mascheroni constant in XVIII century, a chapter of the numerical analysis and the analytic number theory. I shall be focusing my attention particularly on the Italian mathematician Lorenzo Mascheroni (1750-1800). In 1734 Euler discovered that the series of reciprocals of natural numbers differed at the limit of the logarithm for a constant value for which he calculated the first five decimal digits. Unfortunately, the convergence is very slow. Even with 4500 terms the approximation, using the definition, is only good for three decimal digits. Euler carried out most of his scientific activity in Russia and had all his memoirs on this constant published in *Commentarii* of St. Peterburg, where his *Institutiones Calculi Integralis* was also published in 1768-70. This work aroused great interest in Lorenzo Mascheroni, whose *Adnotationes* were then published together with the *Institutiones* in Euler's *Opera Omnia.* Mascheroni was politically involved in the Cisalpine Republic, founded by Napoleon in 1797 at the end of his Italian campaign. This marked the start of the Napoleonic era, which terminated in 1812 with the retreat from Russia two hundred years ago.

1 The origin

The beginning of the history of the function Γ and the constant γ, the names being assigned in the XIX century, constituted a very interesting chapter of the application of differential and integral calculus, and of the theory of the series, outside the fields of geometry and mechanics in which these methods originated. For example, Johann Bernoulli, in 1697, determined the sum of the series of the powers of reciprocals of the natural numbers by means of the integral calculus.

Of great importance for the beginning of the analytic number theory was the correspondence between Euler and Goldbach.

Christian Goldbach (1690-1764) was, in Moscow, the tutor of the Tsarevich. He had suggested the problem of interpolating the factorial to Daniel Bernoulli, and Bernoulli encouraged Euler to propose an interpolating function. Euler wrote to

Goldbach, on 13^{th} October 1729, presenting as the interpolating function the limit:

$$\lim_{n\to\infty} \frac{n!\, n^x}{x\,(x+1)\cdots(x+n)}.$$

The following year, on 8^{th} January 1730, Euler proposed another interpolating function to Goldbach:

$$n! = \int_0^1 \left(\lg \frac{1}{x}\right)^n dx$$

In a letter to Euler of 7^{th} June 1742, Goldbach asked him to demonstrate that every even integer greater than two can be expressed as the sum of two prime numbers. This is one of the oldest unsolved problems in number theory and in all mathematics (it is called Goldbach conjecture).

The correspondence between Goldbach and Euler is very interesting and was printed by Juškevič and Winter in 1965 [1].

The Euler Mascheroni constant, the irrationality and transcendence of which is still an open question, first appeared in one of Euler's memoirs of 1734-35 [2]. When studying harmonic sequences [3]:

$$\frac{c}{a}, \frac{c}{a+b}, \frac{c}{a+2b}, \frac{c}{a+3b}, \dots.$$

Euler went into detail on one of these sequences, the one of the reciprocals of natural numbers:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{n} = \lg(n+1)+\frac{1}{2}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\dots\right)+ \\ -\frac{1}{3}\left(1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\dots\right)+\frac{1}{4}\left(1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\dots\right)+\dots$$

discovering that this one differed, at the limit, from the logarithm for a constant value: for which he calculated the first six decimal digits:

$$1+\frac{1}{2}+\cdots+\frac{1}{n} \approx \log(n+1)+0,577218.$$

Shortly after this, Euler himself introduced a greater precision to the calculus of the constant [2]. Starting from the approximate calculus of:

$$H_{10} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{9} + \frac{1}{10} = 2,92896\ 82539\ 68253\ 9$$
$$\lg 10 = 2,30258\ 50929\ 94045\ 684$$

and from the relation:

$$C = H_{10} - \lg 10 - \frac{1}{20} + \frac{1}{1200} - \frac{1}{1.200.000} + \frac{1}{252.000.000} - \frac{1}{24.000.000.000} + \dots$$

he found, for the C constant, the value (with fifteen exact decimal digits):

$$0,57721\ 56649\ 01532\ 9$$

Euler took up the question again in his treatise on differential calculus, using Bernoulli's numbers for the calculus of the constant [3]. The C constant $= 0,57721\ 56649\ 01532\,5$ was obtained as the sum of the series:

$$\frac{1}{2} + \frac{B_1}{2} - \frac{B_2}{4} + \frac{B_3}{6} - \frac{B_4}{8} + \dots$$

where:

$$B_1 = \frac{1}{6},\ B_2 = \frac{1}{30},\ B_3 = \frac{1}{42},\ B_4 = \frac{1}{30}, \dots .$$

These numbers had been encountered by Jacob Bernoulli in the calculation of the partial sums of natural numbers raised to powers [6].

Once again Euler returned to the C constant in the first volume of the *Institutiones calculi integralis* reporting the calculus of various integrals to the formulae [4]:

$$\int \frac{e^x}{x}\,dx = C + \lg x + \frac{x}{1} + \frac{1}{2}\frac{x^2}{1\cdot 2} + \frac{1}{3}\frac{x^3}{1\cdot 2\cdot 3} + \dots$$
$$\int \frac{dz}{\lg z} = C + \lg\lg z + \lg z + \frac{(\lg z)^2}{2\cdot 2} + \frac{(\lg z)^3}{2\cdot 3\cdot 3} + \dots$$

Euler integrated the power series formally, like all the mathematicians of his day, without considering their convergence, and no greater precision was brought to the calculus of the constant. Still dissatisfied with his results, Euler searched for different solutions to the calculus of the constant (indicated with the letter O):

$$O = \frac{1}{2}\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right) + \frac{2}{3}\left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots\right)$$
$$+ \frac{3}{4}\left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots\right) + \dots 0,57721\ 56649\ 01532\ 5$$

Euler did not consider even this result as definitive. In fact, he wrote: *"Manet ergo quaestio magni momenti, cuiusnam indolis sit numerus isti O et ad quodnam genus quantitatum sit referendus"* [8].

In one of the memoirs published posthumously, Euler provided two more ways to calculate the constant without, however, improving his results [9].

Mascheroni entrusted Tommaso Rossi, private tutor of mathematics and philosophy at Pavia University and *Collegio Ghislieri*, with the study of Euler's constant. Rossi concluded that the constant was finite and Mascheroni accepted this thesis. Rossi did not produce anything new in the calculus of the constant.

Mascheroni took up this calculus in the *Adnotationes* of 1790 in which he started from the results of Euler's differential calculus and integral calculus. He was able to rectify Euler's value for C, reaching the calculus of thirty-two decimal digits, of which only the first nineteen have been revealed to be exact [10]:

$$0,577215\ 664901\ 532860\ 618112\ 090082\ 39.$$

In Mascheroni's *Adnotationes* the study of this constant was the main topic, which was taken up several times. The method of calculus was the same as Euler's and dealt with Bernouilli's numbers reported in the *Institutiones calculi differentialis.* Euler and Mascheroni were the first to tackle this important constant with such insistency: Euler tried to express the constant in other terms (the logarithm of a familiar number) believing, therefore, he was dealing with an irrational number. Mascheroni, on the other hand, felt that it had to be a number expressible with a finite sequence of digits.

There is an ample bibliography on the Euler Mascheroni constant which also includes recent works. An interesting book was published by Julian Havil. It is well written and full of information, but being based on secondary literature a few remarks are required particularly as far as Mascheroni is concerned [11].

Havil only cites Mascheroni once:

"Of course, the desire to extend the accuracy of the estimate was great and, in 1790, the Italian geometer Lorenzo Mascheroni published in Adnotationes ad calculum integralem Euleri *an approximation of γ to 32 decimal places: the estimate then became*

$$0,577\ 215\ 664\ 901\ 532\ 860\ 618\ 1\ldots$$

This was all well and good until 1809, when Johann von Soldner (1766-1809) used $Li(x)$ function to give the value

$$0,577\ 215\ 664\ 901\ 532\ 860\ 6\underline{0}6\ 5\ldots$$

which differs in that underlined 20^{th} decimal place (and after). The matter was resolved (but the confusion not removed) when, in 1812, the inimitable Gauss prevailed on the 19-year-old prodigy of calculation, F.G.B. Nicolai (1793-1846) to check the results. (p.89)"

and Havil adds:

"Mascheroni's permanent contribution to γ's story (apart from making a mistake that led to at least eight subsequent recalculations of the number) was to name it γ (we have seen that Euler originally used C, and O and A have also been used). By such serendipity, its full, accepted name is the Euler-Mascheroni constant. (A more distinguished legacy of Mascheroni is his result that any geometric construction that is possible with straight edge and compass can be achieved with a compass alone.)"

In an extensive and long term research project, on Euler-Mascheroni's constant, Stefan Krämer (Georg-August-Universität, Göttingen) corrected the attribution to Mascheroni of the notation γ: it appears nowhere in the writing of either Euler or Mascheroni and was chosen at a later time because of the constant's connection to the gamma function. For example, the German mathematician, Carl Anton Bretschneider, used the notation γ in 1835.

Lorenzo Mascheroni was one of the most original of the Italian mathematicians in the second half of the eighteenth century, and also a distinguished poet, politician and Rector of the University of Pavia [12].

2 Biographical notes on Mascheroni

Lorenzo Mascheroni (1750-1800) is known to scholars of literature for *L'invito* and for the poem written in his honour by Vincenzo Monti, and to mathematicians for the Euler-Mascheroni constant and the geometry of the compass alone: he was a leading figure in the cultural life of Lombardy during the years of the Hapsburg political reform and the tormented Republican experiences in the years 1796-1799 [13]. First of four children, he was born in Castagneta, a small village a few kilometres from Bergamo Alta, to Giovanni Paolo, trader, and Maria Ciribelli [14]. In 1758 he entered the Seminary of Bergamo, in 1767 he took his ecclesiastical vows and in 1774 he celebrated his first Mass. His scientific and literary talents were quickly recognised and he was enrolled in the *Accademia degli Eccitati*, then summoned to teach philosophy in the *Collegio Mariano* in Bergamo; here he came into conflict with a group of ex-Jesuit prelates. In 1784 he visited the University of Pavia, where he met Gregorio Fontana and Alessandro Volta, and in 1786, following Pietro Paoli's transfer to Pisa, he was called to hold the chair of Algebra and Geometry. Pavia represented a period of intense scientific and teaching activity which gave fruit to considerable publications; for two years he also held the position of Rector of the University. After the French troops had entered Milan on 15th May 1796, Mascheroni aligned himself with Fontana and almost all the teachers of Pavia University in favour of the new democratic governors. With the proclamation of the Cisalpine Republic (1797) he was summoned to form part of the Committee for the Constitution and was then nominated as a member of the Great Council, becoming part of the commission for Public Education: he drew up the *Piano generale di pubblica istruzione* (1798) [15].

Mascheroni also took an active part in other projects like the one which extended the metric-decimal system of weights and measures across the Italian territories. In September 1798 he was sent to Paris for the final approval of this system, and while there he was greatly struck by the news of the end of the Cisalpine. Shortly after this he fell ill and died on 14th July 1800 [16].

As a mathematician, Mascheroni is to be remembered for three important works: the *Nuove ricerche sull'equilibrio delle volte* (1785), the *Adnotationes ad calculum intergralem Euleri* (1790-92), and *La geometria del compasso* (1797).

3 Mechanics and Geometry

The mechanics of vaults and domes constituted one of the most sensitive subjects of the art and science of designing and erecting buildings. Mascheroni referred back to the scientific treatises on the subject, and reminded those practising the art that as recently as 1732 an ammunition magazine in the south of France had collapsed before completion owing to a lack of theoretical knowledge. The studies Mascheroni referred to were the works by Charles Bossut and Claude Antoine Couplet regarding vaults, and again Bossut and Pierre Bouguer on domes. As far as Italy was concerned, Mascheroni referred to the *De fornicum vi et firmitate*, inserted into the second volume of the work by Paolo Frisi (Milan, 1783) and the *Saggi di statica e meccanica* by Anton Maria Lorgna (Verona, 1782). Mascheroni normally made use of the techniques of differential calculus:

"Nascendo l'equilibrio dalla eguaglianza delle forze, noi verremo a considerare il rapporto che possono avere tra loro due forze contrarie facendo seguire per supposizione il moto di due centri di gravità per due linee infinitesime da una parte e dall'altra; l'uno de' quali due moti sia effetto dell'altro. Noi prendiamo il moto in una linea infinitesima, perché in questa supposizione il moto segue con velocità uniforme per tutta la linea, il ché è necessario per calcolare la quantità del moto. In oltre questa supposizione è necessaria per ogni caso, nel quale il centro di gravità di un corpo movendosi muti continuamente direzione" [17].

The *Nuove ricerche sull'equilibrio delle volte* (Bergamo, Locatelli, 1785) are divided into twelve chapters:

"I. On the equilibrium of straight lines, II. On the equilibrium of arches , III. On the thickness of arches, IV. On the planes composed of wedges having the force of arches, V. On the equilibrium of rampant and load-bearing arches, VI. On domes, VII. On load-bearing domes, VIII. On circular planes composed of wedges having the force of domes, IX. On domes with polygonal and oval bases, X. On annular and spiral vaults , XI. On compound arches and vaults, XII. On equilibrium curves with convergent gravity directions".

Mascheroni was not the only theoretician on the equilibrium of vaults and domes: his studies for the main dome of Bergamo's cathedral were of great help in the nineteenth century in the construction of that impressive structure.

Mascheroni's most famous work is *La geometria del compasso* (Pavia, 1797) dedicated to Napoleon Bonaparte. The definition of geometry of the compass alone is best expressed by Mascheroni:

"I would call Geometry of the compass that geometry which, by compass alone without the line, determines the position of the points" [14].

Mascheroni solved the problems which form the basis of geometric constructions with his elegant use of the compass alone:

- Divide the circumference in four and five equal parts.
- Divide a segment in equal parts.
- Find, geometrically, the roots of natural numbers up to ten.
- Given two points of a straight line and two points of another one find the point of intersection.
- Given an arc find its sine, cosine, tangent and secant.
- Circumscribe a square, a pentagon etc., around a circle.
- Find the centre of a given circle.
- Find the sides of a regular polyhedron inside a sphere.
- Duplicate a cube approximately.

One of these problems became unexpectedly famous. Mascheroni did not limit himself only to dedicating his work to Napoleon, as he had also shown it directly to the General who had a great bent for mathematics. On his return to France, Napoleon was the protagonist of a frequently told story: during a conversation with Laplace and Lagrange at François de Neufchateau's home he asked them if they knew how to find the centre of a circle using only a compass. As the two celebrated mathematicians looked hesitant and perplexed, Napoleon took a compass and with six circles determined the desired point, repeating construction n. 142 of *La geometria del compasso*. Their astonishment induced Laplace to exclaim: "We were prepared for anything from you General, except a lesson in mathematics". *La geometria del compasso* was translated into French (Paris 1798) and German (Berlin 1825).

4 Mascheroni and Euler's treatises

The renewal of Italian scientific culture is particularly indebted to Gregorio Fontana for his promotion of translation and editing of works (completed by his ample annotations) which he carried out in Pavia. In just a few years there appeared translations like the *Compendio di un corso di fisica sperimentale*, by Georges Atwood (Pavia, Stamperia del Monastero di S. Salvatore, 1781), the *Trattato elementare di idrodinamica*, by Charles Bossut (Pavia, Stamperia del Monastero di S. Salvatore, 1785), the *Trattato elementare di meccanica*, by Charles Bossut (Pavia, Stamperia del Monastero di S. Salvatore, 1788). Fontana entrusted Ferdinando Speroni with the job of editing a new edition of the *Institutiones calculi differentialis*, by Euler (Pavia, Galeazzi, 1787) and completed the *Lezioni elementari di calcolo differenziale ed integrale*, by abbot Marie (Pavia, Comino, 1793) with an appendix containing a considerable number of examples and applications [15].

The Pavia edition of the *Institutiones calculi differentialis* by Euler was not merely the republication of the original edition in 1755, as it also included an eulogy on Euler written by Condorcet, an unedited appendix by Euler (sent to Gregorio Fontana by Johann Albert, Leonhard's son, expressly for the Pavia edition), notes by the editor (using works by Daniel Bernoulli, Lagrange and Laplace), and an impressive bibliography of Euler's works, including about a hundred unedited dissertations.

The Pavia University study programme reserved a course for Mascheroni, preparatory for mathematics applied to mechanics (taught by Mariano Fontana) and for analysis, taught by Gregorio Fontana. It was during the years in Pavia that Mascheroni endeavoured to keep up with the latest results of infinitesimal analysis, almost all of which led more or less to Euler's work. When talking about his scientific training Lagrange himself confessed he had learnt infinitesimal analysis mainly from studying Euler's *Mechanica* (Petropoli, Ex Typographia Academiae Scientiarum, 2 vols, 1736) [16].

While continuing to publish dozens of memoirs on academic acts, Euler added to these, four treatises: the *Methodus inveniendi lineas curvas maximi minimive proprietates gaudentes* (Lausannae et Genevae, Bousquet, 1744), the *Introductio in analysin infinitorum* (Lausannae, Bousquet, 2 vols, 1748), the *Institutiones calculi differentialis* (Berolini, Impensis Academiae Imperialis Scientiarum Petropolitanae, 1755), and the *Institutiones calculi integralis* (Petropoli, Impensis Academiae Imperialis Scientiarum, 3 vols, 1768-70).

Mascheroni was not a passive reader and took the subject of a memoir from his reading of *Methodus inveniendi* [17].

In the first appendix of the *Methodus inveniendi*, Euler dealt with the configuration of equilibrium of an elastic bar with one end locked in position and the other end loaded with a weight, but he considered it "problema non indignum, in quo geometrae vires suas excerceant" [18]. Mascheroni concluded that it was a spiral arc of the following equations:

$$x = \int \sin \frac{ss}{2aa}\, ds, \quad y = \int \cos \frac{ss}{2aa}\, ds.$$

The work which has rightly brought Mascheroni most recognition is that inspired by Euler's treatises on analysis: it is the *Adnotationes ad calculum integralem Euleri* published in two parts in 1790 and 1792. This modest title collects various very important analytical memoirs and was written in Latin in order to conform to Euler's work and to be easily disseminated throughout Europe.

On 8^{th} January 1790, Mascheroni proudly told his friend and correspondent, Girolamo Fogaccia, about the first part of the work:

"Quanto alla mia edizione di quest'anno, io fo stampare; ossia il Galeazzi di Pavia stampa a suo conto e mi darà dieci copie di tre comenti sopra l'Eulero che saranno stampati in latino della forma del Calcolo differenziale stampato pur da esso, e vi si potranno aggiungere. Il primo è la determinazione della costante C *nell'equazione seguente*

$$\int \frac{dz}{\lg z} = C + \lg \lg z + \lg z + \frac{(\lg z)^2}{2 \cdot 2} + \frac{(\lg z)^3}{2 \cdot 3 \cdot 3} + \dots.$$

Questa costante quanto si voglia l'integrale $\int \frac{dz}{\lg z} = 0$ *quando* $z = 0$ *è stata creduta dover essere infinita dall'Eulero. Io la trovo finita ed eguale alla frazione* $0,57712$ *ecc. sino a trentadue decimali. In secondo luogo l'Eulero ha creduto la serie posta dopo la costante* C *se è reale pel valore di* z *minore dell'unità diventasse immaginaria pel valore di* z *maggiore dell'unità e viceversa. Io la dimostro reale per tutti i casi, e tutti e due i punti sono fiancheggiati da due dimostrazioni. Ma vedrete meglio il tutto quando avrò l'onore di darvene una copia. Il secondo comento sarà l'opuscolo già da me stampato l'anno scorso che serve di comento al capo dell'Eulero nel Calcolo Integrale che vien dopo il capo al quale serve di comento l'opuscolo or citato. Il terzo comento sarà l'opuscolo che già teneva preparato sull'abbaglio del la Grange in una integrazione della quale credo avervi già detto altre volte, e questa sarà comento d'un altro capo del Calcolo Integrale dell'Eulero dove l'Eulero dà una maniera esatta benché non tanto diretta di ottenere quell'integrale. Questi miei tre comenti saranno in latino, e piaceranno almeno ai latinisti di Bergamo, e tanto più quanto il latino sarà abbastanza tondetto"* [19].

In the first *Adnotationes* Mascheroni undertook some of the most difficult problems of infinitesimal analysis of his day, disputing the results with mathematicians like Euler, and Lagrange, not to mention d'Alembert whose discussion he inserted in a famous paradox [20].

In the second *Adnotationes* he examined other points of Euler's work on integral calculus which took him up to the most recent memoirs, he discussed questions proposed by Paoli, and by Gregorio Fontana, who asked to have included in the work the calculation of some integrals the latter had carried out (which are not the most brilliant part of the work). The *Adnotationes* to Euler are Mascheroni's main work in Latin and their drafting comes just one year before his most famous poetical work: *L'invito a Lesbia Cidonia* (Pavia, Comino, 1793). Mascheroni's manuscripts in the "A. Mai" Library in Bergamo still contain several unedited notes which are worthy of attention [25].

It would seem to me that the name Mascheroni should be associated for the contant γ with the great name of Euler for at least three reasons:

— Firstly, Mascheroni, along with Euler, was the only mathematician to study the constant in the XVIII century;
— Secondly, Mascheroni improved Euler's calculus by four correct decimal digits (for fifty years the latter did not go beyond the fifteen digits);
— and thirdly, Mascheroni, by giving credit to Tommaso Rossi, posed the problem of the rationality of the constant.

5 After Mascheroni: an open question

On 20$^{\text{th}}$ January 1799 in Paris, Mascheroni asked his friend Giuseppe Mangili, an anatomist, to bring twenty-four copies of both the *Institutiones calculi differentialis* by Euler and the *Adnotationes* printed in Pavia by Galeazzi, in order to put them on sale at the Parisian librarian Duprat, thus ensuring their international dissemination [26]. In Munich, 1809, Soldner published his theory on the transcendent function of integral logarithm (*Théorie d'une nouvelle fonction trascendante*) calculating 22 digits for the Euler-Mascheroni constant with a different value from Mascheroni's at the twentieth digit: $0,57721\ 56649\ 01532\ 86060\ 6065\ldots$.

In 1813 Gauss, in his famous memoir on the hypergeometric series: *Disquisitiones generales circa seriem infinitam* $1+\dfrac{\alpha\cdot\beta}{1\cdot\gamma}x+\dfrac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma\cdot(\gamma+1)}xx+\ldots$ *pars prior* [27], reported the calculus that his pupil, Nicolai, carried out of the Euler-Mascheroni constant, which was different from the twentieth digit calculated by

Mascheroni and in agreement with Soldner's calculation:

$$\psi_0 = 0,5772156649\ 0153286060\ 6512090082\ 4024310421.$$

In 1857 Lindemann, famous for his demonstration of the transcendence of e and π, obtained a value for the constant in two different ways, with 34 and 24 digits. In 1869 Shanks obtained 59 decimal digits, and in 1871 calculated 110. In 1878 Adams, by calculating the first 62 numbers of Bernoulli, produced the Euler-Mascheroni with 263 decimals digits. This numerical result was by far the best.

Interest in the constant did not, however, lie only in its calculation and the solution to the question of rationality and transcendence: new links were constantly emerging with the theory of numbers and complex analysis.

In 1874 Mertens linked the constant to prime numbers and e and π:

$$e^\gamma = \lim_{m\to\infty} \frac{1}{\lg m} \prod_{p\leqslant m} \left(1 - \frac{1}{p}\right)^{-1}, \quad \frac{6e^\gamma}{\pi^2} = \lim_{m\to\infty} \frac{1}{\lg m} \prod_{p\leqslant m} \left(1 + \frac{1}{p}\right).$$

In 1887 Stieltjes, in an important work devoted to the calculus of $\zeta(2), \dots, \zeta(70)$ (where ζ represents Riemann's function: $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$) with 32 decimals, also obtained the value of the Euler–Mascheroni constant with 32 decimals from the formula (T.J. Stiltjes, *Tables des valeurs des sommes* $\sum_{n=1}^{\infty} n^{-k}$) [28]:

$$\gamma = 1 - \lg \frac{3}{2} - \sum_{k=1}^{\infty} \frac{(\zeta(2k+1) - 1)}{4^k(2k+1)}.$$

The rationality of the Eulero-Mascheroni constant remains an open question even today. It is known in the mathematical literature with the Greek letter γ for its relation with Euler's function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \quad x > 0 \quad (\Gamma(n+1) = n!)$$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \log t\, dt \quad x > 0$$

$$\gamma = -\Gamma'(1) = -\int_0^\infty e^{-t} \log t\, dt = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right).$$

The use of calculators has brought the Euler-Mascheroni constant to the fore and with the method of Bessel's function it has greatly advanced: in 1980 Brent and McMillan calculated 30,100 digits; in 1993 Borwein 172,000 digits; in 1997 Papanikolau 1,000,000; in October 1999 Gourdon and Demichel reached calculation of 108,000,000 digits. It was Papanikolau who proved that if γ is rational the denominator of the fraction must have at least 242,080 digits.

Considering the fact that the transcendence, and, therefore, the irrationality, of e and π was demonstrated in the nineteenth century, the transcendence and the irrationality of γ remains the most important open question in this kind of study. The hope of those engaged in numerical calculus is that Mascheroni was right and that the constant is therefore represented by a finite number of digits [29].

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EDUCATIONAL PROBLEMS OF MATHEMATICAL ANALYSIS

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Abstract. The problems of university courses of mathematical analysis are considered. The idea of different levels of courses depth and rigour is discussed. Several contemporary notions, results and theories (including Henstock–Kurzweil integral and Gordon criterion) that can be naturally included in lecture courses are suggested.

1 Introduction

Moscow school of analysis has the glorious history, and this history is inseparably linked with Moscow State University. D.F. Egorov, N.N. Luzin, A.N. Kolmogorov, D.E. Menshov, M.A. Lavrentyev and dozens of other celebrated MSU graduates continued their careers in Moscow State University and made an inestimable contribution to world mathematics both from the points of science and the points of education.

In 1933 after the reorganization in Moscow State University and the foundation of the Faculty of Mechanics and Mathematics two departments were established at this faculty: "Mathematical Analysis–1" (head — Professor M.A. Lavrentyev) and "Mathematical Analysis–2" (head — Professor V.V. Stepanov). In 1935 these departments were reformed into the Department of Analysis and Theory of Functions (head — Professor M.A. Lavrentyev) and the Department of Functional Analysis (head — L.A. Lyusternik). In 1938 the Department of Analysis and Theory of Functions was reorganized to two departments: the Department of Mathematical Analysis (head — corresponding member of the Academy of Science of the USSR A.O. Gelfond) and the Department of Function Theory (head — corresponding member of the Academy of Science of the USSR I.I. Privalov). Since 1938 the name of the Department of Mathematical Analysis remained unchanged, and the departments of Function Theory and of Functional Analysis were merged in 1943 into the Department of Theory of Functions and Functional analysis (head — corresponding member of the Academy of Science of the USSR D.E. Menshov).

The Department of Mathematical Analysis has a vast experience in not only scientific research, but also in educational topics. The department plays a significant role in the educational structure not only of the Faculty of Mechanics and Mathematics, but of the whole University: its staff members deliver lectures and give exercises at Faculty of Chemistry, Faculty of Biology, Faculty of Geography, Faculty of Bioengineering and Bioinformatics and a number of other faculties. Currently the staff of the department consists of almost 100 lecturers and researchers, including 2 academicians and 20 professors.

2 The mathematical analysis in the educational process

The mathematical analysis is a large and constantly developing field of mathematics that studies functions with methods based on the concept of limits. However, in educational context the term "mathematical analysis" commonly refers to the fundamentals of this field, namely, to the basis of differential and integral calculus and its direct applications.

The educational role of the course of mathematical analysis consists in not only the discussion the basic notions and result of the subject. More importantly, the course plays a key role in forming the mathematical (or, even more general, natural-science) culture. It also shows the interconnections between mathematics and other fields of science, including physics, chemistry, biology, economics, etc. For mathematicians the course additionally shows deep interconnections of mathematical analysis with other fields of mathematics: algebra, logic, geometry and others.

Regardless of the target audience a good course of mathematical analysis solves the following tasks.

- The course shows the basic notions and results of mathematical analysis (definitions, theorems, proofs).
- It teaches to apply these notions and results to various mathematical problems as well as to common-life problems and to problems specific to the target audience (e.g., certain chemical problems if it is the course for the Faculty of Chemistry).
- The course also shows methods and approaches that stand behind these notions and results, and teaches to apply not only ready-made results, but also these methods and approaches (this problem is not as straightforward as the first one, but it is not less important than the first one).
- The course forms the basis of scientific culture, the abilities of logical thinking, rigorous reasoning, simple model construction.

It is worth noting that a truly good course of mathematical analysis, especially in case when the target audience consists of students of mathematical faculties, solves even a deeper problem: along with the basic notions, results and methods that stand behind these notions and results, it also tries to show (meta)approaches, (meta)knowledge that stand behind these methods.

In order to help students to learn how to apply the results, to understand results and methods better, to "feel" the subject, the course of mathematical analysis should include a sufficient amount of seminars (exercises) along with lectures.

In order to achieve its goals, a course of mathematical analysis should generally follow the recommendations stated by acad. P.L. Chebyshev in the XIX-th century. These recommendations include the following.

- A course should include a sufficient amount of comments and explanations; students should understand the course, and the "unconscious remembering" is inadmissible.
- All statements should be brief and at the same time clear, they should not include any non obligatory details, conditions, requirements.
- Along with the rules and results a course should also impart practical skills to apply these rules and results.
- All results should be rigorously proved if possible; at the same time, non-rigorous proofs are not inadmissible. In case if a statement can not be rigorously proved in the frames of the course, this fact should be clearly pointed out.

3 Multi-level approach

The depth and the rigour of the course of mathematical analysis essentially depends on the target audience. It is equally wrong to read too simplified course of mathematical analysis for the mathematicians and too complicated course for, e.g., psychologists or political scientists.

One can distinguish at least four levels of depth and rigour of courses of mathematical analysis:

- courses for students with specializations not in natural and formal sciences;
- courses for students with specializations in engineering and natural sciences;
- courses for students with specializations in mathematics and other formal sciences;
- courses for students with specialization directly in mathematical analysis and close fields of mathematics

(however, the borders between these groups of students and consequently between these levels are sometimes fuzzy).

In case of students with specialization not in natural and formal sciences a widely used approach is to discuss bases of the analysis even without rigorous definitions of some basic notions. For example, acad. S.M. Nikolsky used this approach in the course of mathematical analysis for the students of the Dnepropetrovsk Pharmaceutical Institute, where he discussed the fundamentals of the analysis basing mainly on intuition, without even a formal definition of a limit. Later he followed this approach in popular science books, including books for schools (see, e.g., [1], [2]).

At all higher levels rigorous approach is generally used, but formal definitions and formal reasoning in this case are normally amplified with intuitively clear explanations and motivations (splendid ideas of intuitively clear explanations and motivations of basic concepts of mathematical analysis can be found, e.g., in [3]: this book is especially suitable for students with specializations in natural sciences). However, the depth and the width of a course for various levels is essentially different.

4 Courses development

In spite of the fact that generally the courses of mathematical analysis deal with the classical concepts, the development of mathematics leads to the development of these courses: new results and approaches allow to simplify certain proofs of classical results and to increase the generality without additional complication of the course. The examples that illustrate these ideas are given below.

4.1 Henstock–Kurzweil integral

Generally courses of mathematical analysis use Riemann integral introduced in 1853. It is defined as a limit of Riemann integral sums. For about 15 year T.P Lukashenko, the professor of the Department of Mathematical Anasysis at the MSU Faculty of Mechanics and Mathematics, includes a more general integral into the course he reads for the students of this faculty. This generalization of Riemann integral is known as the Henstock–Kurzweil integral. It was introduced independently by J. Kurzweil (in 1957) and R. Henstock (in 1961). This integral is also defined as a limit of integral sums, but uses a different filter basis on the set of marked partitions.

The Henstock–Kurzweil integral is more general than the Lebesgue integral: it is equivalent to the Perron integral and the Denjoy integral (the latter was studied, e.g., by Luzin, see [4]). However, the definitions of these integral are complicated, so at the Faculty of Mechanics and Mathematics this class of integrals was previously studied only in frames of special courses.

Let us recall the notions required to define the Henstock–Kurzweil integral.

A partition T of a segment $[a,b]$ is an arbitrary finite set of nonoverlaping closed segments $\{\Delta_i\}_{i=1}^n$ which union equals $[a,b]$. A marked partition $\mathbb{T}$ of a segment $[a,b]$ is a pair (t,ξ), where $T=\{\Delta_i\}_{i=1}^n$ is a partition of $[a,b]$ and $\xi=\{\xi_k\}_{k=1}^n$ is a set of points such that $\xi_k\in\Delta_k$ for all $k\in\{1,\,2,\,\ldots\,,\,n\}$.

The definition of the Riemann's integral. A function f is Riemann-integrable on $[a,b]$ and a number I is its integral, if f is defined on $[a,b]$ and for every $\varepsilon>0$ there exists $\delta>0$ such that for every marked partition $\mathbb{T}=\{(\Delta_k,\xi_k)\}_{k=1}^n$ with $|\Delta_k|<\delta$ $(k=1,\ldots,n)$ the following inequality holds:

$$\left|\sum_{k=1}^n f(\xi_k)|\Delta_k|-I\right|<\varepsilon.$$

The definition of the Henstock–Kurzweil integral. A function f is Henstock–Kurzweil integrable on $[a,b]$ and a number I is its integral, if f is defined on $[a,b]$ and for every $\varepsilon>0$ there exists a scale function $\delta(x):[a,b]\to(0,+\infty)$, such that for every marked partition $\mathbb{T}=\{(\Delta_k,\xi_k)\}_{k=1}^n$ with $|\Delta_k|<\delta(\xi_k)$ $(k=1,\ldots,n)$ the following inequality holds:

$$\left|\sum_{k=1}^n f(\xi_k)|\Delta_k|-I\right|<\varepsilon.$$

It is important for the definition of the Henstock–Kurzweil integral that for every scale function $\delta(x):[a,b]\to(0,+\infty)$ there exists at least one marked partition $\mathbb{T}=\{(\Delta_k,\xi_k)\}_{k=1}^n$ with $|\Delta_k|<\delta(\xi_k)$ $(k=1,\ldots,n)$ (otherwise, as one can easily see, every number I is an integral of an arbitrary function). This fact can be easily proved, e.g., using the method of bisection.

It is well known that the Dirichlet function

$$\mathcal{D}(x)=\begin{cases}0, x\notin\mathbb{Q},\\ 1, x\in\mathbb{Q}\end{cases}$$

is not Riemann-integrable on any segment $[a,b]$ $(b>a)$. However, it can be easily shown that this function is Henstock–Kurzweil-integrable, and its integral equals 0. Indeed, let $\{r_k\}_{k=1}^\infty$ be a sequence of all rational numbers of $[a,b]$. For an arbitrary

$\varepsilon > 0$ the scale function $\delta(x)$ can be defined as follows:

$$\delta(x) = \begin{cases} 1, x \notin \mathbb{Q}, \\ \varepsilon \cdot 2^{-k-1}, x = r_k. \end{cases}$$

In this case for every marked partition $\mathbb{T} = \{(\Delta_k, \xi_k)\}_{k=1}^n$ of $[a, b]$ with $|\Delta_k| < \delta(\xi_k)$ $(k = 1, \dots, n)$

$$\left| \sum_{k=1}^{n} \mathcal{D}(\xi_k)|\Delta_k| - 0 \right| = \sum_{k:\xi_k \in \mathbb{Q}} \mathcal{D}(\xi_k)|\Delta_k| \leq 2 \sum_{k=1}^{\infty} \delta(\xi_k) = \varepsilon.$$

Hence, $(K - H) \int_a^b \mathcal{D}(x)\, dx = 0$.

Similarly to the Lebesgue integral, the Henstock–Kurzweil integral allows to integrate function defined almost everywhere on $[a, b]$. Indeed, it can be shown that any function that equals zero almost everywhere is Henstock–Kurzweil-integrable and its integral equals zero; hence, the redefinition of a function on a set with zero Lebesgue measure does not affect Henstock–Kurzweil integrability and the value of the Henstock–Kurzweil integral.

Unlike the definition of the Lebesgue integral, the definition of the Henstock–Kurzweil does not require the introduction to measure theory. However, the notion of the upper measure (and, particularly, zero measure, used in the theory of the Riemann's integral, e.g. in the Lebesgue integrability criterion for Riemann integrability) turns out to be useful in the discussion of this integral: it allows defining measurable functions using Luzin's C-property, and one can show that every bounded measurable function is Henstock–Kurzweil-integrable.

The list of properties of the Henstock–Kurzweil integral that can be proved in a standard course include subsegment integrability, additivity in terms of functions and segments, integrability of all exact derivatives (this property does not hold in case of the Lebesgue integral), differentiability almost everywhere of an integral with a variable upper bound, analogues of the Levi, Fatou and Lebesgue theorems, etc. The details can be found, e.g., in [5].

4.2 The Gordon Criterion

The most contemporary result that is included into the courses of mathematical analysis for the students of the MSU Faculty of Mechanics and Mathematics is the Gordon criterion for the equality of the iterative limits. This criterion was published in 1995 (see [6]), and it is worth noting that in that paper this criterion appeared as a lemma for the theorem dealing with the Henstock–Kurzweil integration. However, the criterion itself is universal and can be applied to limit transitions different from integration.

A similar criterion for the series was introduced by A.A. Markov in 1890 (see [7]). However, Markov imposed an additional condition on the convergence of repeated series. As it follows from the Gordon criterion, this condition can be discarded.

Commonly courses of mathematical analysis include only sufficient conditions for the equality of iterated limits based on the notion of the uniform convergence. However, all these conditions are simple corollaries of the Gordon criterion.

Let us recall the notions required to formulate the Gordon statement.

A non-empty system $\mathfrak{B}$ of subsets of a set M is called a filter basis if $\emptyset \notin \mathfrak{B}$ and for every B_1, $B_2 \in \mathfrak{B}$ there exists $B \in \mathfrak{B}$ such that $B \subset B_1 \cap B_2$. A real-valued function f has a limit b with respect to $\mathfrak{B}$ ($\lim\limits_{\mathfrak{B}} f = b$), if this function defined on at least one element of $\mathfrak{B}$ and for every $\varepsilon > 0$ there exists $B \in \mathfrak{B}$ such that for every $x \in B$ the inequality $|f(x) - b| < \varepsilon$ holds.

Limit of sequence, limit of function, Riemann and Henstock–Kurzweil integral are particular cases of filter basis limits.

The Gordon Criterion. Let $\mathfrak{B}$ and $\mathfrak{D}$ be filter basis of X and Y respectively, and $h(x, y)$ is a real valued function defined on $X \times Y$. Assume that for all $x \in X$ and $y \in Y$ there exist limits $f(x) = \lim\limits_{\mathfrak{D}} h(x, y)$ and $g(y) = \lim\limits_{\mathfrak{B}} h(x, y)$. Then the following assertions are equivalent.

(i) The iterative limits $\lim_{\mathfrak{B}} f(x)$ and $\lim_{\mathfrak{D}} g(y)$ exist and are equal.

(ii) For every $\varepsilon > 0$ there exists $B_\varepsilon \in \mathfrak{B}$ such that for every $x \in B_\varepsilon$ there exists $D_x \in \mathfrak{D}$ with the following property: the inequality $|h(x, y) - g(y)| < \varepsilon$ holds for every $y \in D_x$.

Note that the first assertion is symmetric respect to variables x and y, while the second one is not. Hence, one can simply obtain a similar criterion by simple permutation of x and y in the second assertion. In this case the second assertion acquired the following form. For every $\varepsilon > 0$ there exists $D_\varepsilon \in \mathfrak{D}$ such that for every $y \in D_\varepsilon$ there exists $B_y \in \mathfrak{B}$ with the following property: the inequality $|h(x, y) - f(x)| < \varepsilon$ holds for every $x \in B_y$.

The proof of the Gordon criterion is eaily understandable for students. It is included in some contemporary textbooks, e.g., [8].

The introduction of the Gordon criterion into a course of mathematical analysis simultaneously leads to the growth of the generality and to the simplification of proofs of various theorems concerning uniform convergence and permutation of limit transitions.

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VI. Teaching analysis at universities and schools

Invited session speakers

THE PROBLEM OF A VIBRATING CHORD IN THE HISTORY OF MATHEMATICAL ANALYSIS

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Key words: vibrating chord, partial differential equation, boundary problem

AMS Mathematics Subject Classification: 35-03, 01A50, 01A55

Abstract. The studies on the problem of the vibrating chord became the territory in which were formed many principal notions and important methods concerning the theory of the boundary problems of the equations of mathematical physics, of the general geometrical theory of partial differential equations and the function theory.

1 Introduction

The study of the vibrating chord is a problem which constitutes one of the most important questions in the history of mathematics. It is enough to remember that reflections upon this phenomenon created the beginning mathematical natural science: it thus led Pythagoras to the important formula that «everything is a number». The sense of this formula (which became the basis of his philosophical theory) is that behind the variety of events of the environmental world we have to search for mathematical laws (i.e. number !) which rule these events.

This problem became one of the most famous in the mathematical analysis of the XVIIIth – XXth centuries. Between 1713 and 1715 B. Taylor was the first to start mathematical studies concerning the question of the transverse vibrations of a chord fixed at its extremities. He reduced this problem to the system of two ordinary differential equations $\frac{\partial^2 y}{\partial x^2} = -b^2 y$, $\frac{\partial^2 y}{\partial t^2} = -(ab)^2 y$, integrating them by assuming that the ends of a chord are fixed and at the initial moment that chord coincides with the axis of x. He then received a solution in the form $y = Asinbx \bullet sinaby$ (see [1]).

2 The vibrating chord problem as a problem of the theory of partial differential equations

It was only in 1747 that J. d'Alembert reduced this question to the form equivalent to the partial differential equation of a second order $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ thereby familiar for us. He searched its solution which satisfied with the initial $(u(x, 0) = u_0(x),$

$\frac{\partial u(x,0)}{\partial t} = \nu_0(x))$, and the boundary $(u(0,t) = u(l,t))$ conditions. He wrote the solution of this problem in the following form:

$$u(x,t) = \frac{u_0(x+at) + u_0(x-at)}{2} + \frac{1}{2a} \int\limits_{x-at}^{x+at} \nu_0(\xi)\partial\xi.$$

Thereby the formula is now known under his name. For questions of simplicity we assume that $a = 1$ and $\nu_0(x) = 0$. Without changing the essence of that matter, this makes our presentation more compact. As a result the vibrating cord equation takes the form of

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{1}$$

and d'Alembert's formula becomes:

$$u(x,t) = \frac{u_0(x+t) + u_0(x-t)}{2}. \tag{2}$$

The result of d'Alembert created a strong impression upon his contemporaries and drew their attention to the new field of research inaugurated by d'Alembert, i.e. the theory of partial differential equations development which he presented in 1747–1749 in his works [2, 2, 3] on the problems of aerodynamics and vibration theory (see [3, 6]).

Concerning d'Alembert's results it can be noted that he didn't reduce the problem to the equation (1) usual for us, but to an equivalent form – in the forms of the total differentials. The first partial differential equation can be found in d'Alembert's works in a form almost similar to the modern form. In his book "Traité de dynamique" [4] published in 1743 d'Alembert considered the problem of the oscillation of a thread chord suspended at its extremity. He reduced it to the equation $ddy = [\frac{dy}{ds} - (l-s)\frac{ddy}{ds^2}]dt^2$ – to a form almost similar to the modern one, even though the notations for the partial derivatives had not yet appeared[1] At that time however he was unable to integrate this equation. The method however by which he could integrate this equation (1) as well as a number of other ones [2] were adapted to another form, i. e. in the form of total differentials (see [3]). Thus, he wrote the equation (1) in the following form: by introducing the notations

[1]see We can find modern notations $\frac{\partial y}{\partial t}$, $\frac{\partial^2 y}{\partial t^2}$ etc. in J.A. Cousin [8] – prof. L. Pepe stressed our attention to this – although it became standard only in the XIX c.

$du = pdx + qdt$, $dp = rdx + sdt$, $dq = sdx + wdt$, d'Alembert proposed a problem to find the value of u in a such a way that the expressions

$$rdx + sdtsdx + rdt \tag{3}$$

shall become total differentials.

d'Alembert chose this way to present the problem due to the methods, invented by him (in [4]), which were adapted for this mode of writing — the form of total differentials. Thus for the equation (1) his method will be the following. First of all it was necessary to add the two above expressions (3) . Thus it obtained $(r + s)d(x + t)$. Due to the fact that it is also a total differential we can take $r + s$ as an arbitrary function of $x + t$:

$$r + s = F(x + t). \tag{4}$$

It is thereby necessary to subtract the second expression (3) from the first one above. It thus obtained $(r - s)d(x - t)$, which shall be also a total differential. It can be taken $r - s$ as an arbitrary function of $x - t$:

$$r - s = \Phi(x - t) \tag{5}$$

From (4) and (5) the following is obtained:

$$r = F_1(x + t) + \Phi_1(x - t), \quad s = F_1(x + t) - \Phi_1(x - t),$$

where F_1 and Φ_1 are the arbitrary fu nctions of their variables. Thereby from the integration of the system (3) we can receive p and q , and by the integration of $du = pdx + pdt$, we have the solution:

$$u = \varphi(x + t) + \psi(x - t),$$

where φ and ψ are the arbitrary functions of their variables, and also the solution of the equation (1) with the initial $[u(x, 0) = u_0(x), \frac{\partial(x,o)}{\partial t} = 0]$ and boundary conditions initial $[u(0, t) = u(l, t) = 0]$ is given by d'Alembert's formula (2).

This method proposed by d'Alembert concerning the integration of partial differential equations written in the forms of total differential expressions inaugurated a new field of research, i.e. the theory of partial differential equations. Moreover the impression on his contemporaries was produced by his elegant solution of the vibrating chord problem. Thus the problem became one of the main starting-points for the development of this theory. These studies began to take form to the theory of boundary problems for such equations and took the first steps in their geometric theory (thus the geometric picture of the transformation of its initial form was extremely visible). In the paper cite9 of 1753, published in 1755, L. Euler introduced the notations for the partial derivatives $(\frac{dy}{dx}), (\frac{ddy}{dx^2}), ...$ and the equations and the methods of their integration took the forms closely familiar to us. Euler wrote that [9]: "The vibrating chord problem thus reduced the following: regarding how to find the function y of two variables x and y, satisfying thereby the equation $(\frac{ddy}{dt^2}) = \frac{Fa}{2M}(\frac{ddy}{dt^2})$ and, moreover, the conditions mentioned above (i.e.: the boundary and the initial conditions – S.D.). Thus before taking into consideration these conditions, we can find all possible functions of x and t , which, were substituted in the place of y, thereby satisfying the equation $(\frac{ddy}{dt^2}) = \frac{Fa}{2M}(\frac{ddy}{dt^2})$. The general solution of this problem was given for the first time by Mr. d'Alembert".

This theory was developed at the beginning by two mathematicians — i.e. d'Alembert and Euler, who were competing each other. In the papers [3] and [10], published in 1749 and in 1752 respectively, d'Alembert inaugurated the method to separate the variables. He developed this method, as already mentioned, in the second edition of his book "Treatise on dynamics" (1758) [11]. Moreover, in d'Alembert's book [2] we also find the origins of the method concerning the characteristic changes of the variables (see [3, 6]) even though the meaning of these characteristic changes can be produced only in the equations, written in their ordinary form. This was realized by Euler in the paper [12] published in 1766. By performing such a change in the equation (1), he transformed it to the equation $\frac{\partial^2 u}{\partial t_1 \partial x_1} = 0$ — which now is possible to read in the mathematical textbooks.

Thereby the problem of vibrating chord became an "experimentum crucis" for the development of the most important methods regarding the integration of partial differential equations.

3 The discussion on the vibrations of a chord

As we have mentioned, d'Alembert's famous paper on the vibrations of the chord [2] was published in 1749 in the "Mémoires" of the Berlin Academy of Sciences for 1747. In the next volume i.e. 1748, which appeared in 1750, Euler with his text «Sur la vibration des cordes» [13] responded to it. He proposed his own method (though slightly different from d'Alembert's method) concerning the integration of the equation (1). He said, what is a most important, that the initial function may be any mechanical curve. This arbitrariness, as he said later [14], was limited only by condition that the curve is unbroken (that is to say practically that the function which gives this curve must be continuous according to our comprehension). On the one hand, Euler was based on the physical nature of the problem i.e.: the chord can be put in its initial position in every form which is an inherent arbitrary mechanical curve. On the other hand, the form of the solution (2) allows him to construct a solution in the whole plane x, t with an arbitrary initial data.

The proposed solution by Euler is not classical — it can have gaps not only from the second derivative, but from the first one. Thus essentially he introduced the weak (or generalized) solutions of the equations and in fact by this, it expanded the field of analysis from analytical functions to piecewise-differentiable functions. However, the generalization of the solution was not correctly made by Euler and moreover he didn't even give a definition of such a solution. In some cases, he wrote that this solution satisfied the equation, but he did not explain the meaning of these words (how can the function which have not the second derivative satisfy the equation (1). In other cases, discussing the properties of such solutions, he utilized the purely physical arguments. Euler's incorrectness of reasoning (to make it correct on the level of analysis at that time was not possible) caused a very negative reaction from d'Alembert.

D'Alembert considered that the initial curve (this is his main objection!) should be presented by a "continuous" function. He understood the continuity not in our meaning, but in the ordinary sense for the XVIIIth century: a function should be presented by a single analytic expression. "Dans tous autres cas le probléme ne pourra se résoudre, au moins par ma méthode, et je ne sais même s'il ne dépasse les forces de l'analyse connue", — he wrote in 1750 in his paper [10], published in 1752. Of course (and here he was quite right!), to make rigorous Euler's construction in the framework of the analysis of the XVIIIth century was impossible. However, the concept of "continuous" function (in the sense of the XVIIIth century) was internally inconsistent and therefore the position of the d'Alembert was also defective.

Explaining the essence of his objections, d'Alembert brought them to the two main restrictions (see, e.g., [3, 6, 15]) to the necessity of the "continuity" of the initial function, 2) to the necessity for the initial function to be twice continuously differentiable. At the same time for the knowledge of the mathematicians of that time the second restriction (twice continuously differentiable) should follow from the first restriction ("continuity").

Very soon this discussion was involved with the most prominent mathematicians of that time — Daniel Bernoulli, J.L. Lagrange, P. Laplace, G. Monge, etc. It might be stressed, that most of them accepted d'Alembert's objections: they agreed that, for arbitrary functions (for example, for functions whose first derivative is discontinuous) the proposed methods of integration of the equation (1) were incorrect. At the same time, they tried to justify the use of the "d'Alembert formula" in the case, where the initial function had discontinuities in the second and even in the first derivative. The mathematicians looked for the ways to save Euler's construction. And in fact they comported themselves in such a way on which in the XXth century the mathematicians began to introduce the weak solutions (see [15]). Since Lagrange replaced the equation (1) of the vibrating chord by the corresponding integral equation. Laplace discovered an another way i.e.: the initial curve which has discontinuities in the first derivative could be approximated by a sequence of smooth curves converging to the initial curve. For each of these smooth curves d'Alembert's construction is correct. The limit of the sequence of these solutions provides the desired solution for a non-smooth initial curve.

D'Alembert himself , as a matter of fact, standing on the position of a "classical solution", gradually clarified his position and eventually abandoned the "continuity" of the initial function by reducing his requirements [6, 15–17] for its smoothness: to its twice differentiability. Thus d'Alembert came to the perfect position of the classical solution. In reality as a solution Euler considered the weak (generalized) solution, but he didn't define it correctly. Why, however, did Euler continue to confirm (see, e.g., [14]) the correctness of his position in opposition to d'Alembert's position ? Had he not really seen the incorrectness of his arguments, which was clear for Lagrange, Laplace, and even for Condorcet? It is impossible!

We consider that he could see much further than the others – he felt the validity of generalizing the notion of a solution, although he could not to justify his "premonitions". This exceeded the capacities of the mathematics of the XVIII century. Here d'Alembert was right (we repeat his words of 1750 and already cited by us above) i.e. he declared: «je ne sais même s'il ne dépasse les forces d'analyse connue». The discussion on the problem of the vibrating chord marked one of the main ways of the development of the theory of differential equations, i.e. the introduction of the concept of weak (generalized) solution.

And perhaps one of the main results of this controversy was the following i.e.: during the discussion concerning the vibrating chord destroyed the conviction (established during the centuries) that mathematical structures expressed the properties of the space around us, i.e. its mathematical realizations. The most famous and radical event in the history of the destruction of this conviction is the case of the non-Euclidean geometry. With great difficulty, mathematicians rejected the idea that the Euclidean geometry of three-dimensional of Euclidean space is the geometry of the surrounding physical world. (It might be recalled that even the creator of the non-Euclidean geometry N.I. Lobachevsky tried to justify its "truth" by astronomical observations!). The mathematical analysis in the framework of these ideas constitutes a description of the mechanical processes in the surrounding space. The discussion about the vibrations of a chord, or more precisely about the nature of the arbitrary functions in the solution of the equation of vibrating chords, showed that a physical phenomenon must be distinguished from its mathematical model or models. That is impossible in the discussion of the mathematical question use uncritically the physical arguments as did Euler and d'Alembert.

4 The functional aspects of the dispute

Everyone familiar with the history of the controversy regarding the vibration of the chord could notice that we have forgotten to present one of the main aspects of this discussion, i.e. the functional one. In 1753 D. Bernoulli entered [14] in this controversy. He was based on his physical considerations and he proposed to seek a solution in the form of a superposition of harmonic oscillations — in the form of a trigonometric series

$$y = \alpha \sin \frac{\pi x}{l} + \beta \sin \frac{2\pi x}{l} + \gamma \sin \frac{3\pi x}{l} + ...,$$

where l denotes the length of a chord, $\alpha, \beta, \gamma, \dots$ are the functions of time.

Daniel Bernoulli was based on physical considerations, i.e. the sound of a chord contains a main tone and an infinite set of weaker overtones. Every tone corresponds to the shape of a chord in sinusoid form, therefore, the geometrical figure of a vibrating chord must be a combination of sinusoids (the "principle of superposition of oscillations" of Daniel Bernoulli). Bernoulli believed that any "unbroken" curve can be represented by a trigonometric series.

The above conception provoked a negative reaction on the part of all without any exception the participants of the discussion. Everyone believed that the functions, represented by the trigonometric series, form an very important, as well as a special class of functions contained in the analysis. This situation changed radically after the works of J. Fourier, who opened the world of harmonic analysis (see, e.g. [15]). At its creation an important role accomplished the problem of a vibration chord. During its research appeared the method of separation of variables (d'Alembert, Euler), and the idea to represent the functions by trigonometric series (Bernoulli). The representation of functions by trigonometric series traversed all the history of the theory of functions during the XVIII–XX centuries.

The problem of vibration of the vibrating chord held a special place also in the history of the Moscow school of the function theory. Moreover it is meaningful to remember the famous dissertation of N.N. Luzin "Integral and trigonometric series" (1915) [16]. From the history of the vibrating chord he began his course on the theory of the functions of a real variable. In 1960 the author of this report had a chance to attend at the Faculty of Mathematics and Mechanics of the M.V. Lomonosov Moscow State University a course given by his famous pupil Nina Karlovna Bari

(1901–1961). She also started her lectures stressing the history regarding this discussion. Thus the problem which became a source of insights for Pythagoras, acquired a new life in the mathematical analysis of the New Time (Taylor, d'Alembert, Euler, Bernoulli, Lagrange, Laplace, Monge, Fourier, etc.) and continued to live in the memory and also in the practice of the modern scientific community.

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ICT AND THEIR USE IN EDUCATION

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Abstract. The modern information and communication technologies (ICT) give us powerful tools to percept, record, store, process, examine, transmit, and present the information. We list the ICT, discuss them and their role in education.

Introduction

ICT is the abbreviation for the information and communication technologies. The information technologies (IT) are the technologies for collecting, storing, processing, and presenting the information. The communication technologies (CT) are the technologies of receiving, transforming, and transmitting the signals. There are other definitions. For example, the Information Technology Association of America defines information technology as “the study, design, development, application, implementation, support or management of computer–based information systems”. The term has also been applied more narrowly to describe a branch of engineering dealing with the use of computers and telecommunications equipment to store, retrieve, transmit and manipulate data ([1], Preface). The words “information and communication technology” are often used as an extended synonym for information technology, but is a more specific term that stresses the role of unified communications and the integration of telecommunications (telephone lines and wireless signals), computers as well as necessary enterprise software, middleware, storage, and audio–visual systems, which enable users to access, store, transmit, and manipulate information [2].

The IT and CT are closely related because the information and signal are closely related. The basic term of them is the signal. It can be defined with the mathematical accuracy. The definition of information uses mathematics together with other sciences.

The signal is nothing else but a function of time with special properties. The value of the function at a given time is usually the quantity of electric current in a line of some electric circuit or the difference of electric potentials of two vertices of the circuit. Any signal is equal to zero before some instant called the beginning of the signal. The signals form a linear space. Because the energy is always finite, the Joule–Lenz law implies that any signal is square integrable, i.e., belongs to the space $L^2(R)$. Any function in $L^2(R)$ that is equal to zero at every time before some instant is a signal.

The information is also a function but not necessarily of time. For example, an information can be presented by a picture, i.e., by a function of coordinates x and y. When we perceive such picture, our eyes scan the image and transform it in one or several functions of one variable, i.e. in some signals. The brain analyzes them. Note that the brain is connected to the organs of sense by the cables (nerves) only. Therefore, it cannot perceive anything else but signals.

Every existing object emits many signals. (In particular, they show that the object exists. Therefore, some fishes and animals camouflage their presence by emitting the signals similar to those emitted bysurrounding object.) If the object changes, the signals change too. Such changes of the object can be considered an information, if an equivalence relation exists between signals of different changed objects. Then the objects are called the information carriers. For example, an information can be presented as a file saved on hard drive by special changes of its cylinder magnetism. When we open the file the information is presented in the random access memory by the changes in its electric cells, in the visual card memory by the changes in its electric cells and on the display by the changes in pixels. When we print this file the information is presented by the changes in the paper sheet, because the dots appear on it. All five changes are equivalent and therefore represent one and the same information. In other words, an information is a family of equivalent signals. Having processed each of these signals in its proper way we obtain one and the same signal. These ways are specified by the formats of presentation of information (message) by the signals. The formats of computer files are defined by their extensions like doc, bmp, avi, etc.

The stress we lay upon the fact that the information is represented by changes of an object properties and significance of corresponding formats is justified, e.g., by the practical method to hide an information. There are methods of two kinds for that purpose. The methods of the first kind are given by cryptography. It suggests the use of formats that are not known to third parties. The methods of the second kind are given by steganography — the art and science of producing signals in such a way that no one, apart from the sender and intended recipient, suspects the existence of the signal, and, therefore, of information.

When we teach, we try to transmit the students the signals the most similar to those the real objects we speak about emit. The best way is to show the students the real objects. It is not possible in many cases. Then we show the pictures of real objects. If this is not possible, we show drawings or schemes. Verbal descriptions are often used, especially in mathematics, because its objects are abstract. It is clear that the information that the students get is not exactly that the teacher tries to give them. Therefore, an error of misunderstanding always exists and is considerable. The cognitive abilities of students, in particular, their imagination, reduce the error, but it does not disappear. Moreover, different students perceive the information with different errors, because their cognitive abilities are different. ICT help to reduce such errors, because ICT enable the teachers to create the models of objects under study that emit the signals the most appropriate for creating correct ideas in the brains of students. ICT help the students to perceive the signals, extract the information from them and process it.

The paper is organized as follows. In the next section, we describe how ICT can augment the human senses. This can be important contribution to education because facilitate the perception of information by the students. Therefore, we deal with the ICT role in collecting and receiving information. In Sec. 2 we describe the ICT for recording, storing, processing, transmitting, receiving, and presenting information. In Sec. 3 we draw the first conclusions from what is said in Secs. 1 and 2. We describe there the ideas on augmented (amplified) intellect, man–computer symbiosis, hybrid intellect. Their consequences for education are discussed in the Conclusion.

1 ICT for augmenting the human senses

ICT augment the human abilities to percept and process the information. (Therefore, we can classify the ICT according the abilities they augment.)

All perception involves signals in the nervous system, which in turn result from physical or chemical stimulation of the sense organs. For example, vision involves light striking the retinas of the eyes, smell is mediated by odor molecules and hearing involves pressure waves. Perception is not the passive receipt of these signals, but can be shaped by learning, memory and expectation. Perception involves these "top–down" effects as well as the "bottom–up" process of processing sensory input. The "bottom–up" processing is basically low-level information that's used to build up higher-level information (i.e. shapes for object recognition). The "top–down" processing refers to a person's concept and expectations (knowledge) that influence perception. Perception depends on complex functions of the nervous system, but subjectively seems mostly effortless because this processing happens outside conscious awareness. The teachers, however, should pay much attention to the perception, because it determines the results of learning.

The traditional five senses are sight, hearing, touch, smell and taste. Humans are considered to have at least five additional senses that include nociception (sense of physiological pain), equilibrioception (balance, or vestibular sense), proprioception (kinesthetic sense of relative positions of the body parts), chronoception (sense of time), thermoception (sense of heat and cold), and possibly magnetoception (sense of magnetic field) [3]. There are also six interoceptive senses that are stimulated from within the body. Here we focus on the traditional five senses because they are used in education.

Sight or *vision* is the capability of the eye(s) to focus and detect images of visible light on photoreceptors in the retina of each eye that generates electrical nerve impulses for varying colors, hues, and brightness. There are two types of photoreceptors: rods and cones. Rods are very sensitive to light, but do not distinguish colors. Cones distinguish colors, but are less sensitive to dim light. There is some disagreement as to whether this constitutes one, two or three senses. Neuroanatomists generally regard it as two senses, given that different receptors are responsible for the perception of color and brightness. Some argue that stereopsis, the perception of depth using both eyes, also constitutes a sense, but it is generally regarded as a cognitive (that is, post-sensory) function of the visual cortex of the brain where patterns and objects in images are recognized and interpreted based on previously learned information. This is called visual memory.

ICT augment the human ability to see by magnifying glasses, microscopes, telescopes, X-ray machines that create images using the Roentgen waves, night-sight detectors that create images using the infra-red waves, visible detectors that can see light that is very dim with respect to the background, radars, tomography, and holography. Computers are used for processing electromagnetic signals and for representing them in the frequency range that is perceptible by the human eye(s). Photo and video cameras are used for recording the images and augmenting in this way the human visual memory. Special graphic (bmp, gif, jpeg, tiff, etc) and video (3gp, flv, avi, mpg, vob, etc) formats are designed and used. There are many computer programs for creating and processing images and videos. For example, we mention here the free programs Splashup, iMovie, JayCutMany, PhotoPeachUntil, and Effect Generator.

Pictures are two-dimensional. Therefore, it is sometimes difficult for students to imagine a three-dimensional object shown on picture. Using the rules of perspective, we facilitate the student task. Now we have stereology — a branch of science dealing with determination of the three-dimensional structure of objects based on two-dimensional views of them. The stereograph, otherwise known as the stereogram, stereoptican, or stereo view, when viewed with a stereoscope, appears three-dimensional.

Using hypertext we can make our description closer to the object or process we try to give an idea of. KompoZer, a free web design program, is useful for creating hypertexts. Methods of fast reading accelerate the perception of text descriptions.

Hearing or *audition* is the sense of sound perception. Hearing is all about vibration. Mechanoreceptors turn motion into electrical nerve pulses, which are located in the inner ear. Speech perception is the process by which the sounds of language are heard, interpreted and understood. The process of perceiving speech begins at the level of the sound within the auditory signal and the process of audition. After processing the initial auditory signal, speech sounds are further processed to extract acoustic cues and phonetic information. This speech information can then be used for higher-level language processes, such as word recognition. Speech perception is not necessarily unidirectional. That is, higher-level language processes connected with morphology, syntax, or semantics may interact with basic speech perception processes to aid in recognition of speech sounds. It may be the case that it is not necessary and maybe even not possible for a listener to recognize phonemes before recognizing higher units, like words for example.

ICT augment the human ability to hear with the help of devices that amplify the sounds. Some microphones can detect sound at extremely low volumes. Ultrasound devices detect sounds of very high frequencies. The devices for listening the sky were used during World War II in the systems of defense against the enemy aviation. Echolocators (sonars) use sounds for vision. Some sonar equipment detects sounds of very low frequencies or pitches.

Stereophony, quadraphony and similar methods are used to achieve the high fidelity (HiFi) of sound records and reproducing. Sound-recording devices produce audio files. Several audio formats are designed and used, e.g., wav, aiff, ape, flac, mp3, ogg, etc. The audio files can be created and processed by the computer programs like Audacity.

Sound generators can transform texts in sounds just like we read books to our children. This is useful, e.g., for learning the foreign languages.

Touch or *somatosensory*, also called tactition or mechanoreception, is the process of recognizing objects through touch. It involves a combination of somatosensory perception of patterns on the skin surface (e.g., edges, curvature, and texture) and proprioception of hand position and conformation. People can rapidly and accurately identify three-dimensional objects by touch. This involves exploratory procedures, such as moving the fingers over the outer surface of the object or holding the entire object in the hand. Haptic perception relies on the forces experienced during touch.

ICT augment the human ability to recognize objects through touch using special devices. We deal with some of them when we use touch sensible displays of our mobile phones. On the other hand, a project is to augment human perception by connecting sensors to vibrotactile outputs on the skin [2]. The human skin can detect the pressure of touch, but there are devices that are much more sensitive to the pressure caused by an object than the skin is. For example, there are scales that can measure the weight or pressure caused by small insects that a human could not detect.

Taste or *gustation* is the ability to perceive the flavor of substances including, but not limited to, food. Humans receive tastes through sensory organs called taste buds, or gustatory calyculi, concentrated on the upper surface of the tongue.

ICT augment the human taste using chemical analyzers. There are many chemical devices to detect various compounds that go well beyond what the human tongue can detect. For example, litmus paper can tell if a compound is acidic or base.

Smell or *olfaction* is the other "chemical" sense. Unlike taste, there are hundreds of olfactory receptors, each binding to a particular molecular feature. Odor molecules possess a variety of features and, thus, excite specific receptors more or less strongly. This combination of excitatory signals from different receptors makes up what we perceive as the molecule's smell. In the brain, olfaction is processed by the olfactory system.

To augment his or her smell people use dogs. ICT augment the human smell using gas analyzers. Smoke detectors can "smell" chemicals from burning that we are unable to smell or at levels below our threshold. There are other commercial devices used to detect the "smell" of various chemicals beyond the capabilities of the human nose.

In closing this section, we note that there are projects aimed at establishing a man — computer interaction in which all five senses are used (see, e.g., [3, 6]).

2 ICT for recording, storing, processing, transmitting, receiving, and presenting information

Among the first tools for record of information are pen and paper. ICT gives us the shorthand — methods of rapid handwriting using simple strokes, abbreviations, or symbols that designate letters, words, or phrases. We have also keyboard, scanner, and speech recorders. Position devices of input are mouse and electronic pen(point). Human independent devices for record of information are photo and video cameras and tools of sound record. ICT gave a new form of information record — hypertext, and new form of information storage — database.

The tools of storage are the information carriers. They are human's memory, stone, wood, papyrus, paper, cardboard, canvas, plaster etc. (for texts and painting); wood, plaster, granite, bronze, marble etc. (for sculptures). ICT enlarge the storage tools by films, magnetic stratums, optically inhomogeneous stratums, and by electronic memory of flash cards.

When we store information, archivers are useful for saving capacity of information carriers. There are many archivers. They save the information in different formats such as zip, rar, 7-zip, arj, cab etc.

The main tool for processing the information is the human brain. ICT give it assistants like editors of texts, images, sounds, and videos; computer algebra systems (CAS); the programs that sort, recognize, convert, interpret, translate .etc. Expert systems, search systems and programs for computer added design (CAD) are also useful. Some programs make parts of what is called the artificial intellect.

The information transmission tools are mail, telegraph, telephone, radio communication, and TV systems. Amplitude, frequency and phase modulation, and digital–analog transformation are used to embed information in a signal. The circuit switching was the main method in cable connection. When Internet was designed the packet switching was chosen to be its method of connection. Radio connection systems now use this method also (GPRS means General Packet Radio Service).

There are many applications that work using Internet: e-mail, file transfer, World Wide Web (WWW), Skype etc. The documents that circulate in WWW are in hypertext format. The powerful search engines (Google, Rambler, Yandex etc.) help us to find an information presented in the WWW sites.

In information reception we use demodulation, and analog–digital transformation ICT does not affect them considerably.

ICT for presenting information are markup languages (TeX, RTF, HTML, Postscript, etc.), computer programs for typesetting, spell check, design, and edition, hypertext systems and speech synthesizers. Multimedia and rich media tools are of special importance.

Multimedia includes a combination of text, audio, images, animation, video, or interactivity content forms. Multimedia is usually recorded and played, displayed or accessed by information content processing devices, such as computerized and electronic devices, but can also be part of a live performance. The rich media is an interactive multimedia. It is the most suitable for use in education.

3 Ideas on augmented (amplified) intellect, man — computer symbiosis, hybrid intellect and their consequences for education

In the past devices and tools were used for scientific purposes, for observation and measurement (telescope, thermometer, galvanometer etc.). Later people realized that they strengthen sense organs or expand their possibilities. ICT enable us to save time, accelerate the feedbacks, augment intellect. More and more the engineering become man — computer. A hybrid intelligence appears. The new information medium is spontaneously formed. It is called cyberspace. A new way of evolution of Homo sapiens opens — evolution of human's intellect together with artificial intellect of his computer and together with his communication tools.

Design of devices that amplify the human mental abilities began long before the people start to feel a need in them. The important stake in a history of development of these devices is the V. Bush paper "As We May Think" [4]. It describes a device called Memex. It is a store-house of the information on microfilms and is to be used for extending the human memory.

W.R. Ashby wrote of "amplifying intelligence" in his Introduction to Cybernetics (1956). Related ideas were explicitly proposed as an alternative to Artificial Intelligence by Hao Wang from the early days of automatic theorem provers.

"Man — Computer Symbiosis" is a key speculative paper [8] published in 1960 by J.C.R. Licklider, which envisions that mutually-interdependent, "living together", tightly-coupled human brains and computing machines would prove to complement each other's strengths to a high degree. In Licklider's vision, many of the pure artificial intelligence systems envisioned at the time by over-optimistic researchers would prove unnecessary. (This paper is also seen by some historians as marking the genesis of ideas about computer networks which later blossomed into the Internet).

At the same time D. Engelbart reasoned that the state of our current technology controls our ability to manipulate information, and that fact in turn will control our ability to develop new, improved technologies. Inspired by the Bush paper, he thus set himself to the revolutionary task of developing computer-based technologies for manipulating information directly, and also to improve individual and group processes for knowledge-work. Engelbart's philosophy and research agenda is expressed in the 1962 research report [9].

The tools for amplification of intelligence were developed on the basis of rapid development of micro-, magneto- and optoelectronics and consequently were continuously improved. Now we see that G. Birkhoff was right, when wrote in 1969 that "... we can foresee more and more growing symbiosis of man and machine, in which each partner fulfills tasks, most appropriate for him and it" ([10], Sec.22.1). The machine, which is spoken about here, is the computer.

The system “the man + computer, amplifying his intelligence”, has many titles: symbiosis of man and computer, synergism of man and machine, etc. We prefer the term that V.F. Venda uses in the title of his book [11], — the hybrid intellect.

Conclusion

ICT, which we discussed in Secs. 1 and 2, can be put in the framework of traditional education. Such use facilitates the student perception of what their teachers explain them. Using ICT we can form, shape, and modify objects even if their visualization or sensual perception is impossible for some reason. This is important because the obviousness disappears with expansion of area of known (from great geographical discoveries to Micro world and Mega world).

ICT can accelerate verification of student works, thus reducing the time between the day when a student does the work and the day when the student receives the teacher appreciation and comments. This is important for establishing the suitably fast feedback required for control of the educational process. Using ICT we can create many different but similar tasks in such a way that every student can get its personal task. This enables us to work with every student individually.

The economy of educational time due to the use of ICT allows to expand and to modernize the content of tutoring, approaching it to the modern level of developments of science, engineering and technologies and social relations. ICT give us a possibility to use progressive methods of tutoring (converted scheme, problem tutoring etc.) to demonstrate solutions of problems that are interdisciplinary.

The use of those ICT, which we discussed in Sec. 3, implies considerable changes in education. The educational part of cyberspace complements the traditional educational information medium generated by printed textbooks and manuals. At a stage of purposeful shaping of this medium it becomes a united educational–scientific information medium (UESIM) [12]. It will help us to overcome dissociation and fragmentariness of conventional studies of separate disciplines. It will be a basis for interdisciplinary (entire) vision of the World.

The new object of teaching appeared — a tandem “student + computer” [13]. This sets a new aim of education — formation of hybrid intellect — as outcome of teaching the tandem. This is in agreement with the fact that the most promising of tools, which strengthen abilities of the man up to a level adequate to complexity of tasks, facing to him, is the computer, provided that it is used for immediate and always accessible help (Engelbart’s thesis). The teacher and student can receive necessary immediate and always available help due to mobile access to the Internet.

Having this in mind we create the Mathematical server accessible from mobile phones. Using the phones, the students send the problems to the server. The server solves the problems and sends the solutions back. The software installed on the server is open. Therefore the students can install it on their computers. In this way the server is a model of the computers that are the parts of the tandems.

The server amplifies the student intelligence, namely, such its properties, as

— wholeness of perception (by using methods for analysis of dependencies on $n > 2$ variables);
— memory (by using data bases and search engines);
— ability to solve complicated problems (by using computer algebra system).

The description of the server and recommendations on how to use it in teaching higher mathematics are in [14]. The systems like the server, as well as other systems that augment the human intellect, can considerably change the society. This perspective is described in [15].

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THE LIMIT OF A SEQUENCE IN GEOMETRICAL SITUATIONS

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Key words: limit of a sequence, geometrical example

AMS Mathematics Subject Classification: 97D80

Abstract. In the paper we consider the situations, in which the existence of the limit is obvious, even we know neither the terms nor the formula for nth term of the sequence, but only geometrical construction is well defined. As a starting point we consider the following problem:
Having the given triangle ABC we construct the triangle PQR in such a way, that B is the centre of the line segment AP, point C is the centre of BQ and A is the centre of CR. Can we find the ratio of areas of the triangles ABC and PQR?
Developing the described situation, we can visualize the consecutive steps and to foresee the value of the limit of the sequence of the fractions obtained. In this situation the next step allows to find the nth term of the sequence and using the theory of limits examine our conjecture. The presentation will also include other examples of this type.

1 Introduction

There is no doubt that the concept of the sequence limit is one of the most difficult notions in the curriculum of mathematics at the senior secondary school level. Usually, in accordance with the tradition of mathematics teaching in Poland, the part of curricula dedicated to that theme is included in algebra. In the frame of introducing the main properties of arithmetic and geometric sequences, examples, definition and the main properties of the limit of a sequence are introduced. In that way, the concept of the limit is, in students' minds, associated with the algebraic examples and context. Such way of teaching does not allow constructing, in students' minds, good intuitions connected with the concept of the limit. As a result, students are frequently able to correctly reproduce the definition of the limit of a sequence and to formulate the main properties of that concept, but are completely helpless when the problems appear outside algebra.

In this paper we would like to discuss another way to introduce the concept of the limit, the way based on the examples of geometrical situations. The example we will present below is based on another methodological conception in which we refer to geometrical intuitions before the introduction of numerical examples of limits and procedures connected with processes of forming the concept of the limit of a numerical sequence. In the situation we propose, students have the opportunity to undertake different types of creative mathematical activities, such as formulating new questions, forming hypotheses, making an attempt to verify them (search for a proof or an counterexample), generalizing both results and reasoning process which has led to the results, searching for interesting specific or boundary cases, extending the problems discussed. On frequent occasions those situations form part of multi-stage problems (Klakla,2006), which are special sequences of tasks, problems and didactic situations that combine various types of creative mathematical skills. We also consider geometrical situations in which the existence of the limit is obvious, even if we know neither the terms nor the formula for the nth term of the sequence, and only the geometrical construction is well defined. Let‘s remark that geometrical examples connected with the concept of the limit of a sequence last year are getting larger and larger in modern textbooks for students. Such examples can be found in the new textbook of calculus for mathematics students of pedagogical universities, M. Aslanova and A. A. Fiedorova, 2008, p.15–16.

Starting point

As a starting point we consider the following problem (Problem 1, fig.1):

Having the given triangle ABC we construct the triangle PQR in such a way that B is the centre of the line segment AP, point C is the centre of BQ and A is the centre of CR. Can we find the ratio of areas of the triangles ABC and PQR?

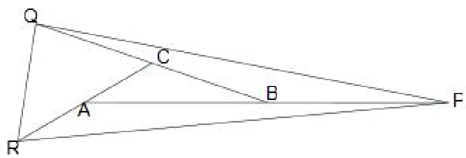

Figure 1

Let‘s underline the open formulation of the problem, neither the suggestion about its existence nor the solution. The problem can be resolved on the level of elementary school students (to solve it, it is enough to know how to find the area of a triangle). For senior secondary school students that problem seems to be difficult, they usually try to use their "strong" mathematical knowledge, for example the theorems concerning trigonometric functions, without referring to elementary formula for the area of a triangle. During the observation of the students’ work we don‘t make any suggestions concerning the existence or the non-existence of the solution of the problem. We asked students for the information about the angles and the sides of the triangle considered, which concerns only the triangle ABC and does not concern any other. The students examined were conscious that there is no such information.

First conclusions

At that moment we asked whether it was possible to solve this problem in that situation? If we suppose so, we must accept that the solution can depend neither on the length of the sides nor on the angles of the triangle. Hence, we can conclude:

If the solution of the problem exists, the ratio of areas of the triangles ABC and PQR cannot depend on the type of the triangle ABC. The ratio we are looking for must be the same for all triangles. So, if we are looking for the value of the ratio of the areas of the triangles ABC and PQR, we can consider a convenient triangle, for example, an equilateral one and to try to resolve the problem in that case.

Second step - special case

We formulate a new problem (Problem 2, fig.2):

Having the given equilateral triangle ABC, we construct the triangle PQR in such a way that B is the centre of the line segment AP, point C is the centre of BQ and A is the centre of CR. Can we find the ratio of areas of the triangles ABC and PQR?

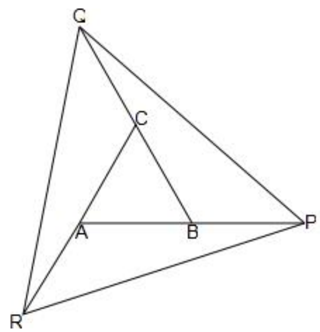

Figure 2

Students try to draw the situation described and spontaneously formulate the observation: *If we start our construction with the equilateral triangle ABC, the triangle PQR is also equilateral.* Students usually need teacher‘s intervention to understand that from observation of the design the unique conclusion is the hypothesis, and we need a proof to accept this observation as a fact. Easy reasoning (triangles ARP, BPQ and CQR are congruent) permit justifying the truth of the hypothesis. Let‘s remark that in fact this is not the solution to the problem given, but a good occasion to ask a new question:

If we apply the same construction to a square, must we obtain a square too?

Third step - square

We formulate a new problem (Problem 3, fig. 3):

Having the given square $ABCD$, we construct the polygon $PQRS$ in such a way that B is the centre of the line segment AP, point C is the centre of BQ, D is the centre of CR and A is the centre of DS. Is the polygon $PQRS$ also a square?

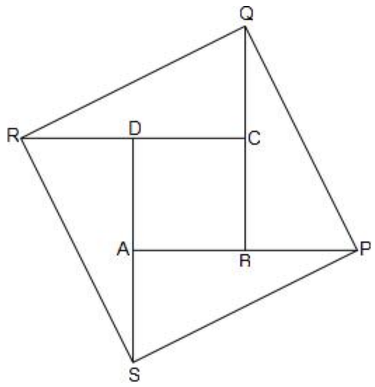

Figure 3

During this step students, after drawing the correct design, are usually able to find the proof, using the precedent reasoning as a pattern. Let‘s remark that the proof that triangles ASP, BPQ, CQR and DRS are congruent is not enough, we need also to check that the angles of the polygon $PQRS$ are rights. This not very difficult reasoning can lead the students to the idea of the **generalization of the problem** and ask the same questions in the case of regular polygon of n sides.

Fifth step - generalization

We formulate a new problem (Problem 4)

Having the given regular polygon P_n we construct another polygon Q_n using the same construction as in precedent cases. Is the polygon Q_n also regular?

The answer is "yes, it is". The detailed reasoning and necessary computations in this case can be made by means of , "as the support", the design of the polygon of 5 or 6 sides, in accordance with the beautiful idea of G. Polya, that "geometry is the art of correct reasoning using incorrect designs".

Sixth step - looking for a limit

Developing the situation described we can visualize the consecutive steps. This is also a good occasion to observe the sequence of figures obtained in all precedent steps (Fig.4), and to extend this observation "using imagination".

It is not difficult to guess that the value of the limit of the sequence of ratios of the areas of the corresponding polygons is equal to 1. The next step allows

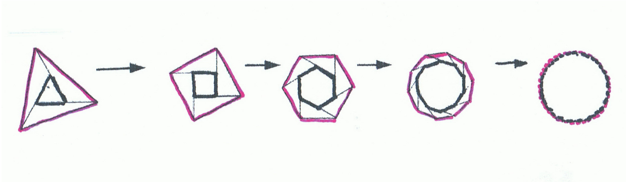

Figure 4

finding the nth term of the sequence and the use of the theory of limits examine our conjecture. For this we can apply the theorem about the ratio of areas of similar figures (as a consequence two regular polygons of n-sides are similar and the ratio of their areas is equal to square of their scale factor). An interesting task consists of finding the number n of sides of the regular polygons in such a way that the ratio of their areas minus one should be less than, given in advance, a small fraction.

Final remarks

The situation described in the paper can be extended in many other directions. For example we can "prolong" the sides of polygons 3-times, 4-times, $\ldots, n$- times and formulate similar questions. We can consider the prolongation of all sides in two directions or consider the similar situations in $\mathbb{R}^3$. Many others possibilities of interesting students' activities and geometrical examples of geometrical situations leading to the concept of the limit of a sequence can be found in the book of V. A. Gusjev and M. Klakla:

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METHODS OF MATHEMATICAL ANALYSIS AND APPLICATIONS

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AMS Mathematics Subject Classification: 97M20

Abstract. The article focuses on the need for the teaching of mathematical analysis in higher technical institutes with applications: to a specialty (or group of specialties), the profile of the university, in physics, other sciences, including the humanities, but with priority to the natural sciences. Classification of professional tasks in terms of complexity, with examples, is given.

The content of the course of mathematical analysis for students of engineering has its own specific profile that is different from the course of mathematical analysis for physical-mathematical professions of classical universities.

Future engineers better perceive mathematics through the application of studied techniques, primarily, to their specialties. In order to enhance students' motivation and compliance with the principle of an optimum combination of fundamental and applied orientation, the authors developed an appropriate methodology of teaching of mathematical analysis in the universities of engineering and technical profile [1].

A course of higher mathematics in a technical college should provide a system of mathematical knowledge and to show the breadth of their possible applications, to give an understanding that math does not study the phenomenon but its mathematical model, learn how to create mathematical models. Essential to achieving this is a purposeful selection of professional tasks, referring to that throughout the course of higher mathematics, students are convinced that they have studied the mathematical methods are applicable to solving specific specialized tasks.

It seems reasonable, constantly in the study of higher mathematics course to acquaint students with the various professional tasks. Without going into the essence of these tasks by entering only the special concepts, it is necessary to demonstrate the solution of classical mathematical problems on these examples.

Previously the authors of this article introduced the classification of professional and other applications on the four levels of difficulty [2]:

— First level - professional analogues of classical tasks and formulas (examples of continuous and discontinuous signals in the basis analysis, signals at the output of differentiating and integrating circuits in the course of the differential and integral calculus, the calculation of the spectral density and the discrete spectrum of various signals in the section of the Fourier series, etc.). The aim is to achieve the ability to reproduce by the pattern of building a mathematical model of a typical professional task.

— Second level - training and professional tasks with the elements of mathematical modeling (eg, an approximation of signals by partial sums of Fourier series). Goal is to build skills in the construction of mathematical model of the professional tasks of medium complexity, its solution by analytical and numerical methods.

— Third level - training and research professional tasks or a lab and coursework (for example, a comparison of approximations to a given signal by decomposition of it on various systems of orthogonal functions). Goal is to build of skills in the mathematical modeling of professional tasks of higher complexity, its decision, analysis, and professional interpretation of results.

— Fourth level - research professional tasks (for example, a study of noise immunity of the posted signal reception by methods of TFKP and random processes). The purpose - to attract gifted students to participate, under the guidance of a teacher in research work.

Thus, in the process of learning math courses it is formed mathematical competence based on vocational and applied orientation (at some of the sources they are defined as professionally-applied mathematical competence (PPMK). In this case the mathematical competences were introduced as mathematical knowledge and skills in the activity approach, i.e. in view of applying them in solving professional tasks and applications.

Already at the first stage of a repetition of the basic elementary functions in practical training lessons and for self-help solving should be used tasks that contain special terminology. For example:

The tasks of the first level:

Graphs of elementary functions

1. When measuring the voltage at the amplifier output for different values of the input voltage was obtained the following table:

U_{bx}	0	5	10	15	-5	-10	-15
U_{bux}	0	20	40	60	-20	-40	-60

Plot the dependence U_{bux} on U_{bx} and find values for U_{bux} at $U_{bx} = 1B$; $U_{bx} = 7B$;$U_{bx} = -3B$.

2. The voltage on the capacitor when charged from a voltage source $U = 2$ changes in law $U_c = U(1 - e^{-1/\tau}$, where τ - time constant. Construct a graph of the function $U_c(t) at \tau = 10^{-6}c$. Find the value U_c at $t = \tau$, and $U = 10B$.

3. The amplifiers are called logarithmic, if their amplitude characteristics are described by a system of equations:

$$\begin{cases} U_{bux} = K_0 U_{bx}, U_{bx} < U_n \\ U_{bux} = K_0 U_n + a ln U bx / U_n, , U_{bx} \geqslant U_n \end{cases}$$

where K_0 is a gain in the linear regime,U_n is an amplitude of input voltage, from which the characteristic becomes logarithmic; a - the coefficient determining the slope of the logarithmic characteristics. Plot the dependence of U_{bux} on U_{bx} at $U_n = 100$, $\alpha = 1$, $K_0 = 1$.

4. Given two AC source with EMF: $e_1 = E_{01}\sin(\omega t + \varphi_1)$; $e_2 = E_{02}\sin(\omega t + \varphi_2)$. Find the sum $e_1 + e_2$ and difference $e_1 - e_2$. Build in single coordinate system the graphs of functions: $e_1 = E_{01}\sin(\omega t + \varphi_1)$; $e_2 = E_{02}\sin(\omega t + \varphi_2)$, $e_1 + e_2$, $e_1 - e_2$.

The tasks of the second level

Differential and integral calculus of functions of one variable

1. The compression x of the spring is proportional to the applied force to it $F = kx$. Let $k = 1000h/m$. Determine the work on the compression of the spring by 5 cm.

2. Determine the work (in joules) in respect of the satellite during the ascent from the surface to a height H km. Mass of the satellite is m t, the Earth's radius is $6380km$, assume the acceleration g due to gravity at Earth's surface set equal to $10m/s^2$.

3. In the workplace salary of each employee Q (rubles) and the number x workers employed in production are related by $Q = L - x^2 - \alpha/x$, where L and α — constant characterizing the production potential of the collective. According to the "golden rule of growth" the salary should be defined so as to take the greatest possible value. Find, when$L = 1500$, $\alpha = 16000$, the specified rule number of workers if it is additionally known that the labor collective has L job places $(N = 15, 18, 25)$.

Vector analysis

1. Determine the vector lines of the magnetic field generated by electric current strength I, flowing in an infinitely long straight wire.

2. The laws of electromagnetic theory formulated in the form of Maxwell's equations. In the case of non-conducting homogeneous and isotropic medium and in the absence of charges and currents, this system has the form

$$\begin{cases} \dfrac{\varepsilon}{c}\dfrac{\partial E}{\partial t} = [\Delta, H] \\ \dfrac{\mu}{c}\dfrac{\partial H}{\partial t} = -[\Delta, E] \\ (\Delta, E) = 0 \\ (\Delta, H) = 0 \end{cases}$$

where E, H are intensity vectors of the electric and magnetic fields,ε, μ are the coefficients of electric and magnetic permeability, c is velocity of light in vacuum (ε, μ – constants).

Using Maxwell's equations and differential operations of the first and second orders, to show that the vectors E, H satisfy the wave equations

$$\frac{\partial^2 E}{\partial t^2} = \nu^2 \Delta E$$

$$\frac{\partial^2 H}{\partial t^2} = \nu^2 \Delta H,$$

where $\nu = \frac{c}{\sqrt{\varepsilon\mu}}$.

3. Intensity vector tube is called a flow of vector field over cross-section of the tube. At the entrance of the ray tube cross section S1 enters the flow of wave energy (e.g. acoustic) intensity I1. Determine the intensity and amplitude of the wave at the exit of the ray tube cross section S2 (field is solenoidal).

Model calculations, which are now practiced in almost all technical universities, should necessarily contain the tasks with professional content

Examples of professional tasks of typical calculations of a various degree of complexity:

1. It is known that the voltage$U$ across the inductance associated with the current rate of change by the relation: dI/dt. In according to Ohm's law the voltage across the resistor R satisfies the relation:$U = RI$, and the voltage across the capacitance C associated with the charge q of the equality:$U = q/C$. Write the differential equation for the oscillator circuit and examine its solutions for different values of the quantities $\alpha = R/2L$ and $\omega_0 = (LC)^{-1/2}$.

2. Find dependence on distance$z$ of the tension electric field E_z, generated by a ring of radius R carrying charge q. The axis oz is perpendicular to the plane of the ring and passes through the center of the ring. Using the techniques of differential calculus to build graph depending E_z on z at the given values of$R$ and q. Answer: $E_z = qz[4\pi\varepsilon_0(z^2 + R^2)^{3/2}]^-1$.

3. Write in differential form, Maxwell's equations, represented in integral form

$$\oint_i \overrightarrow{H} \bullet d\overrightarrow{r} = \frac{\partial}{\partial t} \iint_\sigma \overrightarrow{D} \cdot d\overrightarrow{\sigma} + \iint_\sigma \overrightarrow{I} \cdot d\overrightarrow{\sigma}$$

where l is closed circuit,σ is a surface spanned by this circuit, $\vec{H}$ is a tension of the magnetic field, $\vec{D}$ is an induction of electrical field, $\vec{I}$ is a current density vector.

Answer: $rot\overrightarrow{H} = \frac{\partial\overrightarrow{D}}{\partial t} + \overrightarrow{I}$

4. The force F, acting on a unit positive charge is called an intensity of the electric field and is denoted E. According to Coulomb's law a point charge q creates an electric field of strength $E = q/4\pi r^2 B/m$ at a distance r. Determine the work of the field when you move a unit positive charge from a point, which is r_1 distant from q , to a point in the distance r_2.

Answer: $A = (q/4\pi)[(1/r_1) - (1/r_2)]j$.

The resulting value is called the potential difference between the points, spaced at distances r_1 and r_2 from the charge .

Training research professional tasks (the third level)

Example. Approximation of signals using orthogonal polynomials and special functions.

Analyze the given signal $S(t)$ by approximating polynomials (Laguerre, Legendre, Chebyshev, etc.) to ensure the required accuracy of approximation with a minimum number of members of the Fourier series.

Professional formulation of the problem.

Given the signal pulses $S(t)$ of known shape (rectangular, triangular, Bell, etc.), $F(\omega, r)$ is a function describing the distortion of the environment (damping factor) at a frequency ω and the distance r from the point of emission, $F(\omega)$ is a transfer function of the receiver. Find a distorted form of the signal pulse $S * (t)$.

Mathematical model

The distorted shape of the pulse signal $S * (t)$ - is the inverse Fourier transform

$$F * (\omega) = S(\omega)F(\omega, r)F(\omega),$$

where

$$S(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} S(t) exp(-i\omega t) dt.$$

For many forms of signals, commonly used in electronics, the solution of this problem in an analytical form is great computational difficulties. In these cases, the student must write a program to solve the problem on your computer or use a ready-made software. The input solution algorithm:$S(t), F(\omega, r), F(\omega), r, t$; output: an array of values $S*(t)$, graphic $S*(t)$.

The problems of this type arise in the investigation of noise immunity of signal reception.

The use of Information Technology

A distinctive feature of modern education is an engineer, as seen in the previous example, must be fluent in information technology. The effectiveness of the educational process is the consistent use of computer technology in the application of mathematical methods in solving practical problems. And for the acquisition of computer skills should be studied in all courses (and, of course, all math) to demonstrate and encourage students to perform independent works (current, standard, term, etc.) on a computer.

Examples.

1. Plotting functions.

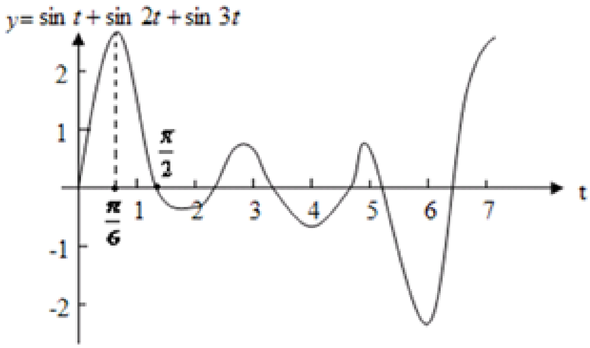

Figure 1

2. Expansion of functions in a Fourier series and illustration of the approximation signal using Fourier polynomials.

3. Obtaining the modulated oscillations

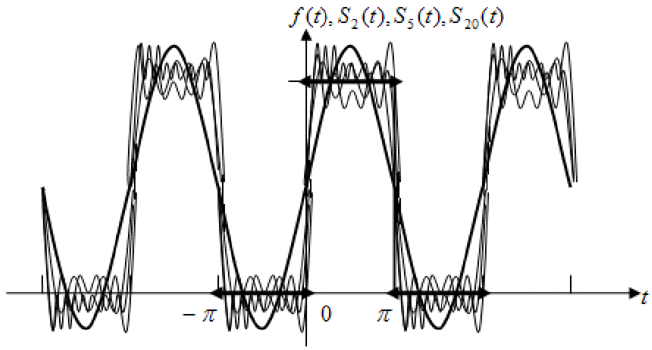

Figure 2

Put the sinusoidal waveform

$$x(t) = F_0 \sin(\omega_0 t + \varphi_0)$$

where the parametersF_0, ω_0, φ_0 are constant, amplitude modulation. Then the modulated vibration takes the form:

$$x(t) = F_0[1 + mf(t)] \sin(\omega_0 t + \varphi_0)$$

here the factor $(1 + mf(t))$ characterizes the effect called modulation, $f(t)$ is modulation function, $|f(t)| \leqslant 1$, and m is modulation depth, with changes in the range from 0 to 1. Let specify a particular type of modulation functions, such as:

$$f(t) = \sin \Omega t \quad \text{(sinusoidal modulation)}.$$

The spectrum of the obtained modulated oscillations is discrete and consists of three spectral lines, corresponding to a frequency ω_0 called the carrier frequency, and two additional frequencies, which appeared as a result of modulation: $\omega_0 - \Omega$ and $\omega_0 + \Omega$. These two frequencies are called side frequencies or satellites. The figure shows a graph of the modulated oscillations.

4. Draw the surface potential distribution in all points of the plane inside and outside the circle of radius 5, which is supported by the stationary distribution of electric potential

$$u = (5, \varphi) = f(\varphi) = 5(cos^2(\varphi) + 3), \quad 0 \leqslant \varphi \leqslant 2\pi$$

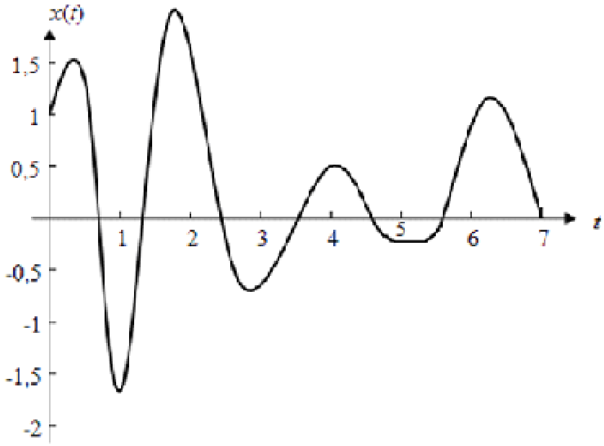

Figure 3

Solve the problem by using the first 20 terms of approximate solutions, computed in the environment of the program Mathematica.

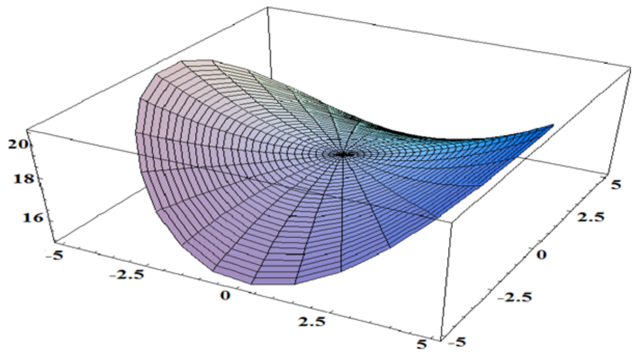

Figure 4

Summary (conclusions):

1. The introduction of professional tasks in the educational process in higher mathematics departments of technical universities (model calculations, laboratory and course work) increase the interest of students in the study of higher mathematics, including mathematical analysis.

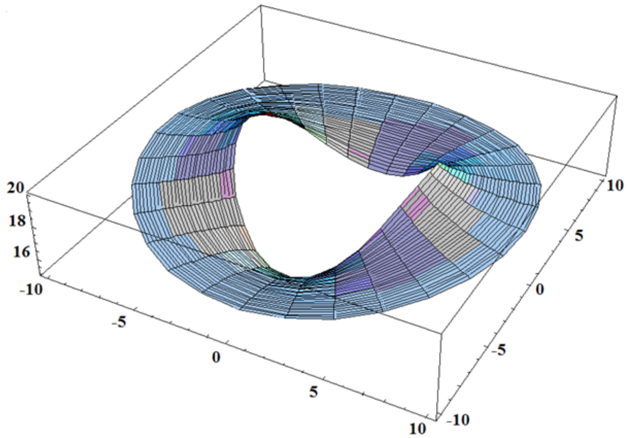

Figure 5

2. The use of personal computers in the solution of professional theoretical and applied tasks gives much greater effect than using them to solve computational problems only.

Both factors lead to increased efficiency of educational process in general.

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PREPARATION OF UNIVERSITY TEACHING PERSONNEL

N. Kh. Rozov

Key words: university personnel teaching, university teachers, scientific and research universities

AMS Mathematics Subject Classification: 97B50

Abstract. If we really want to achieve progress in higher education and science, we certainly need to be aware of the problem concerning establishment of purposeful training of qualified teaching personnel within the walls of universities. And the only way to achieve it is to organise an education program for postgraduate (PG) students who will have scientific preparation as well as guided practice of teaching. Thanks to this practice the PG school will take the fully legitimate and system-forming function.

Today the "Scientific and Research Universities" conception is negotiated and implemented actively. The key aspect of the conception is the universities' top-priorities to be organised. These priorities include:

— attracting students to the scientific and research self-directed work;
— increasing the scientific potential of university teachers;
— stimulating the broad expansion of works of all the research fellows (first and foremost together with students).

Nowadays the “Scientific” indicators in particular are considered to be essential in determination the universities prestige value, in providing a solution of the financing question and so on. The amount of the scientific publications and licences, the research government grants and the awards for scientific achievements, the equipment status of laboratories and the scope of finance for research are all those important points that are considered to be the indicators of today's popular and modern HEI (Higher Educational Institutions) ranking.

The objective of this conference paper is to emphasise the key role of one more fundamental aspect of universities development that is obviously underestimated and even understated. The issue is that a university is first of all an educational institution. The major central tasks of the higher education have always been:

— equipping the big amount of students with education of the best quality,
— training decent specialists,
— training the considerable amount of practitioners (not only certain research scientists) for an effective operation in different aspects of life of the society.

Achieving these tasks is necessary since the “intellectual capital” consists of only the amount of the high-class specialists. Such a brain capital can be considered as the genuine guarantee of successful and dynamic sci-tech and economic development of a country.

No doubt that the involvement of students in scientific work is all-important. However, the participation in the scientific research is impossible without at least minimum preparation and body of basic subject knowledge. A junior student should get such a preparation and body of knowledge during the first years of university studies program. The efficiency of the student's further scientific work depends on the extent of quality and learner-centred definitive training in the very beginning of the university studies and on the qualification and capability of pedagogues to identify and to maintain the potential each of the students.

Therefore, it is necessary to prepare qualified teachers to train qualified specialists.

So what is the definition of the today's notion of a “Higher school teacher”? It is said that a university teacher must have deep knowledge in their subject matter, be involved in research work, be regularly published in magazines, participate in different forums and implement their scientific results as a matter of practice. It is an open secret that the publishing list supplemented by the citation index lies at the root of the policy of HR allocation and promotion at higher educational institutions.

However, it would be an illusion to believe that the university teacher is just the scientific worker, even though the very talented one. Any researcher can deal with slips of paper very easy in a library or elaborate an experiment in a laboratory. But students are too different in terms of intellectual potential and creativity level, professional motivation and personal interests, psychological aspects and chosen educational subject. And the knowledge exposure process of these varied students is not the same as the process of even successful participation in the development of fundamental and applied science. Dealing with students it's important to have some other skills and qualities, and even flair.

Teaching any subject is the genuine and subtle art as well as the most complicated area of human activity where excellent knowledge of subject matter and personal achievements in creative research can't always guarantee the success.

It is also necessary to be capable with all the technological complex of approaches in knowledge transfer and pedagogical process organisation. So *it is essential to know not only what to teach, but also how to teach.* In accordance with up-to-date scientific language this kind of knowledge is possible to identify as the “framework of reference for teaching”.

Such university teachers will not come from without. The university itself should prepare such experts as it's possible to bring up teaching personnel only at higher educational institutions and nowhere else. But how does the preparation of such teaching personnel work? As a rule, young people succeed in disserting apply for teaching. In fact they gained the diverse body of knowledge related to their profession and become promising researchers in the long run. Nevertheless they are not aware of common principles, up-to-date methods and practical approaches of effective teaching in most cases.

There is a wide-spread opinion that any smart young researcher is able to teach everything oneself using "cut and try method" and therefore to make a good teacher. But this idea is considered to be a false statement. Even if we agree with this notion, isn't that inhumane to allow students to appear as "lab rats" only due to out effort to implement the idea of self-education for them as future teachers? The method of "teaching oneself to make a good teacher" can't compete against meaningful and comprehensive guided practice. It's obvious that those who don't want to or unable to teach are impossible to initiate into the art of teaching in any case. But those who do have the vocation for teaching and ready to pass on the torch to the young people should be given a helping hand on their thorny way to making a good teacher.

If we really want to achieve progress in higher education and science, we certainly need to be aware of the problem concerning establishment of purposeful training of qualified teaching personnel within the walls of universities. And the only way to achieve it is to organise an education program for postgraduate (PG) students who will have scientific preparation as well as guided practice of teaching. Thanks to this practice the PG school will take the fully legitimate and system-forming function.

To solve the problem of teaching personnel preparation for higher school the Faculty of Educational Studies (the FES) of the Lomonosov Moscow State University (MSU) was established on Rector's initiative of Victor A. Sadovnichii by MSU Academic Council decree in 1997. Those PG students who are involved in research work and in varied spheres of science at different MSU faculties learn the ropes of the educational work principles here at the FES.

Classes at the faculty start just after the postgraduate entrance and lasts 3 to 4 times a week in the evening. The education program takes 1,5 or 2 years of studying. The MSU PG students get free education. There is no severe requirement concerning in-class attendance. However it is obligatory for students to pass exams or so-called pass-fail exams covering all the subjects of the curricula. Each student chooses their own sequence of studying subjects in the frame of education program. The graduates are awarded the qualification of Teacher or Instructor of institution of higher education and have the right to work as teachers of their major subjects in secondary and high schools and institutions of higher education.

The curricula at the Faculty of Educational Studies of MSU consist of several modules. Two modules are dedicated to psychology and pedagogics. Today the personalised learning method plays a prominent part in teaching. This means that a teacher is aware of psychological tools, is sociable, and knows how to find an approach to an individual learner, how to be on common ground with the audience and to manage it. That is why our audience learns a lot about general psychology as well as social, pedagogical, personalised one. Teaching efficiency can be defined by the knowledge of fundamental principles as well as skills to use necessary teaching techniques in accordance with the specific character of teaching process. All the questions above cover the subject matter of classes dedicated to higher school and general didactics.

Certainly it is not the question of training good specialists of these scientific areas, but it is the question of the primary exposure to foundations of educational work. For this reason classes are conducted according to elaborated and bottom-line curricula that includes minimum of information along with practically oriented coursebooks of certain limit. It is reasonable as our learners are PG students who are involved in scientific work and do not have enough free time. There is no need to disregard the fact that the audience of the Faculty of Educational Studies has an opportunity to attend classes and lectures, learn more about practice and adopt valuable experience from prominent MSU professors and the best instructors as well as secondary and high school teachers.

It is also important to notice that the significance of these two modules is beyond the scope of typical preparation to educational work. Nowadays socialisation of any specialist is becoming more and more wide-spread and faster. Furthermore it is beginning to play a more vital role. All the trends of modern science, production, economic management turn to the "landfills of teamwork and cooperation".

That is the reason why practically oriented PG school should give the opportunity to its students to get the professional psychological and pedagogical education. This is crucial for:

— becoming a part of an already existing team;
— working in this group of people intelligently, creatively;
— managing people fairly and wise;
— being tolerant, patient, sociable, humane;
— being able to keep oneself within bounds,
— to explain obscure things easily and find a way around conflict situations,
— to listen to and be in touch with other people;
— being ready for team work and cooperation.

The third module of education includes studying and discovering up-to-date educational technology (EdTech), foremost computer-based technologies. It's hardly thinkable to imagine teaching any subject at the University of the 21st century without them. Especially if it is about such relevant forms of educational processes as (remote/ distance) e-learning, ongoing training and so on.

The acquirement of practical methods and approaches in teaching a particular subject necessary for the future pedagogue is of great importance. This point is included in the forth module of the curricula which should become a guided practice for PG student (it's possible due to discussion sessions, progress tutorials, guest studies, intro classes and so forth). The educational work is the great and complex art, some kind of a "one-man show". And every class is some kind of masterclass. That is why the future teacher should get to know the basis of acting and oratorical techniques, rhetoric, speech and discussion techniques. PG students get such kind of knowledge within the fifth module.

It is important to mention one more issue. A true teacher shouldn't practice self-denial in getting to know the subject matter and using the modern technique of knowledge transfer. Such a specialist needs to be well-rounded, erudite, cultural person confident in socializing in every sense. This person is a true erudite in different areas of science, technology, culture and art. Therefore during the educational process PG students are allowed to choose one or more electives in order to learn more about the history and achievements of the human civilization. Education at the Faculty of Educational Studies of MSU serves interests of PG students who can either feel a sharp desire for making a good teacher or understand at a good hour that "teaching is not my cup of tea".

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EXTREMAL PROBLEMS: HISTORY AND THE GENERAL APPROACH TO THE THEORY OF NECESSARY CONDITIONS

V. Tikhomirov

Key words: Lagrange principle, calculus of variations, optimization, convex analysis

AMS Mathematics Subject Classification: 49K05 49K15 49K30

Abstract. The paper is devoted to the universal principle for neces- sary conditions in the theory of extremal problems which is called the Lagrange principle

1 Introduction

Extremal problems are formulated initially in terms of the science or the field of applications which gives rise to them, i.e., in terms of geometry, engineering, physics, etc. In order to provide for their mathematical treatment, one has to translate them into the language of mathematics. Such translation is called *formalization.*

To formalize an extremal problem, one has to specify a *function* f (along with its *domain of definition* X, $f : X \to \overline{\mathbb{R}}, \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$) to be minimized or maximized, as well as a *constraint* $C \subset X$. As a rule, constraints are specified by equalities and inequalities.

The problem: «minimize (maximize) f under the constraint C» is written as

$$f(x) \to \min\ (\max), \quad x \in C. \tag{P}$$

(when writing $f(x) \to \text{extr}$ we mean that both problems for maximum and minimum may be considered). If $C = X$ the problem (P) is called *the problem without constraints.*

2 History

There are at least three reasons which stimulated us to solve or investigate extremal problems: pragmatical goals, desire to explain phenomenons of Nature and curiosity, based on tendency of human race to reach the very essence of everything. Among the first problems which were considered in mathematical literature one calls *the classical isoperimetrical problem* on the curve of the fixed length which contain the figure of maximal area.

Let us consider *Example 1* (the Euclid's geometrical problem). *In a given triangle ABC inscribe a parallelogram $ADEF$ ($EF||AB, DE||AC$) of maximum area.*

A motivation for consideration of such problem is a mathematical curiosity. This is a setting of the geometrical problem without formulae. It is easy to show that this problem can be formalized as follows:

$$x(x-b) \to \min, \; 0 \leqslant x \leqslant b,$$

where b is the length of the side AC and x is the length of AD.

Until 17th century extremal problems were solved by individual methods. The first general approach for solution of extremal problems is due to Pierre Fermat .

In 1638 Fermat in a letter addressed to Descartes (see [1]) expressed an idea which on our modern language is possible to translate as follows: *if a smooth function attains a local extremum at a point, then the derivative of the function at the point equals to zero.*

The following problem was discussed in "Mathematical Principles of Natural Philosophy" (1687) (may be the greatest scientific memoir in the history of science). It is called *Newton's aerodynamical problem:*

Find the solid of revolution of given length and width that is subject to least resistance while moving in a (rare) medium.

In a formalization there exists the constraints such as $y'(x) \geqslant 0$ (inequality constraints of derivatives). Such constraints did not consider in the Calculus of Variations. The theory of problems of the Calculus of Variations with such constraints was created only in the 50-60th of the 20th century. This theory was called the Optimal Control.

The following problem was posed by I. Bernoulli in the paper "Problema novum, ad cujus solutionen matematici invitatur" (Acta Eruditorum, 1696): *let two points A and B be given in a vertical plane. Find the curve that a point M, moving on the path AMB must follow such that, starting from A, it reaches B in the shortest time under its own gravity".* It was the first problem of Calculus of Variations.

Wt see that the first problem of the Optimal Control was posed and solved by Newton earlier than the brachistochrone, the first problem of Calculus of Variations.

3 General approach to the theory

The Lagrange principle for necessary conditions.

— In solving extremal problems one can use a unified approach which we call a single general idea which we call the *Lagrange principle.* It may be formulated as follows: *to solve an extremal problem with constrains, one can construct the*

Lagrange function of the problem, and then write down the necessary condition in the similar problem on the extremum of the Lagrange function "as if the variables were independent" (in Lagrange's own words), and finally investigate the relations thus obtained. This idea is the main principle of the first part of the theory of extremal problems. In this section we demonstrate its application to some important classes of extremal problems.

3.1. Problems without constraints. The simplest extremal problem is a *problem without constraints*

$$f(x) \to \text{extr}. \tag{P_1}$$

The first general method of solving (smooth) problems (P_1) in case of one variable was described by P. Fermat (even before calculus was developed). In the modern language it reads: *the main linear part of the increment of f at an extremum point equals to zero.* In this form it remains valid also in the infinite-dimensional case: *if $\widehat{x}$ is a local minimum point of a function f differentiable at the point $\widehat{x}$ then the following equality holds*:

$$f'(\widehat{x}) = 0. \tag{1}$$

The Fermat's idea was expressed in terms of derivatives by I. Newton and G. W. Leibniz, but nevertheless this result is called now *Fermat's theorem.*

3.2. The simplest problem of the calculus of variations. After Fermat, Newton, and Leibniz the theory of extremum made a sudden jump from one variable to infinitely many variables. This happened in 1696 when J. Bernoulli stated the *problem on brachistochrone*, where the argument was an infinite-dimensional object, viz., *a smooth curve* joining two given points of the plane. Later J. Bernoulli proposed to his student L. Euler to find a general approach to problems of brachistochrone type. Euler summarized his results in the memoir "Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes sive solutio problematis isoperimetrici latissimo sensu accepti" published in 1744.

Euler considered the problem

$$\int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))dt \to \text{extr}, \quad x(t_0) = x_0, \quad x(t_1) = x_1, \tag{P_2}$$

which is referred to as the *simplest problem of the calculus of variations.* Here $L = L(t, x, y)$ is a function of three variables called the *integrand* of the problem. The problem of brachistochrone has the following formalization: $J(x(\cdot)) = \int_0^a \frac{\sqrt{1+\dot{x}^2(t)}}{\sqrt{2gx(t)}}\,dt \to \min$, $x(0) = 0$, $x(a) = b$.

A necessary condition for extremum at $\widehat{x}(\cdot)$ in the problem (P_2) is the following *Euler's equation*:

$$-\frac{d}{dt}L_{\dot{x}} + L_x = 0 \Leftrightarrow -\frac{d}{dt}L_{\dot{x}}(t,\widehat{x}(t),\dot{\widehat{x}}(t)) + L_x(t,\widehat{x}(t),\dot{\widehat{x}}(t)) = 0. \qquad (2)$$

3.3. The Lagrange multipliers rule. A general principle for investigation of problems with constraints was first stated by J. L. Lagrange. In his book [2] he wrote:

" On peut les réduire à ce principe générale. Lors qu'une fonction de plusieurs variables doit être un maximum ou minimum, et qu'il y a entre ces variables une ou plusieurs équation, il suffira d'ajouter à la fonction proposée les fonctions qui doivent être nulles, multipliées chacune par une quantité indéterminee, et là chercher ensuite le maximum ou minimum comme si les variables étaient indépendanbes; les équation qu'on trouvées serviront à déterminer toutes les inconnues."

Lagrange considers here a finite-dimensional problem

$$f_0(x) \to \text{extr}, \quad f_i(x) = 0, \quad 1 \leqslant i \leqslant m, \quad \lambda = (\lambda_0, \ldots, \lambda_m). \qquad (P_3)$$

His idea is as follows: compose the function $\mathcal{L}(x,\lambda) = \sum_{i=0}^{m} \lambda_i f_i(x)$ (we somewhat change Lagrange's formulation multiplying the functional itself by an indefinite factor too) and write down the necessary condition in the problem without constraints $\mathcal{L}(x,\lambda) \to \text{extr}$, i.e., apply equation (1) to obtain the relation

$$\mathcal{L}_x(\widehat{x},\lambda) = 0 \Leftrightarrow \sum_{i=0}^{m} \lambda_i f_i'(\widehat{x}) = 0. \qquad (3)$$

(The function $\mathcal{L}(x,\lambda)$ is called the *Lagrange function*, while the numbers $\{\lambda_i\}_{i=0}^{m}$ are the *Lagrange multipliers*.) The result is as follows: *if the problem* (P_3) *satisfies some smoothness conditions then equality* (3) *holds at a local extremum point* $\widehat{x}$. This result is referred to as the *Lagrange multipliers rule.*

Lagrange himself applied the idea of *elimination of constraints* by means of the Lagrange function (not only in finite-dimensional problems, but also in problems of calculus of variations) at least since 1750.

In the books [3] the authors tried to demonstrate the universal applicability of (somewhat extended) Lagrange's approach, according to which a meaningful necessary condition in a problem with constraints can be obtained by *writing down the Lagrange function and deriving then the necessary condition for its extremum "as if the variables were independent".* In [1]–[5] this approach is called the *Lagrange principle.*

We illustrate the application of the Lagrange principle by two examples.

3.4. Lagrange's problem in calculus of variations. Consider the problem:

$$\int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \to \text{extr},\ \dot{x} = \varphi(t, x, u),\ x(t_0) = x_0,\ x(t_1) = x_1, \qquad (P_4)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, f is a function of $n + r + 1$ variables and φ is an n-dimensional vector function of the same variables. The variables x are called the *phase variables*, and u are the control. The problems of the form (P_4) are called the *Lagrange problems of calculus of variations.* Let us apply to them Lagrange's directions.

The Lagrange function here has the form

$$\mathcal{L} = \int_{t_0}^{t_1} L(t, x, \dot{x}, u)\, dt, \quad L = \lambda_0 f(t, x, u) + p(t) \cdot (\dot{x} - \varphi(t, x, u))$$

(and it has been written in this form since Lagrange's time).

Obtaining the necessary condition in the problem for extremum of Lagrange's function without constraints reduces to writing down the Euler equation in x and u. As a result, we arrive at the equations

$$-\frac{d}{dt} L_{\dot{x}} + L_x = 0, \qquad L_u = 0, \qquad (4)$$

which were derived in different forms for long time from brachistochrone in 1696 till forties of the last century (a book summarizing this topic was published in 1939 by the American mathematician G. Bliss).

Assuming that f and φ are sufficiently smooth, one can prove that *if a pair* $(\widehat{x}(\cdot), \widehat{u}(\cdot))$ *affords a local minimum in the problem* (P_4) *considered in* $C^1([t_0, t_1], \mathbb{R}^n) \times C([t_0, t_1], \mathbb{R}^r)$ *(when both nearness of phase coordinates and controls is taken into account; such extremum is said to be weak) then equations* (4) *hold.*

3.5. The optimal control problem

Now let a function f and a mapping φ be defined on the product $[t_0, t_1] \times \mathbb{R}^n \times U$, where U is a subset of $\mathbb{R}^r$. Assume that f and φ are continuous in all variables and smooth in x. Consider the problem of the same form as (P_4) assigning its own number to it:

$$\int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \to \text{extr},\ \dot{x} = \varphi(t, x, u),\ x(t_0) = x_0,\ \ x(t_1) = x_1. \tag{P_5}$$

The problems of type (P_5) are called *optimal control problems*. Their study was begun at L. S. Pontryagin's seminar during 50-s of the last century. In such problems a *strong* extremum is of interest when only nearness of phase coordinates is considered, while nearness of controls is neglected.

The set U in (P_5) may be arbitrary, for example, it may consist of finitely many points. The methods of smooth analysis used to derive equations (2) are inapplicable here. But the Lagrange principle remains valid.

The Lagrange function of the problem (P_5),

$$\mathcal{L} = \int_{t_0}^{t_1} L(t, x, \dot{x}, u)\, dt, \quad L = \lambda_0 f(t, x, u) + p(t) \cdot (\dot{x} - \varphi(t, x, u)),$$

is the same as in the previous example. Consider two problems: first, we fix $\widehat{u}(\cdot)$ and minimize over $x(\cdot)$, then we fix $\widehat{x}(\cdot)$ and minimize over $u(\cdot)$.

In the first case the minimization problem is of simplest problems of calculus of variations type, and a necessary condition in this problem is Euler's equation in x:

$$-\frac{d}{dt} L_{\dot{x}} + L_x = 0. \tag{5}$$

In the second case we get the problem

$$\int_{t_0}^{t_1} L(t, \widehat{x}(t), \dot{\widehat{x}}(t), u(t))\, dt \to \min \quad \text{in} \quad u(\cdot), \tag{P_5'}$$

with $u(t) \in U$. It is easily seen that (under very mild assumptions) a criterion for minimality of $\widehat{u}(\cdot)$ has the form of the following "minimum principle":

$$\min_{u \in U} L(t, \widehat{x}(t), \dot{\widehat{x}}(t), u) = L(t, \widehat{x}(t), \dot{\widehat{x}}(t), \widehat{u}(t)). \tag{5'}$$

Changing signs we arrive at the form in which a necessary condition for the problem (P_5) was stated by Pontryagin's school:

$$\max_{u\in U}(p(t)\cdot\varphi(t,\widehat{x}(t),\dot{\widehat{x}}(t),u)-\lambda_0 f(t,\widehat{x}(t),\dot{\widehat{x}}(t),u)) =$$

$$\max_{u\in U}(p(t)\cdot\varphi(t,\widehat{x}(t),\dot{\widehat{x}}(t),\widehat{u}(t))-\lambda_0 f(t,\widehat{x}(t),\dot{\widehat{x}}(t),\widehat{u}(t)). \quad (5'')$$

The combination of relations (5) and (5″) is known as *Pontryagin's maximum principle.*

An application of the maximum principle to the simplest variational problem leads to Legendre's and Weierstrass' necessary conditions, while its application to the second variation of the functional yields Jacobi's necessary condition.

The Lagrange principle is the main device in the subsequent treatment of concrete problems.

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VI.1 Teaching analysis at universities

(Sessions organizers: L.D. Kudryavtsev, S.A. Rozanova, N.Kh. Rozov)

THE FULL MONGE PROBLEM SOLUTION BASED ON THE LINEAR PROGRAMMING (LP)

A. Andrianov

Key words: L.V. Kantorovich, Monge Problem, Linear Programming (LP)

AMS Mathematics Subject Classification: 01A60, 91-03

Abstract. It is shown the connection between Transportation Problem and Problem of Masses Translocation and how investigation of them come to results which include the solution of Monge Problem as their simple corollary.

1 Introduction

In this article we are going to show the very interesting phenomenon (which has more than two hundred years history) – a situation when a particular method, that has been developed to solve rather specific practical problem, turned out to have a lot greater sphere of applicability. More than that – it has in addition influenced the purely mathematical area by presenting a brand new way to solve classical mathematical problems, that had been stated more than a hundred years earlier. We are talking about the linear programming and its connection with the full solution for the rather famous Monge Problem.

We are going to start with some terminology and two main problems (Transportation Problem and Problem of Masses Translocation) formulation. And then we will return to the core aspect of this article, which is about the connection between this two problems and the Monge Problem (to be more precise, we will show, how to use knowledge, gained from the investigation of those two problems to solve the Monge Problem as their simple corollary).

2 Transportation Problem

As a professor of mathematics and head of the Department of Mathematics at the Institute of Mathematics and Mechanics of Leningrad State University, L.V. Kantorovich was consulted by some engineers from the Laboratory of the Veneer Trust who were concerned with the efficient use of machines. From that practical contact and in connection with analysis of one such question of production organization and planning has sprung his report – work [1] in which the important class of finite-dimensional extremal problems had been studied. Application of classical mathematical analysis methods towards problem of this type had shown themselves to be low-efficient. The investigations that had been carried in purpose to solve this problem, had led eventually to origination of the new mathematical discipline which had been named the Linear Programming.

The problem which is of particular interest in regard with topic of this paper – the problem of translocation of masses has a direct communication with the following simplest linear programming problem, which is widely used now in routine of planning railway, automobile, air and water transportations.

Transportation Problem.

Components of given vector

$$\varphi = (\varphi_1, ..., \varphi_m) \tag{1}$$

(to be more precise, their absolute values) characterize the volumes of production (in case of $\varphi_k \leqslant 0$) or consumption (in case of $\varphi_k > 0$) of some homogeneous product in given m places (for example, cities or stations) with numbers k from set $K = \{1, 2, ..., m\}$. It is also assumed that the total volume of consumption coincides with the total volume of production, in other words:

$$\sum_{k \in K} \varphi_k = 0. \tag{2}$$

The transportation plan is defined by the choice of the matrix

$$\Psi = [\psi_{ij}]_{i,j \in K}, \psi_{i,j} \geqslant 0, \quad i \in K, \quad j \in K, \tag{3}$$

which elements indicate the volumes for transportation from each place i to each place j, which are going to be implemented according to this plan. It is obvious that during the realization of the chosen transportation plan to each place $k \in K$ it will be imported to $\sum_{i \in K} \psi_{ik}$ and export from it $\sum_{j \in K} \psi_{kj}$ units of product which is being consider. And this means that matrix (3) defines feasible transportation plan if folowing balance relationships are satisfied:

$$\sum_{i \in K} \psi_{ik} - \sum_{j \in K} \psi_{jk} = \varphi_k, \quad k \in K. \tag{4}$$

And as it concerns the total cost, associated with the realization of each transportation plan (3), they in the model which is under our consideration are defined by the formula

$$\tau(\psi) = \sum_{i \in K} \sum_{j \in K} r_{ij} \psi_{ij}, \tag{5}$$

where r_{ij} – given nonnegative values which are characterizing the costs of transportation of product unit from place i to place j.

Thereby, the system of feasible transportation plans is defined by a set Ψ_φ of nonnegative solutions (3) of linear equalities system (4). And required, the most economical one, which is called optimal transportation plan $\psi \in \Psi_\varphi$ is characterized by the property that for it total costs (5) reach minimum.

Using general results of linear programming theory (or by direct reasoning) it is easy to verify that the stated above extreme problem is always solvable. And that the feasible transportation plan (3) is optimal when and only when exists such vector $u = (u_1, u_2, ..., u_m)$, that:

$$u_j - u_i \leqslant r_{ij}, \quad i \in K, \quad j \in K, \tag{6}$$

and also $\psi_{ij}(u_j - u_i - r_{ij}) = 0, i \in K$, $j \in K$. The last requirement means that if in feasible plan, which is under consideration, from place i to place j the non-zero transportations ψ_{ij} are being planned then the appropriately inequality (6) has to be executed as an equation.

3 Problem of Masses Translocation

The infinite-dimensional analog (better to say – generalization) for Transportation Problem which is described below, was studied for the first time in 1942 by Kantorovich in paper [2] (see also [3–5]), where Kantorovitch considered the general problem of optimal translocation of mass in compact metric space.

Problem of Masses Translocation.

In this problem we have the finite number of places from the previous one replaced by an arbitrary metric compact K with the metric $r(t, s)$ which characterizes the costs connected with the translocation of unit mass from whatever point $t \in K$ to whatever point $s \in K$. As an analog for vector (1) satisfying condition (2), we have defined on the system B of Borelean sets of compact K countably additive function φ, the positive variation of which $\varphi_+(K)$ coincides with its negative variation $\varphi_-(K)$, in other words such that:

$$\varphi(K) = \varphi_+(K) - \varphi_-(K) = 0. \tag{14}$$

Let us remind here, that for $e \in B$: $\varphi_+(e) = \sup\{\varphi(e') : e' \in B, e' \subset e\}$, $\varphi_-(e) = \sup\{-\varphi(e') : e' \in B, e' \subset e\}$.

For each $e \in B$ values $\varphi_+(e)$ and $\varphi_-(e)$ are interpreted, respectively, as required and existing amount of mass on e. Therefore condition (14) has the same meaning as condition (2) in case of finite number of places.

The masses translocation plan on K is defined by the choice of the finite measure ψ, which is specified upon σ-algebra Ω of Borelean sets of compact $\Lambda := K \times K$. In such case the $e \times e' \in \Omega$ set measure (where $e, e' \in B$) which is designated further as $\psi(e, e')$, indicates the mass amount which is being planned to be translocated from e to e'. Such plan is feasible if it satisfies balance relationships:

$$\psi(K, e) - \psi(e, K) = \varphi(e), \quad e \in B \tag{15}$$

which is the analogue to relationships (4) in Transportation Problem.

The system of feasible translocations ψ, similar to as it is above, will be denoted as Ψ_φ. And total costs connected with implementation of each such translocation ψ here, obviously, will be calculated using double integral:

$$\tau(\psi) = \int\limits_{\Lambda} r(t, s) d\psi(t, s). \tag{16}$$

So the required optimal translocation is characterize by a function $\psi \in \Psi_\varphi$, for which the corresponding value (16) achieves minimum.

Sometimes it is more convenient to think of this problem in a little bit other terms in the following way.

Let R be a metric compact space, although some of following definitions and results can be stated also for spaces of more general type. Let $\Phi(e)$ be mass distribution, that is a function of set:

1) defined for Borelean sets,

2) nonnegative $\Phi(e) \geqslant 0$,

3) absolutely additive: if $e = e_1 + e_2 + ...; e_i \cap e_k = \emptyset$ $(i \neq k)$, then $\Phi(e) = \Phi(e_1) + \Phi(e_2) + ...$

Let $\Phi'(e')$ be another mass distribution, and $\Phi(R) = \Phi'(R) = 1$.

We will define as a masses translocation such a function $\Psi(e, e')$ defined for couples of (B)-sets $e, e' \in R$:

1) nonnegative and absolutely additive by each argument,

2) such that $\Psi(e, R) = \Phi(e)$; $\Psi(R, e') = \Phi'(e')$.

Function $\Psi(e, e')$ is characterizing the amount of mass that is being translocated from set e to set e'. According to this it is also worth mentioning here, that

$\Psi(e,R)=\Phi(e); \Psi(R,e')=\Phi'(e')$. Let $r(x,y)$ be the known continuous nonnegative function – the work necessary for translocation unit of mass from x to y.

We will define as the work necessary for implementing current translocation of given mass distributions the following variable:

$$W(\Psi,\Phi,\Phi')=\int\limits_R\int\limits_R r(x,x')\Psi(de,de')=\lim_{\lambda\to 0}\sum_{i,k} r(x_i,x'_k)\Psi(ei,e'_k),$$

where $\{e_i\}$ where $i=1,...,n$ are disjuncts and $\sum\limits_1^n e_i=R$, $\{e'_k\}$ where $k=1,...,m$ are disjuncts and $\sum\limits_1^m e'_k=R$, $x_i\in e_i, x'_k\in e'_k$, and λ is the maximal of the following numbers diam e_i $(i=1,2,...,n)$ and diam e'_k $(k=1,2,...,m)$. This integral obviously exists.

With the following value $W(\Phi,\Phi')=\inf_\Psi W(\Psi_0,\Phi,\Phi')$ we will designate the minimal work necessary for translocation. Whereas the set of functions Ψ is compact, so it is obvious, that exists such a function Ψ_0 realizing this minimum, another words – such function for which we have: $W(\Phi,\Phi')=W(\Psi_0,\Phi,\Phi')$. Although this function is not unique. Such translocation Φ_0 we will designate the minimal translocation or optimal translocation.

We will also say, that translocation Ψ from x to y does not equal to zero and will label it as $x\to y$, if exists the translocation of mass from x to y – in other words if for any neighborhoods U_x and U_y of points x and y will be $\Psi(U_x,U_y)>0$.

We will designate translocation Ψ potential if exists such function $U(x)$ for which: 1) always $|U(x)-U(y)\leqslant r(x,y)$; 2) $U(y)-U(x)=r(x,y)$, if $x\to y$.

It can be shown that stated extreme problem is always solvable. And that the optimality translocation criterion in this case substantially is not by a much margin different from the one derived from LP general result and given previously for the case of the Transportation Problem. It is provided to be that feasible translocation $\psi\in\Psi_\varphi$ is optimal when and only when exists such a function $u: K\to R$ that:

$$u(s)-u(t)\leqslant r(t,s),\quad t\in K,\quad s\in K, \tag{17}$$

and $u(s)-u(t)=r(t,s)$, if (t,s) belong to a measure ψ support suppψ, in other words if for whatever neighborhoods e_t and e_s of points t and s occur strict inequalities $\psi(e_t,e_s)>0$.

In other words, the minimal translocation can be characterized in the following way (using the feature of peculiar potential existence for such translocation) by folowing theorem which was proved by Kantorovich and was published in 1942 in the article [2].

Theorem. *For translocation Ψ_0 to be minimal it is necessary and sufficient that it is potential.*

Kantorovitch in his article [2] of year 1942 stated that the exploration of the masses distribution space is of a special interest if the $W(\Phi, \Phi')$ value is taken as a distance (for the case, when $r(x, y) = \rho(x, y)$ – the distance).

And this approach seems the most natural in some meaning for working in a metric in current space. Current theorem is of a great interest for us and very useful because it can be successfully applied for solving two practical problems described below.

Problem 1. Transportation Problem.

Problem 2. About the planning of an area. We assume that the lay of land – equation of earth surface $z = f_0(x, y)$ before layout design and $z = f_1(x, y)$ after layout design (with a condition that $\int\int f_0(x, y)dx\ dy = \int\int f_1(x, y)dx\ dy$) – and costs for translocation 1 m^3 of ground from point (x, y) to point (x_1, y_1)) is given. And it is necessary to specify such plan of ground mass translocation with which the total costs for translocation will be minimal.

The Problem of Masses Translocation can be derived (and had arisen in Kantorovich's investigations) as a generalization of considered earlier practical problem about finding the way of connecting places of goods production on the railroad net with places of these goods consumption such that will provide minimal total costs of transportation of these goods. This Transportation Problem had been studied and exhaustive methods for its solution were developed by Kantorovitch conjunctly with Gavurin M.K. As we can see it is obvious that the Transportation Problem is a particular case of described general problem.

Only some time after the publication of his study [2] Kantorovitch had taken notice that the same general problem contains as a particular case one other important problem which had been investigated very much earlier by Gustave Monge (1746–1818) in the paper on "cutting and filling". And aforementioned theorem can be applied with success for its analysis. The formulation in terms of continuous mass distributions has come to be called the "Monge–Kantorovich Problem".

4 The Monge Problem

In 1781 in memorandum [6] outstanding French mathematician G.Monge in connection with question about the most rational ways of soil transportation from earthfill to cannelure had stated the following problem: it is necessary to divide two equivalent volumes into infinitesimal particles and matching them between each other in order to make the being transported particles volumes and their traversed intervals products sum minimal. This study was carried out in conjunction with the French mathematician's work on the moving of soil for building military fortifications.

In connection with this problem study G.Monge developed geometric congruence theory. In regard to the problem itself, he stated the hypothesis (but had not given its rigorous proof) that required mass transport ways are a family of normal lines to certain one-parameter surfaces family.

Many famous mathematicians dealt this Monge Problem later, but rigorous proof for Monge's theorem (at that time only hypothesis stated by Monge) was given only one hundred years later in 1884 by P. Appel in 200-pages memorandum. Although later Appel managed to simplify in a manner his proof but it still remained highly complex and was based on the usage of subtle variation calculus theorems. In spite of this, it is possible to gain Monge hypothesis proof and for essentially wider mass transportation problems class as simple corollary of optimality transportation criterion (the aforementioned abstract theorem), developed (and generalized later in [2]) in connection with Transportation Problem solution in [7].

Lemma. *Let the function* $u : K \to R$ *satisfies the condition (17), and points* t_0, s_0 *and* z_0 *from* K *are such that:*

$$u(s_0) - u(t_0) = r(t_0, s_0) = r(t_0, z_0) + r(z_0, s_0). \tag{18}$$

Then set

$$U_{z_0} = \{z \in K | u(z) = u(z_0)\} \tag{19}$$

situated between two spheres which are incident to z_0 *and with their centers in points* t_0 *and* s_0*. In other words, for any point* $z \in U(z_0)$ *occur the following inequalities:* $r(t_0, z) \geqslant r(t_0, z_0), r(z, s_0) \geqslant r(z_0, s_0)$.

Taking into consideration optimality criterion which was described above, we can assert that Monge–Appel theorem is valid for any mass translocation problem on a convex compact in arbitrary Euclidean or Hilbert space. And if feasible translocation, determinated by the measure $\psi \in \Psi_\varphi$ is optimal then in the capacity of required can be taken one-parameter family of the equipotential surfaces (or in other words level surfaces) $u(z) = C = const$ corresponding to function $u : K \to R$ from optimality criterion and required mass translocation ways have to be normal lines to them.

Really, if such way xy from x to y intercept in the point z the equipotential surface then this surface according to requirement (1) of potentiality have to lie between two spheres (with their centers in points x and y) which are intercepting the point z and thereby have to be normal to way xy.

In other words: really, if we have a translocation being executed from the point t_0 to the point s_0, that is $(t_0, s_0) \in$suppψ, then open segment (t_0, s_0) for any $z_0 \in (t_0, s_0)$ on the strength of the lemma coincides with the corresponding surface (19) normal line.

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ABOUT NECESSARY PRELIMINARY KNOWLEDGE FOR STUDY OF MATHEMATICAL ANALYSIS AND OVERCOMING SOME STEREOTYPES

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Abstract. It would be desirable to pay attention to a number of general principles and conceptions of mathematics in this article. Its knowledge is very important before the beginning of systematic study of mathematics analysis, analytical geometry, discrete mathematics and other mathematical disciplines. Too few attention, in opinion of writer, is paid to this knowledge at school, though while studying these abilities are to be well-known. Certainly, everything discussed in this article is applied to the schools with the deep program in mathematics, which is oriented on the pupils, which wish to study in future mathematical sciences. Nevertheless, at the end of this article some main problems concerned mathematical education will be posed by the author.

Let's note the following. It is necessary to know well the mathematical assertions, the expressions, statements and principles of formal logic. It is important to know, what is a basic concept, axiom, definition, theorem, lemma, inquest, logical necessary condition, sufficient condition, criterion, sign, property, direct theorem, the converse, the opposite theorem, as they correspond one another about their simultaneous equity, about a principle of proof by contradiction.

It is also very important to have a clear understanding about such conceptions as a set, elements of sets, subsets and and basic operations above by sets: association, intersection, difference, about correspondence and one-to-one correspondence, mapping, equivalence of sets, about frequently used quantifiers and denotations for a brief exposition of a mathematical material.

It is still very important to have precise understanding about mathematical expression and its special cases such, as numerical and algebraic expression, rational expression, irrational expression, transcendental expression, fraction, range of permissible values (RPV), the monomial and polynomial. It is important to know, what is equation, inequality, system of equations and inequalities, solution and to solve the equation, inequality, system, to know, what is equivalence, about usage of the formulas in mathematician.

It is rather useful to know a method of a mathematical induction and to know how to apply it both at problem solving, and at the proof of a number of the theorems of school rate.

Many books were written about it in the middle of the 20 century, (some manuals about high school were printed [1], [2]), but, unfortunately, at school a poor notice is paid to the mathematical literature at school.

For better mathematical understanding it is important to pay attention either to writing, or on republishing appropriate manuals devoted to indicated problems.

It would be desirable to mark the offer on revision of the record forms to the answers of solutions of algebraic and trigonometric problems to improve study problem on an exponential function, about a define area of expression $f(x)^{g(x)}$, problem about the complete substantiation of the basic properties of elementary functions (define area, range of values, oddness,periodicity, zeros and intervals of signum constancy, monotonicity, about a number of problems in geometry, in particular, about state of an axiomatics (in the different textbooks on any other), about the different approaches to the attitude of a parallelism (when it is the relation that has a property of a reflexivity or no), about importance of separation of concepts of a segment and its length, corner and its value and their denotations, about an over determination in some definitions like quadrangles (the rectangle that can de fined as parallelogram having even one right angle, the rhombus is possible to define as a parallelogram having equal even it is possible to define two adjacent parties, square can be defined as a rectangle with the even two equal adjacent parties, or as a rhombus, for which even one direct corner, then that all corners for a square are direct and all parties for a rhombus are equal which follows from the properties of a parallelogram), about discrepancies in usage boundary and plane (boundary, integrated with the internal areas) corners and polygons and similar situation concerning space figures, in general about importance to give the much greater notice to bases of geometry and elementary mathematics as a whole.

Restricted volume of the given article does not allow to concentrate one very thing, listed above, especially to give the developed formulas, appropriate definitions and to illustrate them by examples. With all by these problems it is possible to get acquainted, for example, in the books of the writer [3] and [4], concerning the theory of number and different formulas of a method of mathematical inductions, for example, in the book [5].

Therefore we shall stay only on some of the listed problems.

Let's begin from the forms of record of the answers to problems, rather reference for many the allowances after elementary mathematician such as $x \in [a; b]$, $x \in \varnothing$, $n \in Z$. At the description of concepts of set A, its element a in all the books, where the speech goes about sets, the record $a \in A$ means, that a is any let and arbitrary, but fixed element of set A. However, in problems such as "to decide the equation (inequality)" always according to appropriate definition that it means, it is necessary to find all numerical characters, figuring in them, which at substitution in them pay them accordingly in valid numerical equalities and valid numerical inequalities. Therefore with such the points of view we obviously watch mismatching of records of a type $x \in [a;b]$, $x \in \varnothing$, $n \in Z$ with the generally accepted sense records $a \in A$. Moreover, as according to the description empty sets $\varnothing$, as not having of elements, the record $x \in \varnothing$ is received not having any of sense in general.

The truth, in operation of the writer [6] in an example 1 the explanation that is given if to consider in a particular context record $a \in A$ *as expression*, and record $x \in [a;b]$, being laconic record of set all solutions of a certain problem with an absent universal quantifier $\forall$ (which is superseded the words by "anyone" or "for anyone") — *by the formula*, that it in a particular measure can remove the indicated mismatching. But, unfortunately, with introduction of records such as $x \in [a;b]$, $x \in \varnothing$, $n \in Z$ in the school mathematician program such arrangement were not accepted, that is why it is one of reasons of desirtion of resetting to former records accordingly of kind $a \leqslant x \leqslant b$ (it is logical to use record $[a;b]$), verbal "there are no solutions" (it is logical to use and record $\varnothing$) and $n = 0; pm1; \pm 2; ...$ (where record, for example, $n = \pm 1$ is understood as that n can to accept values 1 and -1, certainly, not simultaneously).

Let's consider two examples of problems of entrance examinations on mathematics in MSU 1991 and 1996 illustrating a situation of a undesirability usages in the answers of record $n \in Z$.

In variant of entrance examination after mathematician on economic the faculty MSU in 1991 as the first problem was offered to decide the equation

$$\sin(3\pi \cdot 2^x) = \cos(\pi \cdot 2^x) - \sin(\pi \cdot 2^x).$$

It was natural to make replacement $t = \pi \cdot 2^x$, whence $t > 0$, and to decide then equation $\sin 3t = \cos t - \sin t$. Having decided it, at first we receive

$2^x = \dfrac{1}{2} + n$, $2^x = \dfrac{1}{12} + n$, $2^x = \dfrac{5}{12} + n$, where n accepts anyone whole, but only *non-negative* significance, therefore by final answer to this to problem there will be following values x:

$$x = \log_2\left(\frac{1}{2} + n\right), x = \log_2\left(\frac{1}{12} + n\right), x = \log_2\left(\frac{5}{12} + n\right), \; n = 0;\; 1;\; 2; ....$$

The answer in the directory for acting was in the same way written in the Moscow university in next, 1992. Let's remark, not $n \in N_0$, where $N_0 =$

$\{0;\ 1;\ 2;\ ...\ ;\ n;\ ...\}$ is extended natural number line of numbers (similar frequently is supervised in number line of the books on preparation to entrance examinations in high schools: alongside with record $n \in Z$ it is possible to meet records of a kind $n = 0;\ 1;\ 2;\ ...$, if the takeoff is required solutions of the trigonometric equations, but it is represented rather not logical). However 90 — 95% of the entrants, receiving up to the answer this problem, have written $n \in Z$, in general(common) implying (as before already 15 years learned(taught) at schools), that n is an arbitrary integer. Certainly, it was by a gross error, but the competitive situation demanded for this mistake to not reduce an estimation for the given problem. Was received, that at a failing response, the problem, actually, was set off as decided correctly. Premeditation of the writer by this problems, where the skill was supposed to make takeoff of integer values the parameter n, has failed. But in 1991 acted "cutting off" the program after mathematician for acting in high schools, and special program after mathematician for acting in MSU yet was not.

In variant of entrance examination after mathematician on geological the faculty MSU in 1996 was offered to decide the equation $\sin 5|x| = \sin(-3)$. According to a condition of equality sines is received:
$|x| = \left((-1)^{n+1}\frac{3}{5} + \frac{\pi n}{5}\right)$, owing to a nonnegativity $|x|$ n can to accept only natural values, became to be, the answer of this problem will be the following values x: $x = \pm\left((-1)^{n+1}\frac{3}{5} + \frac{\pi n}{5}\right), n = 1;\ 2;\ 3;\ \ldots$.

Again there was an erratic record $n \in Z$, but as against a situation, circumscribed in the previous example, the problem correctly decided any more was not considered, the existence of more serious program of entrance examinations had an effect after mathematician (in MSU it was entered in 1993) and, as inquest, better preparation of the entrants, rather than in 1991.

Further, we shall stay on a problem of clarification of a define area of expression $f(x)^{g(x)}$. In a school rate of mathematics it is accepted to consider, that $f(x) > 0$, arguments for the benefit of revision such will be indicated however below limitations.

Let's begin from an exponential function $y = a^x$, where a is a constant, x is a variable. Many writers and teachers arrest limitations on a: $0 < a \neq 1$, that is select same limitations, as for the logarithmic function $y = \log_a x$. Apparently, it is possible to explain it by dagging to consider together couples is mutual of inverse functions. Thus the pupils can fast perceive base solutions of the equations and mutual communication of functions. But the operation with functions nevertheless requires by it to not be limited and to consider as much as possible allowable possible values a. In particular, as not each function should be converted (otherwise it is possible to lose, for example, of periodic functions) and it is completely not necessary to be particular for all real values x (for example, the function $y = \log_a x$, where a is a stationary value, which can to accept only values $0 < a \neq 1$, is determined only for $x > 0$), that is necessary to consider and significance $a = 1$, and $a = 0$. As is known, at $a = 1$ at all real x $a^x = 1^x = 1$, at $a = 0$ at all real $x > 0$ $a^x = 0^x = 0$ and 0^x is not determined at anyone $x \leqslant 0$. In some books on elementary mathematician, for example, [7] at consideration of an exponential function $y = a^x$ limitation at first is underlined $0 < a \neq 1$, and then by small-sized font is spoken about cases as $a = 1$, so and $a = 0$, but these cases are considered uninteresting in consequence that these functions on all define areas are stationary values. It should be marked as carelessness concerning not parsed consequences.

So, for example, on a problem on set of positive solutions of the equation $x^x = 1$ some pupils can give the answer "of solutions is not present", that is obvious is coupled from them accustoming to limitation $a \neq 1$ for exponential functions $y = a^x$. Certainly, the solution $x = 1$ here is obvious.

In a situation of entrance examinations, and occasionally local of rounds some mathematical competitions it also can reduce in rather funny situations.

In a variant of entrance examination on mathematician on faculty CS MSU in 1981 it was offered to decide the equation (it there was under the order second problem them to six offered then)

$$\frac{1}{\sqrt{2x-1}} = (2x-1)^{\log_{1/4}(1+7x-2x^2)}.$$

Let's mark, that this problem is competently constituted. The condition $2x - 1 > 0$ follows not because $2x - 1$ stands in a right member of the equation in the basis of a degree, that is why, what still $2x - 1$ stands on the left of equation in an atmosphere of the square root, moreover and in a common denominator. The second limitation: $1 + 7x - 2x^2 > 0$ is a positiveness of expression, from which in a parameter degrees the log in a right member of the equation undertakes. Became to be, RPV of a problem:

$$\begin{cases} 2x - 1 > 0, \\ 1 + 7x - 2x^2 > 0, \end{cases} \Leftrightarrow \begin{cases} x > \dfrac{1}{2}, \\ 2x^2 - 7x - 1 < 0. \end{cases}$$

The equation can be copied by the way

$$(2x-1)^{-1/2} = (2x-1)^{\log_{1/4}(1+7x-2x^2)} \overset{\text{aas}}{\Leftrightarrow}$$

$$\overset{\text{aas}}{\Leftrightarrow}\begin{cases} 2x^2-7x-1<0, \\ x>\dfrac{1}{2}\,, \\ \left[\begin{array}{l} 2x-1=1; \\ 1+7x-2x^2=(1/4)^{-(1/2)}\,, \end{array}\right. \end{cases} \Leftrightarrow \begin{cases} 2x^2-7x-1<0, \\ x>\dfrac{1}{2}\,, \\ \left[\begin{array}{l} x=1; \\ 1+7x-2x^2=2, \end{array}\right. \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2x^2-7x-1<0, \\ x>\dfrac{1}{2}\,, \\ \left[\begin{array}{l} x=1; \\ 2x^2-7x+1=0, \end{array}\right. \end{cases} \Leftrightarrow \left[\begin{array}{l} x=1; \\ x=\dfrac{7+\sqrt{41}}{4}\,, \end{array}\right. \quad as$$

$$\frac{7-\sqrt{41}}{4}<\frac{7-6}{4}=\frac{1}{4}<\frac{1}{2}\,,\Rightarrow\ \frac{7-\sqrt{41}}{4}\notin RPV.$$

In one of examination works, which could be tested by the writer, were correctly decided first, third, fourth, fifth, half sixth problems. In the second problem the solution of an equation $x = 1$ was lost, was not the condition $2x-1 = 1$ is considered. By everything, it has taken place because of accustoming even well prepared pupils to limitation for exponential functions $y = a^x$ $a \neq 1$. In result for this examination operation the estimation "4" was exhibited, there will be no this the mistakes, there would be an estimation "5" according to criteria of exhibiting estimations of that year. Then entering, who had the medal, could be presented to enter on faculty without delivery of remaining entrance examinations, having received it was necessary to hand over an estimation "4" all remaining examinations.

On one of not correspondence of rounds of an olympiad MSU after mathematician "Pokori Vorob'ovy Gory" 2007 (and it concerned the participants of locales of Russia, there is enough expelled from of Moscow) was offered similar on just the considered problem to decide the equation: $\dfrac{1}{\sqrt{3x-4}} = (3x-4)^{\log_2(2-x)}$.

Deciding it completely similarly considered before it to problem, was received, that already in this case to equation satisfied only significance x from a condition $3x-4 = 1$, that is only $x = 5/3$, having lost which, not having considered a condition $3x - 4 = 1$, it was possible in general to not decide a problem and, thereby, to not appear among the winners this olympiad.

Further, we shall consider the problem on an power function $y = x^\alpha$, where α is constant real number, and x is real variable. Here it is possible to establish the fact, that *at anyone* $\alpha \in R$ $x = 1$ switches on in a define area of this function, thus the appropriate significance of the function is equal 1. And for this reason It be necessary to consider an exponential function $y = a^x$ at $a = 1$. Became to be, rather illogically, that frequently in different books on elementary mathematician the function $y = x^\alpha$ is determined on extreme to measure for all real $x > 0$, in particular(personally), at $x = 1$, and the exponential function $y = a^x$ at $a = 1$ for some reason is not present. Follows, however to mark, that at $\alpha > 0$ the hipping degree is considered also functions at $x = 0$ by significance 0. The last circumstance will be agreed with the following number of the facts:

1) according to definition of a natural degree *of any number a*

$$a^n \overset{def}{=} \begin{cases} a, & \text{if } n = 1, \underbrace{aa\cdots a}_{n}, \text{ if } \ n \geqslant 2, \end{cases}$$

whence, in particular, at $a = 0$ $\forall$ natural n $a^n = 0$,

2) If $\alpha = \dfrac{m}{n}$, where m and n are natural numbers (that is α is rational positive number), That, distributing on a case $a = 0$ definition of a degree with rational by positive parameter $a^\alpha = a^{m/n} \overset{def}{=} \sqrt[n]{a^m}$, let's receive, that at anyone positive rational α $a^\alpha = 0^\alpha = 0$, 3) If on a case irrational positive α for $a = 0$ To distribute definition of significance a^α at $0 < a < 1$ (this such number b, which satisfies for anyone rational positive α' and α'' such, that $\alpha' \leqslant \alpha \leqslant \alpha''$ to inequalities $a^{\alpha''} \leqslant b \leqslant a^{\alpha'}$), that we shall receive, as at anyone irrational positive α $a^\alpha = 0^\alpha = 0$, 4) At last, at anyone fixed $\alpha > 0$ the function $y = x^\alpha$ has the right limit in a point $x = 0$

$$\lim_{x\to 0+0} x^\alpha = 0.$$

The indicated circumstances quite justify consideration exponential functions $y = a^x$ at $a = 0$, particular at $x > 0$ and accepting on an interval $(0, +\infty)$ unique significance 0. As at anyone $a \neq 0$ and anyone b, at which is determined a^b, $a^b \neq 0$, according to appropriate definitions of a degree and to property of a degree $a^{-b} = 1/a^b$ the appropriate property is natural was to expect and from $a = 0$, but then it would be reduced to feasibility of the operation $1 : 0$, that actually can not be executed, to be it became clear, why $\forall\alpha < 0 0^\alpha$ is not determined. Uncertainty 0^0 will be corresponds, for example, the following circumstances: on the one hand, according to with property of a degree $a^{\alpha-\beta} = a^\alpha : a^\beta$ would be received, that $0^0 = 0^{2-2} = 0^2 : 0^2 = 0 : 0$, that at least *uniquely* is not feasible, that is why 0^0 can not to be uniquely determined, on the other hand, from a calculus it is known, that

$$\lim_{x\to 0+0} x^x = 1, \text{ at } \ a > 1 \ \lim_{x\to 0+0} \left(\frac{x}{a^{\frac{1}{x}}}\right)^x = \lim_{x\to 0+0} \frac{x^x}{a} = \frac{1}{a} \in (0, 1),$$

$$\lim_{x\to 0+0}\left(\frac{x}{a^{\frac{1}{x}}}\right)^{-x} = \lim_{x\to 0+0}\frac{a}{x^x} = a \in (1,+\infty),\ \lim_{x\to 0+0}\left(\frac{x}{a^{\frac{1}{x^2}}}\right)^{x} = \lim_{x\to 0+0}\frac{x^x}{a^{\frac{1}{x}}} = 0,$$

became to be, uncertainty of a kind 0^0 at calculation of a limit

$$\lim_{x\to 0+0}(f(x))^{g(x)},\ \text{where}\quad \lim_{x\to 0+0} f(x) = \lim_{x\to 0+0}\frac{x}{a^{1/x}} = \lim_{x\to 0+0}\frac{x}{a^{1/x^2}} = 0,$$

$$\lim_{x\to 0+0} g(x) = \lim_{x\to 0+0}(\pm x) = 0$$

can be any non-negative number.

Now we shall consider the problem on a define area of expression $f(x)^{g(x)}$. It is a general case for as exponential, and degree functions. In a school rate of elementary mathematics it is accepted to consider, that $f(x) > 0$. Frequently it happens is coupled to submission of expression a^b, for example, as $a^b = e^{b\ln a}$.

However inquisitive pupils note mismatching this situation, for example, with the formula of solutions of the trigonometric equation $\sin x = a$, where $|a| \leqslant 1$ with a factor, figuring in it, $(-1)^n$. In consequence with it it is necessary to muse above a problem of a define area expressions $f(x)^{g(x)}$ as set of all values x, conceding calculation of expression a^b, at which it certainly accepts real values. This expression is determined at all $a > 0$, if $a = 0$, $b > 0$ and $a^b = 0$, if $a < 0$, $b = m/(2n-1)$, where m is whole, n is natural number, the greatest common divisor GCD $(|m|;(2n-1)) = 1$ in particular, at $n = 1$ $b = m$ is an integer. Thus availability of a uneven denominator in an exponent b, and in general(common) availability of a condition GCD$(|m|;(2n-1)) = 1$ appear rather essential. So, for example, for want of these conditions, but at preservation of feasibility of the basic properties of degrees of numbers it is easy to confront with the contravention, for example, of a kind $-1 = \sqrt[3]{-1} = (-1)^{1/3} = (-1)^{2/6} = ((-1)^2)^{1/6} = 1^{1/6} = \sqrt[6]{1} = 1$. Therefore this problem can have double solution: for preservation of the basic properties degrees of positive numbers — to refuse non positive values expressions $f(x)$, indicative-degree expression, costing in the basis $f(x)^{g(x)}$, or, as it is done in rates of a calculus, the function $y = x^\alpha$ at $\alpha = m/(2n-1)$, where m is whole, n is natural number, GCD$(|m|;(2n-1)) = 1$ in particular, at $n = 1$ $b = m$ is the integer can be determined at negative values x by an even image at m even and uneven image at m uneven, namely, $y = x^\alpha = |x|^\alpha$ at m even and all $x \neq 0$, if $\alpha \leqslant 0$ and all x, if $\alpha > 0$ and $y = x^\alpha = (\mathrm{sgn} x)|x|^\alpha$ at m uneven and all $x \neq 0$, if $\alpha \leqslant 0$ and all x, if $\alpha > 0$, where

$$\mathrm{sgn} x = \begin{cases} 1, \text{ if } x > 0,\\ 0, \text{ if } x = 0,\\ -1, \text{ if } x < 0,\end{cases}$$

thus *for all* x (not only for positive, as it is considered in the school program after mathematician), at which is determined the function $y = x^\alpha$, it will be reprezented by the way $y = \sqrt[2n-1]{x^m}$. The mentioned above contravention is removed, for example, so: *on definition* we count $(-1)^{2/6} = (-1)^{1/3} = \sqrt[3]{-1} = -1$, and equality of a type $(-1)^{2/6} = ((-1)^2)^{1/6}$, any more not having a place at negative the basis of a degree, that the significance of the radical sixth will be corresponds also that degrees from $(-1)^2 = 1$ can be not necessarily positive, but then it *is not meant by a radical.* And if to talk about expression z^α for complex values $z \neq 0$ and α, from a function theory complex variable (FTCV) it is known, how many values it has and how many from them are real numbers.

Taking into account, that all circumscribed hippings of an power function on a case non positive values of its argument by the school program after mathematician they are skipped, though in [1] and number of other books after elementary mathematician are considered, for makers of examination problems introductory tests is necessary, that positiveness of the basis of a degree $f(x)$ in indicative-degree expression $f(x)^{g(x)}$ would follow from certain conditionals: or as it was done(made) in the considered above examples problems 1981 and 2007, or instead of expression $f(x)^{g(x)}$ to consider expressions or $\dfrac{1}{(\sqrt{f(x)})^{(-2g(x))}}$, or $a^{g(x)\log_a f(x)}$ at constant a, which satisfies to inequalities $0 < a \neq 1$.

Finishing an exposition of a problem about exponential and exponential functions, it is important to mark, that the formula of a derivative exponential function $y = a^x$ $y' = a^x \ln a$ is fair *at anyone* $a > 0$, including at $a = 1$ (that, unfortunately, by overwhelming majority of the writers is skipped), and formula for a derivative exponential function $y = x^\alpha$ $y' = \alpha x^{\alpha-1}$ is valid *for all* $x \neq 0$ (in a situation, when exponential function is determined on set $x < 0$) and for $x = 0$ at $\alpha > 1$.

About an identification of characters ∞ and $+\infty$.

In the mathematical literature it is pleased frequently about $+\infty$ is omitted the sign $+$ there, where its presence would be essential, as, as it will be visible below, as against a situation of an identification for final values $+a = a$ for ∞ and $+\infty$ it any more so. Usually character ∞ without any of the sign can characterize limitlessness any of set, especially without gated in in it of ordering elements, for example, we speak: “the indefinitely expelled point”, meaning by its character ∞, “infinite-dimensional space”, in functional the analysis the space of functions meant L_∞ is considered. Frequently, comparing final values finding “inside” some unrestricted set with perpetuity, they write, that $< \infty$. However in case of numerical direct, equivalent *ordered* to set of all real numbers, signs $+\infty$ and $-\infty$ inform on limitlessness by means of a limiting improper point. Thus the sign ∞ still arrests limitlessness, but it will characterize again disarray of a unrestricted subset real numbers. With reference to real number x (final), expressing coordinate of a point on numerical direct, correct are parities $-\infty < x < +\infty$.

Let's reduce definitions: an indefinitely major sequence, (about it speak, that it has "an infinite limit", indefinitely major sequence of the positive sign, indefinitely major sequence of the negative sign and limiting symbolics, appropriate to them. Accordingly

$$\forall A > 0 \exists N(A) : \forall n > N \Rightarrow \ |x_n| > A,$$

$$\forall A > 0 \exists N(A) : \forall n > N \Rightarrow \ x_n > A,$$

$$\forall A > 0 \exists N(A) : \forall n > N \Rightarrow \ x_n < -A,$$

$$\lim_{n\to+\infty} x_n = \infty, \lim_{n\to+\infty} x_n = +\infty, \lim_{n\to+\infty} x_n = -\infty.$$

From reduced definitions immediately follows, that if

$$\lim_{n\to+\infty} x_n = +\infty, \text{ that } \lim_{n\to+\infty} x_n = \infty, \text{ if } lim_{n\to+\infty} x_n = -\infty, \text{ that } \lim_{n\to+\infty} x_n = \infty,$$

$$\text{however from } \lim_{n\to+\infty} x_n = \infty, \not\Rightarrow \lim_{n\to+\infty} x_n = +\infty, \text{ and } \text{ from } \lim_{n\to+\infty} x_n = \infty, \not\Rightarrow$$

$$\lim_{n\to+\infty} x_n = -\infty.$$

Let's consider an example of a numerical sequence, which general member x_n looks like $x_n = (-1)^n \cdot n$. Arresting arbitrary $A > 0$, having taken in quality $N(A) = [A] + 1$, ($[A]$ is the whole part of number), we shall receive, that $|x_n| = n > A$, however, definitions of an indefinitely major sequence both positive, and negative sign for this sequence will not be executed. Became to be, ∞ and $+\infty$, and also ∞ and $-\infty$ *are not equivalent*, that is why *usage is undesirable* the denotations for intervals numerical direct (a, ∞), $(-\infty, \infty)$ (basically, then it was possible to use and anywhere not meeting the denotations (∞, a), $(\infty, +\infty)$), and *it is necessary* accordingly $(a, +\infty)$, $(-\infty, +\infty)$ also *usage is undesirable* the denotations for improper integrals and Laurent series

$$\int_a^\infty f(x)dx, \qquad \int_{-\infty}^\infty f(x)dx, \qquad \sum_{n=-\infty}^\infty f_n(z),$$

basically, then it was possible to use and anywhere the meeting denotations

$$\int_\infty^a f(x)dx, \qquad \int_\infty^{+\infty} f(x)dx, \qquad \sum_{n=\infty}^{+\infty} f_n(z),$$

and *it is necessary* accordingly

$$\int\limits_{a}^{+\infty} f(x)dx, \quad \int\limits_{-\infty}^{+\infty} f(x)dx, \qquad \sum_{n=-\infty}^{+\infty} f_n(z).$$

More valid in similar situations is the denotation for the sum of a number(line) (sum ordered on numbers — to natural numbers) elements of a numerical sequence u_n and its limit

$$\sum_{n=1}^{\infty} u_n, \lim_{n\to\infty} u_n \quad \text{instead of accordingly} \quad \sum_{n=1}^{+\infty} u_n \;\text{ and }\; \lim_{n\to+\infty} u_n$$

(though in a number of the books, for example, [8] these denotations prevail), as on set of all natural numbers ∞ and $+\infty$ *are equivalent*.

Let's stay on a problem of the used denotations for a situation, when one set is a subset other.

On the one hand, two signs $\subset$ and $\subseteq$, then are used $\subset$ is the sign of strict including means following parity between by sets A and B: if both they non blank, $A \subset B$, if $\forall a \in A \Rightarrow a \in B$ and $\exists b' \in B : \; b' \notin A$, such set A is named *own* as a subset of set B, for any nonempty subset $A \;\; \varnothing \subset A$; $A \subseteq B$ is the sign of weak including between sets A and B means only, that if they non blank, $\forall a \in A \Rightarrow a \in B$, thus the situation as is possible(probable) that A is own by subset of set B, and that can $A = B$ (that is these the sets consist of the same elements). Such approach is stated, for example, in the book [9]. It will be corresponds with used attitudes between by expressions both $<$, and $\leqslant$. Thus $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$, for any set A (empty and not empty) $\varnothing \subseteq A$ and $A \subseteq A$, including $A \subset A$ *does not take place* (by analogy with falsehood of an inequality between numbers $a < a$). At such approach at research on signum constancy or on on any of monotonous the function $y = f(x)$ it is natural to consider, that it takes place on a certain set $X \subseteq D[f]$, where $D[f]$ is a domain of a function $y = f(x)$ (that is it is quite possible and $X = D[f]$).

On the other hand, the sign $\subset$ is used only. At such approach already for any set $A \;\; A \subset A$ (that will not be corresponds any more with incorrect of the attitude $a < a$) and in a case, when A is own it is necessary to write a subset of set B two parities: $A \subset B$ and $A \neq B$. But this approach in the mathematical literature recently is used more often, probably, it is coupled to the natural mechanism of a generality mathematical language, that reputes usage for a different sort the denotations as small as possible of any characters, but it results to not quite to natural situations, about what is told above. Became to be, first approach with by usage of two signs $\subset$ and $\subseteq$ nevertheless it is more preferable.

Is possible still to mark different approaches to introduction of complex numbers in school mathematician and rates of maximum mathematics, accordingly as expressions $a+bi$ with not clear in the beginning by parity "+", where almost $i = \sqrt{-1}$, and as the ordered couples of real numbers (a, b), above which the attitudes(relation) $=$, $\neq$, operations of addition are entered, multiplying on particular rules, identification of a couple $(a, 0)$ with by real number a and on definition $i = (0, 1)$, then already *is proved*, that $i^2 = -1$ and $(a, b) = a + bi$. Probably, it is better to start to study complex numbers at once it is correct.

At an exposition by that, bound with scalar and vector product vectors frequently are overlooked(forgotten) "trivial" (actually "special") cases, when at least one of vectors is a zero (point). Then not the corner and is determined, became to be,its value between vectors, and if the result of a vector product of two vectors appears zero vector, the appropriate triple of vectors does not have any orientation "right" or "left". Unfortunately, it is pleased widespread(pleased distributed) oversight of many writers.

In some allowances on a technique of teaching of elementary mathematics, and in a number of the allowances for acting in high schools and after maximum mathematician it is possible to meet a nomenclature at the denotation of intervals numerical direct such as $[a; b[$ instead of $[a; b)$. It is necessary precisely to know, that is valid, one time (in 1975–1985) such denotation ($[a; b[$) in a school rate of mathematics dominated, but then, fortunately were cancelled, and again of steel to be used *round* brackets instead of the inverted square brackets.

Well and, at last, mentioned in the beginning necessity to distinguish section (as geometrical object) and its numerical characteristic is the length, corner (as geometrical object) and its numerical characteristic is the value (measure), are usually considered a degree and radian measure and their denotation. In a particular period (approximately with 1975 for 1983) in school rate of geometry and was, that was precisely mirrored in geometrical problems of entrance examinations after mathematician in high schools those years. Unfortunately, then, and to this day these concepts of steel again to be identified (is exacter, to be mixed).

Simple example of one of such situations of a problem of entrance examination in MSU 2009.

The parties of a delta circuit are equal 3, 5 and 7. To find the value the greatest corner of a delta circuit.

More correct formula would be such:

"The lengths of the parties of a delta circuit are equal 3, 5 and 7. To find the value of the greatest corner of a delta circuit".

Though it was possible and so (that is characteristic of problems United State Examination (USE) after mathematician).

The parties of a delta circuit are equal 3, 5 and 7. To find the greatest corner delta circuit.

Deciding this problem, it is necessary to designate tops of a delta circuit, for example, For A, B and C, then let $|AB| = 3$, $|BC| = 5$, $|AC| = 7$, greatest corner in a delta circuit ABC will be a corner opposite greatest it(him) to the party AC, that is $\angle ABC$, on a cosine law for this purpose delta circuit

$$\cos\widehat{ABC} = \frac{|AB|^2 + |BC|^2 - |AC|^2}{2|AB| \cdot |BC|} = -\frac{1}{2},$$

became to be, degree value of a required corner $\widehat{ABC} = 120°$.

Summarizing, it would be desirable to pay serious notice that, explicating a modern mathematical science, each scientific mathematician to the theorist, the applied mathematician it is necessary to not forget about existence of huge quantity of problems in initial mathematical derivation of children, youth, and adult people, whom there is necessary qualified mathematical knowledge.

Exists an opinion, that at initial study of mathematics to give children at once composite for them an axiomatics it is not necessary, and it is necessary to use visualization and whenever possible to state a material simply. Yes it, in general, so, but if and to leaving school all this in such style will be stated, then in serious high schools will be trained extremely difficultly.

Before the methoditions, teachers, and number of the scientists the life strongly puts a serious problem of construction of an initial rate of mathematics so that it was initially studied as a serious science, would not require then studying over and whenever possible made more derivated everyone, who should collide with mathematics.

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MATHEMATICAL CULTURE AND ITS POSITION IN MODERN SOCIETY

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Key words: mathematical culture, professional orientated teaching aids, mathematical language and leaver's mentality, the oral examination in mathematics for university entrants, elementary mathematics studies, student's scientific research work, returning the ?teach book – book of mathematical problems" system

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Abstract. The state of the mathematical culture of the modern society and proposals for improving the mathematical culture of the students and pupils are discussed.

First of all, mathematics attracts attention with logical grace, uncompromisingness and profound connections with other sciences such as technical, natural, social sciences and humani-ties. The result doesn't depend on the place and time. But to suggest some new, "to invent the proof" (J.-A. Diedonne), we are in need of imagination, intuition. It suddenly dawned upon us, but not the exact computation and the logic. It is not known of what attitude of mind and associations the idea of mathematical or supplied problem solution arises. If there are more associations then we have more culture medium ground for ideas birth.

Mathematical culture is a constituent part of general human culture, so the level of mathematical culture of our students has an effect on the level of culture of the whole modern society. But mathematical culture of professional mathematic, engineer, and human specialist are different in broadmindedness and profundity of thought (fig.1).

Any civilized society is very attentive to training of specialists. It is related to the branch of education too. The problem is under attention in our country. But analysis of methods for solving the problem during last 50 years shows that results are not always successful. Different adaptations, modernizations and optimizations lead to the loss of the high quality of education.

Socio-economic processes, economic state of modern society, attitude of the state to science and education have great influence on the development of mathematical culture. The concept of "social order of the society" for training of specialists is absent now. As mathematical and professional knowledge too of final-year students of technical universities are unclaimed.

Many socio-humanities specialized fields, which are opened in technical universities now as a machine-building industry brunch, insufficiently use mathematical

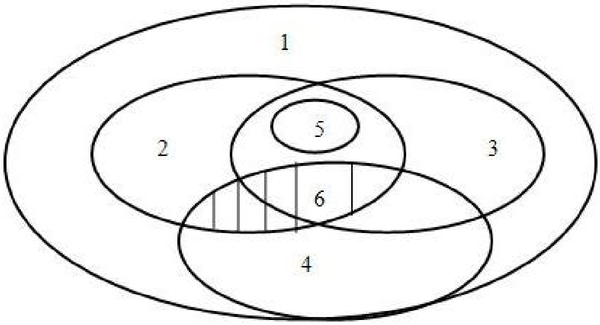

Figure 1. 1 — human culture; 2 — mathematical culture of mathematician; 3 — average mean of society member's culture; 4 — special technical knowledge; 5 — mathematical culture of humanities; 6 — mathematical culture of engineer

knowledge. The egress is to teach students to use worked-out mathematical models before, than to receive it themselves.

Any productive activities may be realized if it is called for only. Cognitive activities needs, building and investigation of real processes with mathematical modeling are became aware of the aim subjectively. The achievement of the aim gives reasons for cognitive activities.

Professional orientated teaching aids reasonable to use to give reasons for mathematics' study. Highly skilled professional mathematics, orientated towards that subject, must prepare these teaching aids. The aim of these teaching aids is to raise the standard to students' mathematical competence with using mathematics for solving professional problems. Such type text-books for the first year students, who are studied mathematics, but don't study special subjects are prepared to illustrate the using of mathematics and to show the necessary of its teaching.

Development of mathematical language and leaver's mentality is the basis of mathematical culture. Leavers and students too must be able to use special mathematical terminology, to put their thoughts clearly and briefly. According to S.A. Rozanova, these abilities are formed with systems "culture — language — cognitive processes", "textbook's language — teacher's language — pupil's language".

The introduction of the "common state examine" (CSE) has negative role to form mathematical mentality, leads to mathematical culture's loss. Common state examine has a test form, so it is a problem how to perform mathematical mentality with the system "teach book — book of mathematical problem", when the pupil makes "ticks" and doesn't say a word, or he tries to remember with agonizing doubts in what task he must multiply and where he must square a number to receive right answer.

One of test's defects is that we form an estimate with "using of results" and we don't consider the way of its solving: with calculations, reasoning or at random method. As a rule we can not see the type of mistakes made by the pupil, so we can not make "the process's control" and to show worked mistakes. The test control doesn't allow taking account psychological state of the pupil too.

Improvement of mathematical culture of students as future specialists gives them the possibility to add to their knowledge themselves, to modernize their professional knowledge. So they will be able to complete with others and be professionally mobile specialists, which are working for the benefit of mankind.

Improvement of mathematical culture of individual leads to the increasing of his average culture level mean and so to the increasing of the culture level of the whole society.

The mathematical culture level will increase, if the oral examination will be recovered for university entrants who are entering as physics and mathematics at least. The oral examination in mathematics was on a level with examination in written form before. And if it will return for entering in technical universities than university entrants, their parents and teachers and education ministry officials will be panicstricken because the failure of such examination will be ensured.

The "teach book — book of mathematical problems" system was very useful. The teach book contained some definitions, theorems and their proofs. Methods of problem's solutions were considered in the teach book too. We were teaching to mediate on problems and to prove our statements. The book of mathematical problems contained problem's statements, answers and some explanations, i.e. it developed some practical knowledge. The refusal of the "teach book — book of mathematical problems" system leads to that the pupil doesn't know what he may accept without the proof, what he must do himself and what is clear with the definition.

School mathematical education level of the first year students is appreciated with the enter control, as usual. The enter control is test which contains ten problems, their difficulty level is equal to the first part of the common state examine (CSE). The low level was equal to three right answers from ten. Unfortunately, 38% of the first course students of Kazan Research Technical University couldn't answer rightly to three questions. So we need to conduct elementary mathematics studies additionally.

We are talking about elementary mathematical incompetence, but not about mathematical culture in that case. The new problem arises: to repeat the course of the elementary mathematics by the first course students.

Russian mathematical education was great before traditionally. Leading Russia scientists anxiously state their opinion about the mathematical education repeatedly. For example, V. Arnold (1937–2010) wrote: “Mathematical illiteracy is more destructive than the campfire of inquisitor”.

High school teachers are given the task: to reconstruct knowledge of basis of elementary mathematics, which are used during the high mathematics studying, to introduce the using of mathematical knowledge to their professional work. The role of student’s scientific research work (SRW) is great to draw them into sciences study. SRW includes the preparation and addresses student’s scientific meetings of different level. Mathematical culture leads to the rise of increase in average mean of society member’s culture.

So, we suggest:

- returning the oral examination in mathematics for university entrants who are entering as physics, mathematics and engineers, at least;
- conducting elementary mathematics studies additionally during the first year studies at universities;
- developing of student’s scientific research work;
- returning the ‘‘teach book -- book of mathematical problems’’ system, using special teach books on the elementary mathematics for first-year students.

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ON STABILITY BY LYAPUNOV AND ASYMPTOTICAL STABILITY

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Abstract. Under discussion is the relationship between two main definitions in the theory of stability by Lyapunov: the Lyapunov's stability and the asymptotical stability

In the theory of stability by Lyapunov there are two main definitions. In the first, a solution $x = \psi(t), t \geqslant t_0$ of a differential equation $\dot{x} = f(t,x)$ is called stable by Lyapunov if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each solution $x(t), t \geqslant t_0$ of the equation: $|x(t_0) - \psi(t_0)| < \delta \Rightarrow |x(t) - \psi(t)| < \epsilon, t \geqslant t_0$. (If the solution $x = \psi(t), t \geqslant t_0$ does not satisfy to this condition it is called *unstable*). In the second, the solution $x = \psi(t)$ is called asymptotically stable if it is stable by Lyapunov, and there exists $\delta_1 > 0$ such that for each solution $x(t), t \geqslant t_0$ of the equation: $|x(t_0) - \psi(t_0)| < \delta_1 \Rightarrow \lim_{t\to\infty} |x(t) - \psi(t)| = 0$. Of course, two conditions of asymptotical stability must be independent. But it is not easy to find an example of *unstable* solution of a differential equation which satisfies to the second condition of the definition of asymptotical stability. I found in literature only one example of this kind in [1, example 2, Fig. 13, page 161]. It is represented in the form of phase portrait for a system of two equations $\dot{x} = p(x,y)$, $\dot{y} = q(x,y)$ without analytical definitions of functions $p(x,y)$, $q(x,y)$ but in assumption that they are continuous on the t,x plane.

An equation with analytically defined functions which has *unstable* solution that satisfies to the second condition of the definition of asymptotical stability can be constructed by virtue of functions: $x_a(t) = e^{a\left(1-t^2\right)^2} - 1$, where a is a parameter ($a \in [0,1]$), $t \in [-1,1]$ is independent variable, and $x_b(t) = e^{\left(b^2-t^2\right)^2} - 1$, where b is a parameter ($b \in [1,\infty)$), $t \in [-b,b]$ is independent variable. Families of all such functions we denote by $\{(x_a(t), t \in [-1,1]) : a \in [0,1]\}$, and by $\{(x_b(t), t \in [-b,b]) : b \in [1,\infty)\}$. It will be useful the evident inequalities: $\dot{x}_a(t) < 0, t \in (0,1), a \in (0,1]$ and $\dot{x}_b(t) < 0, t \in (0,b), b \in [1,\infty)$ (1). Graphs of some of functions $x_a(t)$, $x_b(t)$ are illustrated in Fig.1.

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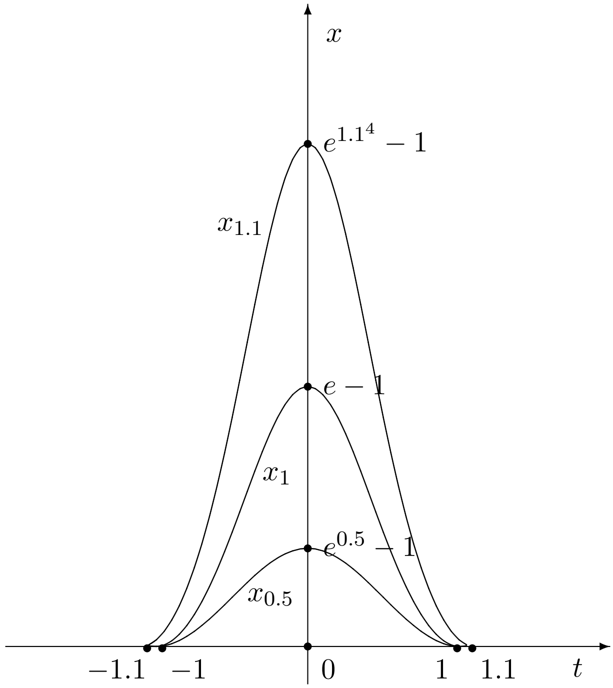

Figure 1. for parameters $a = 0$, $a = 0.5$, and $a = 1$ the functions $x_0(t)$, $x_{0.5}(t)$, and $x_1(t)$ are defined for all $t, -1 \leqslant t \leqslant 1$; the graph of $x_0(t)$ coinsides with the closed interval [-1,1]; for parameter $b = 1.1$ the function $x_{1.1}(t)$ is defined for all $t, -1.1 \leqslant t \leqslant 1.1$.

Denote by A the union of all graphs of the functions which belong to the family $\{(x_a(t), t \in [-1,1]) : a \in [0,1]\}$. A is a closed bounded set of points in the t, x plane, and the boundary of A consists of all points which belong to graph of the function $x_0(t) = 0, t \in [-1,1]$ or to graph of the function $x_1(t) = e^{\left(1-t^2\right)^2} - 1$, $t \in [-1,1]$. It is clear that for fixed but arbitrary point $t_* \in (-1,1)$ we have: $0 \leqslant a_1 < a_2 \leqslant 1 \Rightarrow x_{a_1}(t_*) < x_{a_2}(t_*)$. Therefore, for fixed but arbitrary point $(t_*, x_*) \in A$ there exists only one parameter $a_* \in [0,1]$ such that the point (t_*, x_*) belongs to graph of the function $x_{a_*}(t)$, $t \in [-1,1]$. In order to find this a_* we must find $a \in [0,1]$ which satisfies to the equation $x_a(t_*) = e^{a\left(1-t_*^2\right)^2} - 1 = x_*$ (2). It follows from (2) that $a_* = \frac{\ln(x_*+1)}{(1-{t_*}^2)^2}$ (3). Then from (3) we get $x_{a_*}(t) = e^{a_*\left(1-t^2\right)^2} - 1$, $t \in [-1,1]$, and $\dot{x}_{a_*}(t_*) = \frac{-4t_*(x_*+1)\ln(x_*+1)}{(1-{t_*}^2)}$ (4).

Define the function $U : A \to R$ by formulas: $U(t,x) = \frac{-4t(x+1)\ln(x+1)}{(1-t^2)}$, $t \in (-1,1)$, and $U(\pm 1, 0) = 0$. The function U is continuous at a point $(t,x) \in A$,

$t \in (-1,1)$. We find from (4) that the value of $U(t,x)$ at a point $(t,x) \in A$, $t \in (-1,1)$ is equal to slope of the tangent at the point (t,x) to graph of the function $x_a(t)$, $t \in [-1,1]$ for some $a \in [0,1]$ (5). We shall prove presently that $U : A \to R$ is continuous on A. For this purpose we find the partial derivative $\frac{\partial U}{\partial x} = \frac{-4t(\ln(x+1)+1)}{(1-t^2)}$, $(t,x) \in A$, $t \in (-1,1)$. If $(t,x) \in A$, $t \in [0,1)$, then $\frac{\partial U}{\partial x} \leqslant 0$. Therefore, for fixed but arbitrary point $\tau \in [0,1)$, and arbitrary point $(\tau,x) \in A$ we find: $0 \leqslant x \leqslant x_1(\tau) \Rightarrow U(\tau,x_1(\tau)) \leqslant U(\tau,x) \leqslant U(\tau,0)$ (6). Now suppose that $\tau \to 1$. From (5) we get: $U(\tau,0) = \dot{x}_0(\tau) = 0$, $\tau \in [0,1) \Rightarrow \lim_{\tau\to1} U(\tau,0)) = \lim_{\tau\to1} \dot{x}_0(\tau) = 0$, and $U(\tau,x_1(\tau)) = \dot{x}_1(\tau), \tau \in [0,1) \Rightarrow \lim_{\tau\to1} U(\tau,x_1(\tau)) = \lim_{\tau\to1} \dot{x}_1(\tau) = 0$. Therefore, we get from (6) that $\lim_{(\tau,x)\to(1,0)} U(\tau,x) = 0$. So, $U : A \to R$ is continuous at the point $(1,0) \in A$. Since $U(-t,x) = -U(t,x)$, $(t,x) \in A$, the function U is continuous at the point $(-1,0) \in A$ and on A.

Denote by B the union of all graphs of the functions which belong to the family $\{(x_b(t), t \in [-b,b]) : b \in [1,\infty)\}$. B is a closed unbounded set of points in the t,x plane, and the boundary of B consists of all points which belong to graph of the function $\tilde{x}_1(t), t \in (-\infty,\infty)$, where $\tilde{x}_1(t)$ is equal to $x_1(t)$ if $t \in [-1,1]$, and is equal to 0 if $|t| \geqslant 1$ ($\tilde{x}_1(t)$ is the differentiable extension of $x_1(t), t \in [-1,1]$ to $(-\infty,\infty)$). It is clear that for fixed but arbitrary point $t_* \in (-\infty,\infty)$ we have: $max(1,|t_*|) \leqslant b_1 < b_2 \Rightarrow x_{b_1}(t_*) < x_{b_2}(t_*)$. Therefore, for fixed but arbitrary point $(t_*,x_*) \in B$ there exists only one parameter $b_* \in [1,\infty)$ such that the point (t_*,x_*) belongs to graph of the function $x_{b_*}(t)$, $t \in [-b_*,b_*]$. In order to find this b_* we must find $b \in [1,\infty)$ which satisfies to the equation $x_b(t_*) = e^{\left(b^2-t_*^2\right)^2} - 1 = x_*$ (7). It follows from (7) that $b_* = \sqrt{{t_*}^2 + \sqrt{\ln(x_*+1)}}$ (8). Then from (8) we get $x_{b_*}(t) = e^{\left({b_*}^2-t^2\right)^2} - 1$, $t \in [-b_*,b_*]$, and $\dot{x}_{b_*}(t_*) = -4t_*(x_*+1)\sqrt{\ln(x_*+1)}$ (9). Define the function $V : B \to R$ by formula $V(t,x) = -4t(x+1)\sqrt{\ln(x+1)}$. V is continuous on B, and it is clear from (9) that the value of $V(t,x)$ at a point $(t,x) \in B$ is equal to slope of the tangent at the point (t,x) to graph of the function $x_b(t)$ for some $b \in [1,\infty)$.

Consider the differential equation $\dot{x} = W(t,x)$ (10), where $W = U$ on A, $W = V$ on B and $W(t,x) = -W(t,|x|)$, $t \in (-\infty,\infty)$, $x \leqslant 0$. Since U is continuous on A, V is continuous on B, W is continuous on the t,x plane. Define functions $\tilde{x}_a(t)$ (resp. $\tilde{x}_b(t)$) in the next way: $\tilde{x}_a(t) = x_a(t)$ if $|t| \leqslant 1$, and $\tilde{x}_a(t) = 0$ if $|t| \geqslant 1$ (resp. $\tilde{x}_b(t) = x_b(t)$ if $|t| \leqslant b$, and $\tilde{x}_b(t) = 0$ if $|t| \geqslant b$). Functions $\tilde{x}_a(t)$ and $\tilde{x}_b(t)$ are differentiable extensions of $x_a(t), t \in [-1,1]$ and $x_b(t), t \in [-b,b]$ to $(-\infty,\infty)$. Solutions of the equation (10) defined on the maximal interval $(-\infty,\infty)$ are functions which belong to the family $\{(\pm\tilde{x}_a(t), t \in (-\infty,\infty)) : a \in [0,1]\}$ or to the family $\{(\pm\tilde{x}_b(t), t \in (-\infty,\infty)) : b \in [1,\infty)\}$ (11). Each point (t,x) with coordinates $|t| < 1, x = 0$ or with coordinates $t \in (-\infty, \infty), |x| > 0$ belongs to only one graph of the solution defined on $(-\infty, \infty)$. But if $t_0 \geqslant 1$, then each point (t, x) with coordinates $|t| \geqslant t_0, x = 0$ belongs to all graphs of solutions $\pm\tilde{x}_a(t), t \in (-\infty, \infty)$, $a \in [0, 1]$, and to all graphs of solutions $\pm\tilde{x}_b(t), t \in (-\infty, \infty)$, $b \in [1, t_0]$ (Fig.1).

We shall prove presently that the solution $\tilde{x}_0(t) = 0, t \in (-\infty, \infty)$ considered on $[t_0, \infty)$ is *unstable* if $t_0 \leqslant -1$, and is stable by Lyapunov if $t_0 > -1$ (12). Suppose that $t_0 \leqslant -1$. Then for $\epsilon = \frac{e-1}{2} > 0$ and for each $\delta > 0$ the solution $\tilde{x}_1(t)$, $t \in (-\infty, \infty)$ considered on $[t_0, \infty)$ satisfies to conditions: $\tilde{x}_1(t_0) = 0 < \delta$, $\tilde{x}_1(0) = (e-1) > \epsilon$. In case $t_0 \leqslant -1$ the statement (12) is proved. Suppose that $t_0 \in (-1, 1)$. Let ϵ be a fixed but arbitrary positive number. Since $\lim_{a \to 0} \tilde{x}_a(0) = \lim_{a \to 0} x_a(0) = 0$, there exists the parameter $a(\epsilon) \in (0, 1]$ such that $0 < \tilde{x}_{a(\epsilon)}(0) < \epsilon$. If $\delta = \tilde{x}_{a(\epsilon)}(t_0) > 0$, then for all solutions $\pm\tilde{x}_a(t)$, $t \in (-\infty, \infty)$ considered on $[t_0, \infty)$ we have: $|\pm\tilde{x}_a(t_0)| = \tilde{x}_a(t_0) < \delta \Rightarrow |\pm\tilde{x}_a(t)| = \tilde{x}_a(t) < \epsilon, t \geqslant t_0$ (Fig.1). In case $t_0 \in (-1, 1)$ the statement (12) is also proved. In case $t_0 \geqslant 1$ the statement (12) is evident because it is enough to take $\delta = \epsilon > 0$ and to remember (1), (11).

The constructed equation (10) has the required properties. Indeed, suppose that $t_0 \leqslant -1$. Then all solutions $\pm\tilde{x}_a(t)$, $t \in (-\infty, \infty)$, $\pm\tilde{x}_b(t)$, $t \in (-\infty, \infty)$ of the equation (10) considered on $[t_0, \infty)$ have zero limits at infinity, and according to (12) the solution $\tilde{x}_0(t) = 0, t \in (-\infty, \infty)$ considered on $[t_0, \infty)$ is *unstable*.

It is known that if a differential equation satisfies to conditions of the theorem that guarantees continuous dependence of its solutions on initial moments of time, then stability by Lyapunov of some solution of the differential equation does not depend on initial moments of time [1, page 162].

By virtue of the constructed equation we can continue this theorem in the next way: if conditions of the mentioned theorem for some differential equation are not fulfilled, then stability by Lyapunov of its solution can depend on initial moments of time.

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ON UNIFORM AND NONUNIFORM CONVERGENCE OF PROPER PARAMETER-DEPENDENT INTEGRALS

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Abstract. It is shown that the notion of the uniform convergence is quite relevant and plays an important role not only for improper parameter-dependent integrals but also for proper parameter-dependent integrals. Examples of uniformly and nonuniformly convergent proper parameter-dependent integrals are given.

1 Introduction

Our article is devoted to parameter-dependent integrals. Such integrals play a very important role in mathematics itself and in its applications.

It is known that many special functions (gamma- and beta- Euler functions, cylindrical functions, a probability integral, etc.) may be represented in the form of parameter-dependent integrals. In mathematical physics integral transformations (Fourier integral transformation, Laplace integral transformation, Mellin integral transformation, etc.) find various applications. All these integral transformations are defined by parameter-dependent integrals.

In theory of parameter-dependent integrals, several types of theorems play an extremely important role. Among them are the theorems in which conditions are formulated at the fulfilment of which: 1) an integral specifies a continuous function of a parameter; 2) passing to the limit with respect to a parameter under the integral sign is possible; 3) differentiation of an integral with respect to a parameter under the integral sign is possible; 4) integration of an integral with respect to a parameter under the integral sign is possible. All these theorems are actively used in practice at studying specific parameter-dependent integrals, as well as at analysing functions given by these integrals.

A notion of a uniform convergence of an integral by a parameter is used in formulations of many of the theorems stated above. However, for reasons unknown to us authors of all mathematical analysis books and textbooks use this notion only in relation to improper parameter-dependent integrals.

In our article we show that the notion of the uniform convergence has a wider field of application than it is generally considered and that this notion is quite relevant and plays an important role not only for improper parameter-dependent integrals but also for proper parameter-dependent integrals.

Let us note here that under the term "proper integral" we mean a usual definite integral (in other words, a one-dimensional Riemann integral).

2 Definition of uniform convergence

Let there be given a function $f(x,\ p)$ that is continuous (as a function of two variables x and p) on the set

$$\Pi = (a,\ b) \times P = \{\langle x,\ p\rangle \in \mathbb{R}^2 \mid a < x < b,\ p \in P\}, \tag{1}$$

where $-\infty \leqslant a < b \leqslant +\infty$ and P is an arbitrary interval.

We consider the integral

$$I(p) = \int_a^b f(x,\ p)\, dx, \tag{2}$$

depending on the parameter $p \in P$.

If $a = -\infty$ or (and) $b = +\infty$, then the integral $I(p)$ is an improper integral at any value of the parameter $p \in P$.

If $a,\ b \in \mathbb{R}$, then at some values of the parameter p the integral $I(p)$ may be a proper integral (at these values of the parameter p the function $f(x,\ p)$ is bounded on the interval $(a,\ b)$), and at other values of the parameter p the integral $I(p)$ may be an improper integral (at these values of the parameter p the function $f(x,\ p)$ is not bounded on the interval $(a,\ b)$).

In any case, if the integral $I(p)$ is an improper integral, then only the points a and b may be particular points of this integral (since at any fixed value of the parameter $p \in P$ the function $f(x,\ p)$ is continuous as a function of the variable x on the integration interval $(a,\ b)$).

Below we give a general definition of the uniform convergence, which is applicable both to improper parameter-dependent integrals and to proper parameter-dependent integrals, as well as to integrals that are improper at some values of the parameter and are proper at other values of the parameter. In other words, we do not demand in the following definitions that the integral $I(p)$ must necessarily be improper integral at all values of the parameter $p \in P$.

Definition 1. The integral $I(p)$ is called *uniformly convergent* with respect to the parameter p on the interval P *at the point* a, if:

1) The integral $I(p)$ is either proper or convergent improper integral at any value of the parameter $p \in P$.

2) $(\forall \varepsilon > 0)\ (\exists a_1 \in (a,\ b))\ (\forall a_2 \in (a,\ a_1))\ (\forall p \in P)\colon \left|\int\limits_a^{a_2} f(x,\ p)\,dx\right| < \varepsilon.$

Definition 2. The integral $I(p)$ is called *uniformly convergent* with respect to the parameter p on the interval P *at the point* b, if:

1) The integral $I(p)$ is either proper or convergent improper integral at any value of the parameter $p \in P$.

2) $(\forall \varepsilon > 0)\ (\exists b_1 \in (a,\ b))\ (\forall b_2 \in (b_1,\ b))\ (\forall p \in P)\colon \left|\int\limits_{b_2}^{b} f(x,\ p)\,dx\right| < \varepsilon.$

Definition 3. The integral $I(p)$ is called *uniformly convergent* with respect to the parameter p on the interval P, if:

1) The integral $I(p)$ converges uniformly with respect to the parameter p on the interval P at the point a.

2) The integral $I(p)$ converges uniformly with respect to the parameter p on the interval P at the point b.

Remark 1. Condition 2) of Definition 1 is equivalent to the condition

$$\lim_{c \to a+0} \sup_{p \in P} \left| \int_a^c f(x,\ p)\,dx \right| = 0. \tag{3}$$

(If $a = -\infty$, then the record $c \to a + 0$ should be understood as $c \to -\infty$.)

Condition 2) of Definition 2 is equivalent to the condition

$$\lim_{c \to b-0} \sup_{p \in P} \left| \int_c^b f(x,\ p)\,dx \right| = 0. \tag{4}$$

(If $b = +\infty$, then the record $c \to b - 0$ should be understood as $c \to +\infty$.)

Remark 2. If the integral $I(p)$ is not uniformly convergent with respect to the parameter p on the interval P, then the integral $I(p)$ is said to converge nonuniformly with respect to the parameter p on the interval P.

Remark 3. If the integral $I(p)$ converges uniformly (nonuniformly) with respect to the parameter p on the interval P, then for brevity we shall speak further in the following way: "The integral $I(p)$ converges uniformly (nonuniformly) with respect to the parameter $p \in P$."

Remark 4. Let us pay attention to one terminological subtlety.

If the integral $I(p)$ is a proper one, then it is said that the integral $I(p)$ exists, and it is not said that the integral $I(p)$ converges. The term "convergent integral" is used (at least usually) only for improper integrals.

Another situation takes place with the notion "uniformly convergent integral". This term, as follows from the definitions given above, may be used for any (improper and proper) integrals.

Such terminology may seem strange, however, it is rather convenient, as practice shows. It is scarcely reasonable to introduce any new term like a "uniformly existing proper integral" for proper parameter-dependent integrals (especially as one and the same parameter-dependent integral at some values of the parameter may be proper and at other values of the parameter may be improper).

3 Continuity of parameter-dependent integrals with respect to parameter

As we have already mentioned above, the notion of the uniform convergence plays an extremely important role in the theory of parameter-dependent integrals.

Since the objective of our article is to show that the notion of the uniform convergence is actual and plays an important role not only for improper parameter-dependent integrals but also for proper parameter-dependent integrals, we shall restrict ourselves to one theoretical question, namely, the theorem on the continuity of a parameter-dependent integral on a parameter.

Let us give a standard formulation of the theorem on the continuity of a proper parameter-dependent integral on a parameter. This formulation may be found in all (or almost in all) rather full mathematical analysis courses.

Theorem 1. *Let the following conditions be fulfilled:*

1) $a \in \mathbb{R}$ *and* $b \in \mathbb{R}$.

2) The function $f(x,\ p)$ *is continuous (as a function of two variables* x *and* p*) on a rectangle closed from the left and from the right* $\Pi = [a,\ b] \times P = \{\langle x,\ p\rangle \in \mathbb{R}^2 \mid a \leqslant x \leqslant b,\ p \in P\}$.

Then the function $I(p)$ *is continuous on the interval* P.

Now, let us formulate a theorem on the continuity of a parameter-dependent integral on a parameter in which the notion of the uniform convergence is used.

Theorem 2. *Let the following conditions be fulfilled:*

1) The function $f(x, p)$ is continuous (as a function of two variables x and p) on a rectangle open from the left and from the right $\Pi = (a, b) \times P = \{\langle x, p\rangle \in \mathbb{R}^2 \mid a < x < b, p \in P\}$.

2) The integral $I(p)$ converges uniformly with respect to the parameter $p \in P$.

Then the function $I(p)$ is continuous on the interval P.

In the formulation of Theorem 2 it is not required that the points a and b are finite, i.e., these points may be either finite or infinite.

To give a complete picture, let us formulate one more theorem. This theorem may be considered to some extent as a combination of Theorems 1 and 2.

Theorem 3. *Let the following conditions be fulfilled:*

1) $b \in \mathbb{R}$.

2) The function $f(x, p)$ is continuous (as a function of two variables x and p) on a rectangle open from the left and closed from the right $\Pi = (a, b] \times P = \{\langle x, p\rangle \in \mathbb{R}^2 \mid a < x \leqslant b, p \in P\}$.

3) The integral $I(p)$ converges uniformly with respect to the parameter $p \in P$ at the point $x = a$.

Then the function $I(p)$ is continuous on the interval P.

In the formulation of Theorem 3 it is not required that the point a is finite, i.e., this point may be either finite or infinite.

4 Examples of uniformly and nonuniformly convergent proper parameter-dependent integrals

We have formulated three theorems on the continuity of a parameter-dependent integral on a parameter. To tell the truth, all these theorems may be found in mathematical literature.

However, for reasons unknown to us authors of all mathematical analysis books and textbooks place Theorems 2 and 3 only in a chapter devoted to improper parameter-dependent integrals. Whereas, in a chapter devoted to proper parameter-dependent integrals, authors restrict themselves to Theorem 1 and don't formulate Theorems 2 and 3 and even don't introduce the notion of the uniform convergence.

Below we consider two simple examples. These examples show clearly that both Theorems 2 and 3 and the notion of the uniform convergence itself may be applied to proper parameter-dependent integrals to a no lesser extent than to improper parameter-dependent integrals.

Example 1 (the example of a proper parameter-dependent integral that converges nonuniformly). Let us consider the integral

$$I(p) = \int\limits_0^1 \frac{2xp}{(x^2+p^2)^2}\,dx, \quad p \in \mathbb{R}. \tag{5}$$

The integrand $f(x,\ p) = \dfrac{2xp}{(x^2+p^2)^2}$ is continuous (as a function of two variables x and p) on a rectangle open from the left and closed from the right $\Pi = (0,\ 1] \times \mathbb{R} = \{\langle x,\ p\rangle \in \mathbb{R}^2 \mid 0 < x \leqslant 1,\ p \in \mathbb{R}\}$.

At $p = 0$, the integrand $f(x,\ p)$ is identically equal to zero, and it means that the integral $I(p)$ is a proper one. At any fixed value of the parameter p, $p \neq 0$, the integrand $f(x,\ p)$ is continuous as a function of the variable x on the interval $[0,\ 1]$, and it means that the integral $I(p)$ is also a proper one.

Thus, the integral $I(p)$ is a proper one at all values of the parameter $p \in \mathbb{R}$.

Let us show that the proper integral $I(p)$ converges nonuniformly with respect to the parameter $p \in \mathbb{R}$ at the point $x = 0$. Let $c \in (0,\ 1]$. As a result of simple calculations we find:

$$I(p,\ c) = \int\limits_0^c \frac{2xp}{(x^2+p^2)^2}\,dx = \ldots = \begin{cases} \dfrac{c^2}{p(c^2+p^2)} & \text{if } p \neq 0; \\ 0 & \text{if } p = 0. \end{cases} \tag{6}$$

From this it follows that $\sup\limits_{p\in\mathbb{R}} |I(p,\ c)| = +\infty$ at any $c \in (0,\ 1]$.

Thus, condition (3) is not fulfilled, and it means that the integral $I(p)$ converges nonuniformly with respect to the parameter $p \in \mathbb{R}$ at the point $x = 0$.

The value of the integral $I(p)$ equals to

$$I(p) = I(p,\ 1) = \begin{cases} \dfrac{1}{p(1+p^2)} & \text{if } p \neq 0; \\ 0 & \text{if } p = 0. \end{cases} \tag{7}$$

We see that the function $I(p)$ is discontinuous at the point $p = 0$. It is not surprising because the integral $I(p)$, as mentioned above, converges nonuniformly with respect to the parameter $p \in \mathbb{R}$ at the point $x = 0$. If the integral $I(p)$ converged uniformly with respect to the parameter $p \in \mathbb{R}$ at the point $x = 0$, then, in accordance with Theorem 3, the function $I(p)$ would be a continuous function of the parameter $p \in \mathbb{R}$.

Example 2 (the example of a proper parameter-dependent integral that converges uniformly). Let us consider the integral

$$I(p) = \int\limits_0^1 \frac{2xp^5}{(x^2+p^4)^2}\,dx, \quad p \in \mathbb{R}. \tag{8}$$

The integrand $f(x,\ p) = \dfrac{2xp^5}{(x^2+p^4)^2}$ is continuous (as a function of two variables x and p) on a rectangle open from the left and closed from the right $\Pi = (0,\ 1] \times \mathbb{R} = \{\langle x,\ p\rangle \in \mathbb{R}^2 \mid 0 < x \leqslant 1,\ p \in \mathbb{R}\}$.

At $p = 0$, the integrand $f(x,\ p)$ is identically equal to zero, and it means that the integral $I(p)$ is a proper one. At any fixed value of the parameter p, $p \neq 0$, the integrand $f(x,\ p)$ is continuous as a function of the variable x on the interval $[0,\ 1]$, and it means that the integral $I(p)$ is also a proper one.

Thus, the integral $I(p)$ is a proper one at all values of the parameter $p \in \mathbb{R}$.

Let us show that the proper integral $I(p)$ converges uniformly with respect to the parameter $p \in \mathbb{R}$ at the point $x = 0$. Let $c \in (0,\ 1]$. As a result of simple calculations we find:

$$I(p,\ c) = \int\limits_0^c \frac{2xp^5}{(x^2+p^4)^2}\,dx = \ldots = \begin{cases} \dfrac{c^2 p}{c^2+p^4} & \text{if } p \neq 0; \\ 0 & \text{if } p = 0. \end{cases} \tag{9}$$

At $p \neq 0$, we evaluate the module of the integral $I(p,\ c)$ by using the inequality of arithmetic and geometric means:

$$|I(p,\ c)| = \frac{c^2|p|}{\dfrac{c^2}{3}+\dfrac{c^2}{3}+\dfrac{c^2}{3}+p^4} \leqslant \frac{c^2|p|}{\dfrac{4}{\sqrt[4]{27}}\,c^{3/2}|p|} = \frac{\sqrt[4]{27}}{4}\,c^{1/2}. \tag{10}$$

It is obvious that inequality (10) is also true at $p = 0$ (since at $p = 0$ the integral $I(p,\ c)$ equals to zero).

It follows from evaluation (10) that $\sup\limits_{p\in\mathbb{R}} |I(p,\ c)| \leqslant \dfrac{\sqrt[4]{27}}{4}\,c^{1/2}$ at any $c \in (0,\ 1]$. Therefore, $\lim\limits_{c\to 0+0} \sup\limits_{p\in\mathbb{R}} |I(p,\ c)| = 0$.

Thus, condition (3) is fulfilled, and it means that the integral $I(p)$ converges uniformly with respect to the parameter $p \in \mathbb{R}$ at the point $x = 0$.

The value of the integral $I(p)$ equals to

$$I(p) = I(p,\ 1) = \frac{p}{1+p^4}, \quad p \in \mathbb{R}. \tag{11}$$

We see that the function $I(p)$ is a continuous function of the parameter $p \in \mathbb{R}$. It should be in such a way in accordance with Theorem 3, since the integral $I(p)$, as mentioned above, converges uniformly with respect to the parameter $p \in \mathbb{R}$ at the point $x = 0$.

Remark 5. Let us note that in Example 2 we have proved the continuity of the proper parameter-dependent integral on the parameter by using the notion of the uniform convergence and Theorem 3.

The following question arises naturally: is it possible to prove the continuity of the function $I(p)$, $p \in \mathbb{R}$, in Example 2 not using the notion of the uniform convergence and not using Theorem 3, but applying Theorem 1 only?

The answer to this question is negative.

In Theorem 1 the integrand $f(x,\ p)$ must be continuous (as a function of two variables x and p) on a rectangle closed from the left and from the right $\Pi = [0,\ 1] \times \mathbb{R} = \{\langle x,\ p\rangle \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1,\ p \in \mathbb{R}\}$. However, in Example 2 the integrand $f(x,\ p) = \dfrac{2xp^5}{(x^2+p^4)^2}$ is not continuous on this rectangle.

Indeed, let us approach the point $\langle x,\ p\rangle = \langle 0,\ 0\rangle$ along a parabola $x = t^2$, $p = t$, $t \in (0,\ 1)$. We have $f(x,\ p) = f(t^2,\ t) = \dfrac{1}{2t}$, therefore $\lim\limits_{t\to 0+0} f(t^2,\ t) = +\infty$. We see that the function $f(x,\ p)$ is not bounded in a neighborhood of the point $\langle x,\ p\rangle = \langle 0,\ 0\rangle$. Consequently, the function $f(x,\ p)$ cannot be defined at the point $\langle x,\ p\rangle = \langle 0,\ 0\rangle$ so that the resulting function would be continuous at this point.

Thus, it is not possible to prove the continuity of the function $I(p)$, $p \in \mathbb{R}$, in Example 2 by means of standard Theorem 1 on continuity of a proper parameter-dependent integral on a parameter. To prove the continuity of the function $I(p)$, $p \in \mathbb{R}$, we need to use the notion of the uniform convergence and Theorem 3.

5 Conclusions

The examples given by us confirm that the notion of the uniform convergence is actual for proper parameter-dependent integrals to a no lesser extent than for improper parameter-dependent integrals. Therefore, in our opinion, the notion of the uniform convergence should be introduced for an arbitrary parameter-dependent integral, not specifying what kind of integral is it (proper, improper or proper for some and improper for other values of the parameter).

The author would be very grateful to the readers for any comments or remarks on the subject of this work.

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GENERALIZED NPS-APPROACH FOR EDUCATION QUALITY RATE

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Key words: NPS-approach for education, generalized NPS coefficient, analysis of NPS coefficient dynamics

AMS Mathematics Subject Classification: 97B40, 97D40

Abstract. In the article it is suggested the model for estimation quality of education on the base of introduced by authors new concept of generalized NPS coefficient with m-valued scale and arbitrary weights. Approaches to analysis of dynamics generalized NPS coefficient in the case of finite entire assembly is suggested on basis of using Student's distribution, Behrens-Fisher test and normal approximation. Comparative analysis of these approaches is given. Practical recommendations for choice of the approach to analysis of dynamics generalized NPS coefficient are given. Criterion of significant difference of NPS coefficients is formulated.

1 Introduction

This article is dedicated to mathematical model of client oriented functioning structures quality rate, particularly educational institution. Main point is related with NPS (Net Promoter Score) coefficient use, which was introduced by Reichheld [1] and was used till now only for analysis of such client oriented structures like insurance, telecommunication companies, banks etc. In works of Reichheld NPS-approach wasn't used in education, though author marked possibility of such using. In the course of the works with NPS coefficient we sorted out disadvantages of classical NPS-approach [2], which can bring to wrong administrative decisions, wrong estimates of education level. New customer loyalty metric is presented in current article — generalized NPS coefficient. Results of Reichhold show that answer on key question of company recommendation is mainly correlated with success of a company. For education process we can offer such question: "How you rate your recommendation of education course (institute, department, speciality) within the m-digit scale?" Let us correspond the weight α_j, $-1 \leqslant \alpha_j \leqslant 1$, to the answer of j-type. It is natural to suppose that $\alpha_i \leqslant \alpha_j$ if $i < j$. As a entire assembly it is considered the set of company clients (all university students etc.).

2 Definition of Generalized NPS coefficient

Definition 1. Generalized NPS coefficient is called

$$NPS(\alpha_0, \dots, \alpha_{m-1}) = \sum_{j=0}^{m-1} \alpha_j \cdot p_j \tag{1}$$

Here p_j is an unknown probability of j-type subject appearance in entire assembly of volume N. It is convenient to consider the generalized NPS coefficient as average value $NPS(\alpha_0, \dots, \alpha_{m-1}) = \frac{1}{N}\sum_{i=0}^{N} X_i$ of independent and equally distributed random values X_i, which are determined by the following way: $X_i = \alpha_j$, if i-st subject of entire assembly has j-type. As an estimate of probability p_j observed frequency is used $\omega_j = m_j/n$, where n is a number of questioned subjects (sample volume), m_j is number j-type subject in sample. Replacing in (1) probability p_j with observed frequency ω_j, we receive unbiased and consistent estimate $(NPS)^{\wedge}_{gen} = \sum \alpha_j \cdot \omega_j$ for generalized NPS coefficient.

Making regular samples for measurement of generalized NPS coefficient, you can study specific tendency, make conclusions and administrative decisions. NPS is a sort of indicator, if dynamic is negative then additional analysis will be needed to find out problem causes. Thereby, usage of NPS-approach supposes to compare NPS values, received results of different questionings. As you can consider NPS as an average value, then comparing NPS of two different independent samples is based on using methods of comparing average values. Main approaches for dynamic analysis of generalized NPS coefficient are based on usage of Students distribution, usage of Berence-Fisher test, on approximation of normal distribution. Let us consider the modification of these approaches for finite entire assembly case and make comparative analysis.

Lemma 1 (see [3]). *If σ^2 is a variance of finite entire assembly, then variance of sample average $\overline{X}$ for simple random sample equals to $D(\overline{X}) = \sigma^2 \cdot (1-f)/n$, where $f = n/N$ — part of selection, N — volume of entire assembly.*

Let $A = \{X_i, i = 1, \dots, N_A\}$ and $B = \{Y_j, j = 1, \dots, N_B$ — two given entire assemblies volume N_A and N_B (clients sets of two different companies, sets of company clients that are considered in different moments, graduates of two different years etc.). Let $f_A = n_A/N_A$ and $f_B = n_B/N_B$ — parts of selection, where n_A and n_B — volumes of samples. Denote unknown probability of j-type subject appearance in totalities A and B through p_j^A, p_j^B. The random values $\{X_i, i = 1, \dots, N_A\}$ (as, $\{Y_j, j = 1, \dots, N_B\}$) are supposed to be equally distributed $P\{X_i = \alpha_j\} = p_j^A$

(so, $P\{Y_i = \alpha_j\} = p_j^B$), $j = 0, 1, \ldots, m-1$. Let $\mu_A = NPS_{gen}^A$ and $\mu_B = NPS_{gen}^B$ be generalized NPS coefficients of finite collections A and B. Goal of comparing NPS of two different independent entire assemblies A and B is a goal of comparing mathematical expectation value based on selective data. As an estimation of generalized NPS coefficients of totalities A and B we take sample averages: $(NPS)_C^\wedge = \overline{X}_C = n_C^{-1} \sum_{i=1}^{n_C}$, $C = A, B$. Let $\sigma_A^2 = \left\{ \sum \alpha_j^2 p_j^A - \left(\sum \alpha_j p_j^A \right)^2 \right\}$ and similarly, σ_B^2 are variances of entire assemblies A and B. Sample variance S_A^2 is determined by the formula $S_A^2 = \frac{n_A}{n_A - 1} \left\{ \sum \alpha_j^2 \omega_j^A - \left(\sum \alpha_j \omega_j^A \right)^2 \right\}$, where ω_j^A — observed frequency of j-type subject appearance. Similarly sample variance S_B^2 may be found. Note, that sample variances S_A^2 and S_B^2 are unbiased variance estimates.

3 First approach for comparison generalized NPS coefficient

Assume that entire assemblies A and B with unknown average μ_A and μ_B are infinite and normally distributed. In case where variances are unknown but equals $\sigma_A^2 = \sigma_B^2 = \sigma^2$, goal of constructing confidence interval for difference $\delta = \mu_A - \mu_B$ is decided in the following way [4]. Value $d = \overline{X}_A - \overline{X}_B$ has normal distribution with parameters $M(d) = \mu_A - \mu_B$, $D(d) = \sigma_A^2/n_A + \sigma_B^2/n_B$. Each of values $(n_A - 1) \cdot S_A^2/\sigma_A^2$ and $(n_B - 1) \cdot S_B^2/\sigma_B^2$ have χ^2-distribution with $n_A - 1$ and $n_B - 1$ degree of freedom. As samples are independent then value $\left[(n_A - 1) \cdot S_A^2 + (n_B - 1) \cdot S_B^2\right]/\sigma^2$ has χ^2-distribution with $K_{St} = n_a + n_b - 2$ freedom degrees. In that case it is considered the combined unbiased estimate of variance $\left[(n_A - 1) \cdot S_A^2 + (n_B - 1) \cdot S_B^2\right]/(n_A + n_B - 2)$. Easy to check $M\left(S_p^2\right) = \sigma^2$. Statistics

$$T = \frac{(d - \delta) \cdot \sigma^{-1} \left(1/n_A + 1/n_B\right)^{-1/2}}{\sqrt{S_p^2 \cdot \sigma^{-2}}} = \frac{d - \delta}{S_p \cdot \sqrt{1/n_A + 1/n_B}} \qquad (2)$$

is ratio of standard normal random value and radical of unbiased estimate of variance. Since S_A^2 and S_B^2 don't depend from $\overline{X}_A$ and $\overline{X}_B$, then denominator and numerator are independent. Further the value $S_p^2 \cdot \sigma^{-2} \cdot (n_A + n_b - 2)$ has χ^2-distribution with $(n_A + n_b - 2)$ freedom degrees. It follows that T has t-Student's distribution with $K_{St} = n_A + n_b - 2$ freedom degrees.

Decline the assumption about normality and infiniteness of entire assemblies. In virtue of lemma 1 instead of (2), $T = (d - \delta)/\left[S_p\sqrt{(1 - f_A)/n_a + (1 - f_B/n_b)}\right]$ statistics is considered, that has distribution similar to t-Student's distribution with $K_{St} = n_A + n_b - 2$ freedom degrees. Confidence interval covering difference $\delta =$

$\mu_A - \mu_B$ with reliability $\gamma = 1 - \alpha$ is given by

$$\big(d - t_{1-\alpha/2}(k) \cdot S_{St}(n_A, n_B), d + t_{1-\alpha/2}(k) \cdot S_{St}(n_A, n_B)\big) \tag{3}$$

Here $S_{St}(n_A, n_B) = S_p\sqrt{(1 - f_A)/n_A + (1 - f_B/n_B)}$, $t_{1-\alpha/2}(k) = t_{1-\alpha/2}(n_A+n_B-2)$ — quantile of order $1 - \alpha/2$ for Student's distribution with $k = n_A + n_B - 2$ freedom degrees. Strictly speaking as it was mentioned before in such approach generalized totalities are supposed normally distributed with equal variances. However in practice this test is used not only for normal distributed values. This is related to the point that t-test (especially for large samples) "steady enough" to deflection of researched entire assemblies from Gauss's. It is needed to understand that true values of significance level and strength of a test will differ from specified ones. In practice test is pretty often used without assumption about variance's equality of researched entire assemblies.

4 Second approach for comparison generalized NPS coefficient

First consider case of infinite totalities A and B. In case where variances are unknown and not equal $\sigma_A^2 \neq \sigma_B^2$, goal of comparison two averages normally distributed totalities is known as a problem of Berence-Fisher [4]. Since exact solution of symmetrically relying from selective data can't be found, different approximations are used in practice. Statistics

$$Z = [d - \delta]/S(n_A, n_B) \tag{4}$$

where $S(n_A, n_B) = \sqrt{S_A^2/n_A + S_B^2/n_B}$, is a ratio of Gauss's random value with zero average and radical of independent unbiased estimate it's variance. An estimate $S^2(n_A, n_B)$ has not χ^2-distribution and therefore distribution of random value Z isn't a Student's distribution. Though, with any n_A and n_B random value Z has approximately Student's distribution with K_{BF} freedom degrees [4], where $K_{BF}(n_A, n_B) = D^2(d)\left\{D_A^2/(n_A - 1) + D_B^2/(n_B - 1)\right\}^{-1}$. Here $D_A = \sigma_A^2/n_A$, $D_B = \sigma_B^2/n_B$. Described test with $\sigma_A^2 \neq \sigma_B^2$ is also known as Berence-Fisher test. In practice Berence-Fisher test is used not only for normally distributed values and for freedom degrees number the following approximate value is used

$$\hat{K}_{BF}(n_A, n_B) = S^4(n_A, n_B)\left\{V_A^2/(n_A - 1) + V_B^2/(n_B - 1)\right\}^{-1} \tag{5}$$

Here $V_A = S_A^2/n_A$, $V_B = S_B^2/n_B$. For the case of finite entire assemblies because of lemma 1 instead of (4) is considered the following statistics: $Z =$

$\{d-\delta\}/S_{finite}(n_A, n_B)$, where $S_{finite}(n_A, n_B) = \sqrt{(1-f_A)V_A + (1-f_B)V_B}$. In formula (5) expressions V_A and V_B should be replaced by $W_A = (1-f_A)V_A$ and $W_B = (1-f_B)V_B$:

$$\hat{K}_{BF,finite}(n_A, n_B) = S^4_{finite}(n_A, n_B)\left\{W_A^2/(n_A-1) + W_B^2/(n_B-1)\right\}^{-1} \quad (6)$$

Confidence interval that covers difference $\delta = \mu_A - \mu_B$ with given reliability $\gamma = 1-\alpha$ is the following

$$\left(d - t_{1-\alpha/2}(k)\cdot S_{finite}(n_A, n_B), d + t_{1-\alpha/2}(k)\cdot S_{finite}(n_A, n_B)\right), \quad (7)$$

where $t_{1-\alpha/2}(k)$ — Student's distribution quantile with $k = \hat{K}_{BF,finite}(n_A, n_B)$ freedom degrees of order $1-\alpha/2$.

5 Third approach for comparison generalized NPS coefficient

Easier approach can be suggested that is based on approximation of normal distribution. If samples are independent and have large enough value (at least 30 each) then sample averages are distributed approximately normally and sample variances are good enough estimates of general variances. More precisely, let $X = \{X_1, \ldots, X_n\}$ is a sample of some distribution with unknown average μ with finite (unknown) variance σ^2, S_n^2 — sample variance. Then random value distribution $Z_n = \sqrt{n}\cdot\left(\overline{X}-\mu\right)/S_n$ tends as $n\to\infty$ to standard normal distribution. In this way statistics $Z = \{d-\delta\}/S_{finite}(n_A, n_B)$ has approximately normal distribution with zero average and unit variance. The asserted statement now follows that for specified confidence probability $\gamma = 1-\alpha$ confidence interval for theoretical difference $\mu_A - \mu_B$ is the following form

$$\left(d - c_{1-\alpha/2}\cdot S_{finite}(n_A, n_B), d + c_{1-\alpha/2}\cdot S_{finite}(n_A, n_B)\right), \quad (8)$$

Here $c_{1-\alpha/2} = \Phi^{-1}(1-\alpha/2)$ — quantile of standard normal distribution of order $1-\alpha/2$.

6 Analysis of dynamics generalized NPS coefficient

Theorem 1. *It is fair the inequality*

$$c_{1-\alpha/2} < t_{1-\alpha/2}(n_A+n_B-2) < t_{1-\alpha/2}\left(\hat{K}_{BF,finite}(n_A, n_B)\right), \quad (9)$$

Proof follows from the following lemmas.

Lemma 2. *Let $T_k(x)$ — Student's distribution function with k freedom degrees, $\Phi(x)$ — standard normal distribution function. Then for $x > 0$ the inequality $T_k(x) < \Phi(x)$ is fair and in addition if $k < m$, then $T_k(x) < T_m(x)$.*

Corollary 1. If $k < m$, then $t_{1-\alpha/2}(m) < t_{1-\alpha/2}(k)$, $c_{1-\alpha/2} < t_{1-\alpha/2}(k)$.

Lemma 3. *Number of Student's distribution freedom degrees $k = \hat{K}_{BF,finite}(A,B)$ in Berence-Fisher test does not exceed $n_A + n_B - 2$:*

$$\hat{K}_{BF,finite}(A,B) \leqslant n_A + n_B - 2. \tag{10}$$

Moreover, $\hat{K}_{BF,finite}(A,B) \geqslant \min(n_A - 1, n_B - 1)$.

Remark 1. If $N_A = N_B$, $n_A = n_B$, $S_A^2 = S_B^2$, then in inequality (10) equality is reached. From theorem 1 we have the following statement.

Corollary 2. Assume that surveys A and B are taken in same conditions: $n_A = n_B$, $f_A = f_B$. Then confidence interval of Berence-Fisher approach is as well as confidence interval of Student's distribution approximation (first approach), is wider then confidence interval of normal approximation.

Remark 2. Confidence interval of Berence-Fisher approach always wider then confidence interval of normal approximation.

In general case $S^2_{finite}(n_A, n_B) \neq S^2_{St}(n_A, n_B)$. Can we say at once which one of two variances is larger? Let us denote $n_A = n$, $n_B = kn$, $S_A^2 = S^2$ and $S_B^2 = \beta S^2$, $1 - f_A = \nu$ and $1 - f_B = \xi\nu$.

Lemma 4. $S^2_{finite}(n_A, n_B) - S^2_{St}(n_A, n_B) = \left[S^2\nu/(kn(n + nk - 2))\right] \cdot (1 - \beta)\{n(k^2 - \xi) - (k - \xi)\}$. *If $f_A = f_B$, then* $S^2_{finite}(n_A, n_B) - S^2_{St}(n_A, n_B) = [\nu/(n_A n_B(n_A + n_B - 2))]\,(n_B - n_A)\left(S_A^2 - S_B^2\right)\{n_A + n_B - 1\}$.

Corollary 3. Assume that $f_A = f_B$. If $(n_B - n_A)(S_A - S_B) \leqslant 0$, then confidence interval in normal approximation is the most narrow.

Formulate some practice recommendation basing on comparing method choice. Denote lengths of confidence intervals (3), (7) and (8) for given reliability $\gamma = 1 - \alpha$ through:

$$\Delta_{St} = 2t_{1-\alpha/2}(k) \cdot S_{St}(n_A, n_B),\ k = n_A + n_B - 2, \tag*{(3)*}$$

$$\Delta_{BF} = 2t_{1-\alpha/2}(k) \cdot S_{finite}(n_A, n_B),\ k = \hat{K}_{BF,finite}(n_A, n_B), \tag*{(7)*}$$

$$\Delta_{normal} = 2c_{1-\alpha/2} \cdot S_{finite}(n_A, n_B), \tag{8)^*$$

Consider the most common situation in practice: let questionings A and B be in similar conditions: $f_A \approx f_B$, $n_A \approx n_B$. Then $S_{finite}(n_A, n_B) \approx S_{St}(n_A, n_B)$. Ratio of confidence intervals values (3), (7) and (8) in the case identify with the help of f quantiles ratio. First compare $(3)^*$ and $(8)^*$.

Bring in ratio error $\lambda(k) = [t_{1-\alpha/2}(k) - c_{1-\alpha/2}]/c_{1-\alpha/2}$. For given reliability $\gamma = 1 - \alpha$ and given ratio error λ define the Student's distribution freedom degrees $k(\alpha, \lambda)$ in such way that with $k > k(\alpha, \lambda)$ inequality $\lambda(k) < \lambda$ will be fair. For example let reliability level $\gamma = 90\%$ is given, and ratio error for defining confidence interval $\lambda = 5\%$ suits us. If $k = n_A + n_B - 2 > 20$, then first approach (usage of Student's distribution) in essence doesn't differ from 3rd approach (usage of normal approximation). Thereafter (because of lemma 3), if $\min(n_A - 1, n_B - 1) > 20$, then all three approaches are equivalent.

7 Methodical recommendations

Developed approach allows correctly from mathematical point of view to compare generalized NPS coefficients corresponding two different interviews of students. At the beginning it is selected one of suggested methods of comparison. If n_A and n_B are not "large enough" (as it was described before), then it would be wiser to use first or second approaches. Which one to choose? If you have grounds to thing that variance of entire assemblies differ a lot (you can check it with the help of Fisher-Snedecor test), then it is better to use the most "conservative" approach — Berence-Fisher test. After choosing the approach for comparing NPS values that refer to two independent questionings A and B, we can say that NPS_{gen}^{A} essentially differs from NPS_{gen}^{B} for given level of significance $\alpha = 1-\gamma$, if difference of estimates $(NPS)_A^\wedge$ and $(NPS)_B^\wedge$ doesn't get into the proper confidence interval. If NPS_{gen}^{A} essentially differs from NPS_{gen}^{B}, then we can talk about changing of education quality.

NPS-technology for estimation quality of education includes: a) development of key question and number scale for estimating of student's answers; b) carrying out two different student's interviews; c) revelation of dynamics generalized NPS coefficient for given confidence level; d) development of recommendations for improvement quality of education (correction of education, argumentation efficiency of education model etc.) in the case of significant decrease of generalized NPS coefficient. Algorithm for revelation of dynamics generalized NPS coefficient includes selection of confidence level; construction distribution of interviewed students depending on answer to the key question for each enquiry; construction of estimates for NPS coefficients; construction of confidence interval for difference of NPS coefficients; finding significant increase or decrease of NPS coefficient. Significant increase of generalized NPS coefficient is indicator of validity of selected strategy and tactics for education; significant decrease NPS is indicator of existence of problems in realizable model education. Further work on finding the problem in education model supposes conversation with students, detailed questionnaire survey, analysis of student's works.

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INTERACTION DIALECTICS OF LINEAR AND CIRCULAR METHODS IN TEACHING MATHEMATICS ANALYSIS

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Key words: mathematical analysis in higher school, linear and circular teaching methods

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Abstract. Two methods of presenting mathematical analysis: a linear one leading to subsequent explaining the material from axioms to supplements, and a circular method using a systematic way of coming back to the learnt material.

Mathematics analysis as a study discipline in higher schools with mathematics trend has a long history of its teaching and established approaches to the content presentation.

A linear method has been the main principle in delivering lectures. This method is a basic one in other special mathematics courses with an advanced teaching process. The use of the method depends on the main mathematics education problems. Here it allows to develop strict thinking in students' learning the subject matter. The method helps to point out fundamental concepts clearly, fixing basic statements proof of theorems and searching mathematics application of the latter both within and beyond mathematics proper. The structure of the present mathematics education has so far allowed solving this problem at the expense of good fundamental teaching the subject in the secondary school and the number of hours in the curriculum for the physics - mathematics speciality at higher schools. This approach has been supported by good teaching manuals, particularly, in the works of L.D. Kudryovtsev [1, 2, 2] and B.P. Demidovich [3]. We strongly recommend these books as basic ones in learning mathematics analysis by the students.

The present situation in the secondary and higher school education has changed though the main task of mathematics education is the same and the conditions of its solving are different.

1) The basic level of mathematics qualification amongst the first-year students has clearly become worse. This is the view by lecturers not only in provincial higher schools but elsewhere in the country.

Hardly any lecturer at a higher school can deny this present situation and be pleased with it. Consequently, it has to carry out revising control tests and to organize additional classes in elementary mathematics at the beginning of the academic year. Each higher school is completely free in arranging such classes: they can be "compulsory" or voluntary depending on the budget of the college. Of course, such classes can only partially liquidate the students' knowledge gap in elementary mathematics.

The first-year students lack knowledge in formula, definitions and theorems at the beginning of the academic year. They can hardly discuss and prove theorems. Especially it's evident at classes in trigonometry while solving transcendental inequality and geometric problems. A basic knowledge in mathematics is becoming less, though it doesn't contradict to positive results at final unified state examinations in mathematics. The break in learning mathematics for the first-year students is almost 12 weeks, during this period the school-leaver usually does not think about the subject. It's arguable but new information without supporting it with special training reduces the man's memory level at 80% a week. But this statement is well in accordance with our experimental data. Mathematics operates with abstract concepts the pupil does not deal with in every day life while successful gaining information in mathematics by the learner is only possible through systematic training. What is more, the reduction of the number of hours in mathematics at the secondary school leads to lessening the stock of knowledge and as a result it worsens the level of mathematics problems skills. Thus, by the beginning of the academic year the majority of the first-year students does not know elementary mathematics.

"The presidential board on improving the conducting the examination suggests withdrawing it from under monitoring by educational departments, i.e. the Ministry of Education and Science and the Russian Educational Supervision. The members of the board believe that it will allow to a more objective results of tests for educational departments are inclined to estimate better themselves" ([3], http://www.aif.ru/society/news/89981/, "About the decline of trust to the USE results").

Our commentary: It is well known that one of the arguments about the advantage of the USE introduction was a statement that the school teachers can't be trusted in arranging final exams as they value themselves. Now the same is true about the Federal Ministry, it can't be trusted either. Where can this continuous statement lead to?

Is a successful educational reform possible in Russia?

A live radio broadcast "Komsomolskaya Pravda" (participants-the director of the education development institute at HSE Prof. Irina Abankina, the editor of the education department in "KP" Alexander Milkus, journalist Helen Afonina and radio listeners). H. Afonina: "Let's sum up the results of our voting. Only 2% believe that successful educational reform is our country possible and 98% are sure, alas, that this is impossible in Russia".

Number of the task	Undertook the solution in 2009	Have solved fully in 2009	Undertook the solution in 2010	Have solved fully in 2010	Undertook the solution in 2010	Have solved fully in 2010
C-1	38,74%	28,65%	60,01%	38,11%	67,23%	31,70%
C-2	35,34%	17,38%	20,45%	10,56%	25,18%	17,32%
C-3	7,60%	1,53%	25,21%	4,14%	42,56%	7,82%
C-4	4,04%	0,95%	3,75%	0,31%	9,00%	1,66%
C-5	7,17%	1,11%	6,49%	1,03%	12,30%	1,75%
C-6	-	-	4,27%	0,37%	4,79%	0,49%

Reorientation of the Physics and Mathematics faculties in training specialists of the engineering profile with advanced learning mathematics, but the number of hours in the curriculum is reduced about twice (e.g. the qualification "applied mathematics and computer science"), while the material volume study according SGS is practically the same as in the former specialization "mathematics".

The latter is preserved only at the largest universities. Is it good or bad? In teaching and education, as a rule, there are no definite answers. On the one hand, it's good as in such higher schools large scientific powers are concentrated. On the other hand, the answer is "no", for: a) not all applicants can join higher schools, b) there is a differentiation according to academic study results in each group and depending on the life-level adaptation in a large city (this differentiation does take place at all education levels and at all colleges, no matter where they are situated), c) not all graduates of central higher schools are inclined to return home after the graduation.

It is worth marking the presence of objective factors which is well-known in an educational co-society: there is a rapid growth of new information in science which is to be inherited by a new generation. Consequently, we'll have to reject the teaching of some study material in favor of new scientific facts. In mathematics it means to give a priority to teaching skills or to an art of proof or discussion. For this it is not necessarily to study all the theorems with a full proof. We understand the complexity and problematic of realization this view.

Thus, the objective necessity of combining the circular and linear methods in teaching mathematics analysis is up-to-date.

The principal of linear presentation the theory has become a traditional methodological principal delivering mathematics courses at mathematics and Physics and Mathematics faculties in Russian universities. We would like to point out again that our principle means a successive presentation of the study material (partly, mathematics analysis) and beginning with the description of indefinite concepts, formulating axioms and a strict proof of following theorems and mentioning the supplements of corresponding mathematics theories.

Our teaching experience at the university is a proof of usefulness and necessity of such an approach in delivering lectures on math analysis. The method under discussion enables to arrange a successive and gradual complexity of abstract concepts. It teaches the students to understand the necessity of proving their statements and makes it clear in doing so of the inner math analysis structure as a science. The linear method helps to arrange the representation from primary concepts till latest achievements of the science. Here the lecturer supposes that a student can cope with the fixation of the lecture in his notes and the memory and simultaneously to keep up with the lecturer's discourse. Not every freshman is able "to work" at the lecture this way. Usually a school-leaver of a profile mathematics class at school can cope with this problem as he studies the course of algebra and the start of analysis at school with the help of Vilenkin N.Ya. text-book [9]. This manual includes approximately 150 theorems and 300 definitions which the pupil studies in 10-11 grades. A school-leaver of the comprehensive secondary school is but a rare Physics and Maths student. Here there is a good text-book by Mordkovich A.G. [8]. It consists of 33 theorems and 82 definitions. A corresponding calculation is undertaken by the authors of the present work and certainly it is of an approximate character. We regarded those mathematical statements which are named as theorems and definitions by the authors of the manuals themselves. By the way, L.D. Kudryavtsev text-book [1] comprises 300 theorems and 400 definitions meant for the students during the first semester. If we transfer the situation into an arithmetic language, we can say that it is necessary to have applicants for university with the score not less than 80 to 100 for organizing a normal teaching process. Unfortunately, as our experience shows, there are 2-3 students in each group.

Note: It goes without saying that it is impossible to present all the material with the help of the linear method. It's not by chance that already in Ancient Greece before the beginning of philosophic disputes the discussing sides came to an agreement about what is known, i.e. to take it as the point which does not need proof. The same is true in any handy text-book where in the foreword the writer usually does the same with the reader about what is definite and he describes this material in short and introduces only some remarks, e.g. in the text-book [1] it is paragraph 1: "Sets and functions. Logical symbols".

It's necessary to underline the fact that an increase of the text-book material takes place in the mathematics analysis course itself. For this we may compare:

1) A two-volume textbook in math analysis by Kudryavtsev L.D. published 25 years ago [3], a 3-volume edition of the same manual published 5 years ago;

2) a three volume by G.M. Fikhtengolts;

3) a two-volume manual by V.A. Zorich (each volume comprises more than 600 pages);

4) one-volume collection of problems by B.P. Demidovich on mat analysis (4460 problems) [2] and a three-volume collection of problems on the same subject by a group of authors headed by Kudryavtsev L.D. (6157 points with approximately 10 thousand problems [3] and a two-volume collection of problems by another group of authors from Lomonosov Moscow State University).

Hence, the linear method cannot be realized because of the week "readiness" of the applicants as well as for the reason of the theory and problems value which is to be studied and solved by the first and second year students.

The circular method presents the study material at one level of strictness and sometimes the same material is given at a more advanced level of strictness, systematization and generalization. This method is usually used in social and natural scientific disciplines. It is done so in teaching mathematics as well. Thus, the function of the true (real) variable on the basis of true numbers axioms is used in mathematics analysis.

The generalization of a true number to a complex number concept takes place in a complex analysis. A complex-valued function from a complex variable concept is introduced, a corresponding differential and an integral calculus is built. A more strong generalization takes place in the functional analysis where functional spaces of different nature are studied. Here the functions themselves are the elements and one space changes into another with the help of operators which generalize the function concept. Secondary school text-books on mathematics contain many theorems, though they are not called as theorems, but a statement is formulated and it is justified either schematically ("on fingers") or it is illustrated geometrically or graphically, or checked in examples. These same theorems are studied in math analysis course but by a more strict proof, i.e. a circular method takes place.

Thus, a linear method of teaching organically combined with a circular one. This is why one can mention proportions in which these approaches should be combined. It is hardly possible to suggest categorical statements. One may speak about the experience of presenting separate themes in math analysis and in reading a special course on "Boundary-Value Problems of Analytical Functions Theories". One of the principal possibilities in solving this contradiction might be seen in the realization of the "circular" teaching the subject matter principal described, for example, in the article by Professor Kirillov A.I. [10]. In this article a circular principle is mostly illustrated by examples from the field of teaching physics. We have tried to realize this principle while reading the lectures in a special course "Boundary-value problems of analytical functions" in the third year at the mathematics faculty of the Chuvash State University. Usually this course is conducted in accordance with a famous monograph by F.D. Gakhov [11] about a linear principle: the integral qualities of Cauchy type are studied thoroughly and on their basis the Riemann boundary-value problem is regarded and singular integral equations are studied. Following the circular principal idea, we presented first the material without proof but with necessary explanations of the mentioned qualities of Cauchy type after which we presented a whole solution of the Riemann problem for the simply connected field with Holder coefficients in boundary conditions. Only after that we came back to the proof of all integral Cauchy type qualities that had been omitted in the initial presentation of the study material. Such an order of studying has given, on the one hand, the possibility already in the first stage of specialization to present the course theses directly connected with the themes of scientific researches at the chair and on the other hand the lecture had a free choice: which theorem should be proved and which ones to be given to the students for self-learning, i.e. organize the students" self-work.

Besides, in this way of presenting the special course a possibility to include the results of research published in scientific journals works, or obtained by the lecturer of the chair, into the study material.

The role of student's self-work:

Let's start with a historical example. Leonard Euler displayed his outstanding mathematical capabilities in the early youth and the destiny presented him with an excellent teacher Iohgann Bernulli. "At that time there were no manuals on higher mathematics while Bernulli had no time to teach Leonard individually. And he found a unique right method which was later highly appreciated Euler himself: he offered the youth to read mathematics memoirs and come home to him on Saturdays to analyze together the points which were not clear.

During several years Euler spent every Saturday afternoon at Bernulli's home. Many years later he remembered that having analyzed with his tutor the question that was not clear he tried to get at clearness in many other problems: several remarks or loading questions of the scientist were enough for the curious student to solve other ones".

Isn't it an ideal scheme for the student's self-work! A curious student and an outstanding scientist and a teacher, reading valuable books on mathematics, the atmosphere of mutual trust between the student and the tutor. Only some remarks and leading questions, not analyzing deeply every detail at consultations and the rest can be achieved by the student himself. What can be alternative? We can't but add: use modern information means.

It's necessary to draw attention to one more modern specific feature. Nowadays it is necessary to consult all the students, both strong and weak ones. Here the character of consultation is changing, the number of consultation hours is different, it is necessary to use different kinds of consultations. Thus, the problem of the lecturer's teaching activities and the character of obligatory consultations to the students, the question of including this type of activities into the main load of all the lecturers is of prime importance.

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PERIODIC AND ALMOST PERIODIC SOLUTIONS OF THE DIFFUSION EQUATION

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Key words: diffusion-heat conductivity equation, wave equation, Cauchy problem, periodic and almost periodic functions, Poisson formula, D'Alembert-Euler formula, Fourier series.

AMS Mathematics Subject Classification: 97I70, 97I30

Abstract. The methodical questions arising by the account of one special section of «Partial differential equations» are discussed. The new methodical approach for the finding of Cauchy problems solutions of diffusion-heat conductivity equation and wave equation where the right parts of these equations and initial values are periodic or almost periodic functions of spatial variables.

National Research Technological University MISiS realises the course «Partial differential equations» for the students of physical and chemical specializations and a specialization 'Applied mathematics'. The course lasts one semester. In this course we are discussing in detail the method for constructing solutions of initial boundary value problems for the diffusion equation and the wave equation in spatially limited areas in the form of decomposition on the basis consisting of own functions of the differential operator. In the present our paper we want to turn the attention of instructors to the cases where the tracing of solutions may be essentially simplified. That are the cases where the right parts of the equations and initial values are periodic or almost periodic functions of spatial variables.

Let's consider Cauchy problem for the one-dimensional diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - a^2 \dfrac{\partial^2 u}{\partial x^2} = f(x,t), \quad -\infty < x < +\infty, \quad t > 0. \\ u(x,o) = 0, \quad -\infty < x < +\infty. \end{cases} \tag{1}$$

As is well known the solution of this problem is given by Poisson formulae

$$u(x,t) = \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{\partial \tau}{\sqrt{t-\tau}} \int_{-\infty}^{+\infty} f(\xi,\tau) e^{\frac{-(x-\xi)^2}{4a^2(t-\tau)}} d\xi. \tag{2}$$

In our lectures course we obtain this formula by means of formal Fourier transformation and show then that by corresponding restrictions to the function $f(x,t)$ formula (2) gives the classical solution of the problem (1).

The calculation of this bulky integral in cases when $f(x,t)$ is periodic or almost periodic function of spatial variables can be successfully replaced by much simpler calculations. On the practical pursuits we turn the attention of our students to the usefulness of preliminary analysis arising situation. We should not use the ready stamps, if we can solve the problem easier. It is very essential for future investigators.

Let $f(x,t)$ – continuous,2π-periodic on a variable function. Let's expand it in a trigonometric Fourier series

$$f(x,t) = \frac{a_0(t)}{2} + \sum_{k=1}^{\infty}(a_k(t)\cos kx + b_k(t)\sin kx).$$

Remark that the solution of a problem (1) is possibly also$2/pi$-periodic on a variable x function and it can be try to found in a form

$$u(x,t) = \frac{A_0(t)}{2} + \sum_{k=1}^{\infty}(A_k(t)\cos kx + B_k(t)\sin kx).$$

Here $A_k(t)$,$B_k(t)$ are unknown functions for definition of which after taking account of differential equation and initial condition the following Cauchy problems for the ordinary differential equations turn out

$$\begin{cases} A_0' = a_0(t), \\ A_0 = a_0(t), \end{cases}$$

$$\begin{cases} A_k' + a^2k^2A_k = a_k(t), \\ A_k(0) = 0, \end{cases}$$

$$\begin{cases} B_k' + a^2k^2B_k = b_k(t), \\ B_k(0) = 0, \end{cases} \qquad k = 1, 2, ...$$

The solutions of these problems can be calculated and written in the integral forms:

$$A_0(t) = \int_0^t a_0(s)ds$$

$$A_k(t) = \int_0^t e^{-a^2k^2(t-s)} a_k(s)ds, \quad B_k(t) = \int_0^t e^{-a^2k^2(t-s)} b_k(s)ds, \quad k = 1, 2, ...$$

Thus

$$u(x,t) = \int_0^t \left\{ \frac{a_0(s)}{2} + \sum_{k=1}^{\infty} e^{-a^2k^2(t-s)} (a_k(s) \cos kx + b_k(s) \sin kx) \right\} ds.$$

Even easier is the solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - a^2 \dfrac{\partial^2 u}{\partial x^2} = f(x,t), \quad -\infty < x < +\infty, \quad t > 0, \\ u(x,o) = \varphi(x), \quad -\infty < x < +\infty, \end{cases} \tag{3}$$

where $/varphi() - 2/pi$ — periodic function. Let's expand it in a trigonometric Fourier series

$$\varphi(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} (c_k \cos kx + d_k \sin kx).$$

We will construct the solution of a problem (3) in the form of a similar series with the unknown coefficients depending from a variable t

$$u(x,t) = \frac{C_0(t)}{2} + \sum_{k=1}^{\infty} (C_k(t) \cos kx + D_k(t) \sin kx).$$

As a result we will receive

$$u(x,t) = \frac{c_0}{2} + \sum_{k=1}^{\infty} e^{-k^2 t} (c_k \cos kx + d_k \sin kx).$$

The following example in our opinion is especially conclusive.

Example 1.

$$\begin{cases} \dfrac{\partial u}{\partial t)} - a^2 \dfrac{\partial^2 u}{\partial x^2} = e^{-2t} \sin x, -\infty < x < +\infty, \quad t > 0, \\ u(x,o) = 0, \quad -\infty < x < +\infty. \end{cases}$$

Application of the Poisson formula (2) leads to integral

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int\limits_0^t \frac{e^{-2\tau} d\tau}{\sqrt{t-\tau}} \int\limits_{-\infty}^{+\infty} \sin \xi\tau) e^{-\frac{(x-\xi)^2}{4(t-\tau)}} d\xi.$$

Calculation of the last integral is rather laborious. Therefore, we will go the other way.

According to the offered method, we will construct the solution of a problem in the form

$$u(x,t) = B(t) \sin x.$$

For definition $B(t)$ we have a problem

$$\begin{cases} B' + B = e^{-2t}, \quad t > 0, \\ B_0 = 0, \end{cases}$$

with solution

$$B(t) = e^{-t} - e^{-2t}.$$

Hence

$$u(x,t) = (e^{-t} - e^{-2t}) \sin x$$

Now we consider the situations where the right-hand parts and initial values are almost periodic with respect to spatial variable. Restrict us by finite number of terms.

Let now in a problem (1)

$$f(x,t) = \sum_{k=1}^{n} \cos(\alpha_k x + \beta_k) e_k(t), \quad \alpha_k > 0 \quad (k = 1, ..., n).$$

If the numbers $\alpha 1, \ldots, \alpha_n$, are rationally commensurable, then f represents periodic function, as above. If $\alpha 1, \ldots, \alpha_n$ are rationally incommensurable, then f is almost periodic function with an almost period $2l$. In this case, the solution of the problem is also almost periodic function with the same almost period as f.

In fact try to find the problem (1) solution in the form

$$u(x,t) = \sum_{k=1}^{n} u_k(t) \cos(\alpha_k x + \beta_k). \tag{4}$$

where the functions-coefficients must be define. After substitution (4) in (1) and equating the coefficients we obtain the system of initial problems

$$\begin{cases} u_k' + (a\alpha_k)^2 u_k = e_k(t), \\ u_k(0) = 0, \quad k = 1, 2, ..., n. \end{cases}$$

The solutions of this problems may be rewritten by formula

$$u_k(t) = \int_0^t e^{-(a\alpha_k)^2(t-s)} e_k(s) ds, \quad k = 1, 2, ..., n.$$

Thus the solution of the problem (1) has now the expression

$$u(x,t) = \sum_{k=1}^{n} (\int_0^t e^{-(a\alpha_k)^2(t-s)} e_k(s) ds) \cos(\alpha_k x + \beta_k).$$

It is almost periodic function with respect to x, if the numbers $\alpha 1, \ldots, \alpha_n$ are rationally incommensurable.

The same is truly for the problem with trivial right part and initial function

$$\varphi(x) = \sum_{k=1}^{n} A_k \cos(\alpha_k x + \beta_k), \quad \alpha_k > 0 \quad (k = 1, ..., n).$$

Demonstrate it by the simplest example.

Example 2.

$$\begin{cases} \dfrac{\partial u}{\partial t} - a^2 \dfrac{\partial^2 u}{\partial x^2} = f(x,t), \quad -\infty < x < +\infty, \quad t > 0, \\ u(x,o) = \cos(\sqrt{2}x) + 5\sin(\pi x), \quad -\infty < x < +\infty. \end{cases}$$

We will construct the solution of a problem in the form

$$u(x,t) = u_1(t)\cos(\sqrt{2}x) + u_2(t)\sin(\pi x).$$

For definition of unknown functions, we have the Cauchy problems:

$$\begin{cases} u_1^{'} + 2a^2 u_1 = 0), \\ u_1(0) = 1, \quad t > 0. \end{cases}$$

$$\begin{cases} u_2' + (\pi a)^2 u_2 = 0, \\ u_2(0) = 5, \quad t > 0. \end{cases}$$

Their solutions are

$$u_1(t) = e^{-2a^2 t}, \quad u_2(t) = 5e^{-(\pi a)^2 t}.$$

Consequently

$$u(x,t) = e^{-2a^2 t} \cos(\sqrt{2}x) + e^{-(\pi a)^2 t} \sin(\pi x).$$

Similarly, we can consider the Cauchy problem for the wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = f(x,t), \quad -\infty < x < +\infty, \quad t > 0 \\ u(x,o) = \dfrac{\partial u(x,o)}{\partial t} = 0, \quad -\infty < x < +\infty. \end{cases} \tag{5}$$

As is well known the solution of problem (5) is given by D'Alembert-Euler formula.

D'Alembert-Euler formula leads to the following integral expression:

$$u(x,t) = \frac{1}{2a} \int_0^1 d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau) d\xi.$$

The calculation of such tipe of integrals is not so vast as for integral of (2) tipe. But our method permits to solve our problem in the case of a special right-hand side of the equation essentially fast.

Example 3.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = a(t)\cos(\alpha x + \beta), \alpha > 0, \quad -\infty < x < +\infty, \quad t > 0 \\ u(x,o) = \dfrac{\partial u(x,o)}{\partial t} = 0, \quad -\infty < x < +\infty. \end{cases}$$

We construct the solution of a problem in the form

$$u(x,t) = A(t)\cos(\alpha x + \beta).$$

For definition $A(t)$ we have a problem

$$\begin{cases} A'' + \alpha^2 A = a(t), \quad t > 0, \\ A(0) = A^{'}(0) = 0 \end{cases}$$

with the solution which can be written down as

$$A(t) = \frac{1}{\alpha} \int\limits_0^t a(s) \cdot \sin(\alpha(t-s)) ds.$$

Remark. Aforesaid is transferred on the case when in a problem (1) $x \in R^l$, $l \geqslant 2$, and f — periodic (almost periodic) function on each of spatial variables $1, 2, \cdots, k.$

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THE KNOWLEDGE AND QUALITY MONITORING SYSTEMS OF MATHEMATICAL EDUCATION

A. Novikov

Key words: educational process, examination forms and contents, knowledge quality, inverse effect of examination, entrance tests, the Unified State Examination system lacks

AMS Mathematics Subject Classification: 97D80, 97D10

Abstract. There is studied the influence of examination form and contents on the quality of educational process in mathematics at secondary school and in higher school, it is underlined that intermediate (terminal) examinations are an important stage of educational process, and test forms of examination contradict the idea of knowledge ordering during an examination.

A great number of publications are devoted to the research of objectivity of various forms of intermediate and final knowledge testing pupils and students. Thus there has practically always been neglected an important question: ***in what way does any knowledge testing system influence educational process quality*** *and, as a consequence, secondary school leavers or higher school* ***graduates quality of education****?*

Undoubtedly, an examination form and the contents of examination tasks make a great huge impact on the quality of educational process. The Feedback of examination with educational process can be both positive, contributing improvement of educational process quality, and negative having profound negative consequences for an education system. The article analyzes the experience of the national system of secondary and higher school mathematical education under the given point of view.

Examination forms in the system of secondary and higher school mathematical education may be divided into two groups. The first includes control tests which though show the results of training for some period, but thus period is intermediate. They are terminal examinations in higher school, final examinations in 4th and 9th forms of secondary school. The second group comprises control tests the ultimate goal of which is streaming of examinees according to their points. The further destiny of a person depends on the results of examinations in the given group as they define his or her launching positions in their future occupation. Those are entrance and final examinations in higher school, entrance examinations for master's degree programmers and postgraduate study. Though, the last two examinations, probably, have an exceptional position and form another, third group.

Let's discuss at first the influence of examination forms on educational process and then the influence of examination tasks on it contents.

The first group examinations are an important ***stage of educational process***. The primary goal of an examination in this group consists in ordering of the knowledge received by pupils or students in a given subject during or a certain period of time. To achieve the highest positive result pupils and students' knowledge testing here should be done in oral or in written-oral forms. This condition is especially important to follow for terminal examinations in higher school. The examiner in this case has an opportunity to make a better acquaintance with every student of the stream, to estimate the students' level of knowledge of all terminal material and each theme apart, to analyze the committed mistakes, and also to have or opportunity to see methodical errors of a lecture course and seminar classes. Such conversations of the teacher and the student make a great stimulating impact on creative, gifted persons and have a powerful educational influence.

The written form of examination in this examinations category has but a weak positive influence on educational process. Such an examination form transforms it from an important stage of educational process into a passionless control vehicle of results. In this case a major link of full-fledged educational process – a professional and simply human dialogue of the teacher with the pupil drops out. "Unfortunately, we lose culture of mathematical speech (the cancellation of oral examinations in algebra and geometry at school, the Unified State Examination, various tests in higher schools). The competent expert must be able to argue on the object of research" [1].

Studying mathematics is impossible without regular, thoughtful and hard work of the pupil (student) with the textbook, without the independent problems solution. However, actively introduced both in secondary and higher school test technologies of training and knowledge testing form in the trainees' consciousness a simplified view about the ways of no genesis. Instead of deep, system-level knowledge of theoretical material they have fragmentary notions, instead of the ability to solve challenges and to receive thus esthetic pleasure from beautiful solutions they have multiple solutions of basically simple, problems of the same type. The test form of knowledge check by definition imposes considerable restrictions on the contents of questions and the result form. It essentially reduces the creative potential of test materials.

Despite the above – mentioned drawbacks, written and test forms of carrying out examinations in mathematics have got recently a considerable popularity. It concerns both secondary, and higher school. We will emphasize, the given statement concerns not only and even not so much the innovations introduced from above, but the transformations occurring directly in educational institutions. In is necessary to look for the explanation of this in phenomenon in those processes which occur in the education sphere. The transformation of the education sphere into the sphere of services, the decrease in prestige of teaching work and the mass character of higher education have led to the situation when of two forms of carrying out an examination (oral, on the one hand, and written or test – on an the other) even more often the preference is given to the second form demanding less time and fewer emotions.

Unfortunately, also at the state level a unreasonably great part is assigned to the development and introduction of test technologies in to, the educational process. At the third international conference "Functional spaces. Differential operators. The general topology. Problems of mathematical education", taking place in Moscow, the representatives of the National accreditation agency in the education sphere from Ioshkar-Ola informed that more than 600 higher schools in Russia use the Internet-examination not only for State Final Examination, but also for terminal examinations. However, in each semester in higher school they study one or two concrete courses of the discipline "Mathematics" and consequently terminal examinations are to help students to systematize their knowledge in these courses. The Internet-examination contains tasks on all sections of the discipline "Mathematics". It is intended for "... evaluating the students level of knowledge and the ***basic training of*** **educational institutions'** ***graduates*** in accordance with requirements of the State Educational Standards (SES)" [2], i.e. the Internet-examination pursues the aims of testing **permanent** ***knowledge*** of senior students who finished studying the discipline "Mathematics".

The experience of carrying out the Internet-examination as a terminal examination has shown that it:

— does not give an objective elevation of students' knowledge;
— sharply reduces the motivation of large groups of students to serious, hard work as even "poor" students after getting to know the so-called zero variant, type at Internet- examination at least than 80 points;
— breaks the integrity of educational process, as incomplete there is the process of studying some disciplines important for applications (in engineering universities) - complex variable theory and probability theory.

Undoubtedly **the system of two examinations in mathematics – written and oral was optimal** both at the objectivity degree, and according to the stimulating back influence on the educational process. The Written final examination for the course of secondary school and entrance exam in higher school allowed to test practical skills of solving problems and doing sums, and together with the oral examination also the mathematical culture of the examinee. At an oral examination there was checked not only the volume and depth of theoretical knowledge, but also the applicants' ability to make independent logic conclusions that allowed to give an objective evaluation of creative and informative potential of the applicants'.

The abolition of the oral examination has led to such a situation that the greatest part of present-dag schools-leavers and students of higher schools have a poor knowledge properties of elementary functions; they cannot accurately formulate statements, and having formulated them, do not understand what is given in the theorem statement and what it is required to prove.

The general **negative** property of the new forms of knowledge check, and especially test forms, consists in the fact that they form students' simplified view of success and make the latter to cram for the evolution of elementary problems. In such conditions it is difficult to train a highly skilled expert. The formal attitude to the learning in mathematics, deprived of an emotional and esthetic component, will allow to prepare highly skilled experts neither in natural-science fields of knowledge, nor in technical disciplines.

Let's discuss now **the influence of examination tasks contents on educational process**. We will have air discussion using the current experience of final examinations at secondary school and entrance examinations to the higher school. We will analyze the contents tasks of the centralized testing and the Unified State Examination (USE).

A positive quality of the centralized testing was the thing that the examination "Mathematics – I" contained tasks on 22 themes which on the whole covered all basic sections of the program in mathematics for the secondary school course. It obliged school to teach all the sections of the school course at an adequate level. The tasks of the centralized testing, in particular, included *the fundamentals of vector algebra and its applications, applications with arc- trigonometric functions, the method of plane and in space coordinates.* The USE does not have tasks an these themes. The school has quickly reacted to this signal: the volume and depth of study of the named themes have essentially decreased. For higher schools, and especially technical, the most painful is studying trigonometry at an insufficient level (once again!).

The level of tasks in mathematics in the centralized testing allowed on the whole to check the applicant's degree of readiness for his or her studying at higher school *with the standard requirements to the knowledge of mathematics.* Another matter is that *the possibility to guess answers and the deformed system of the estimation of test examination results, deprived its results of objectivity.* For 4 done problems from 30 ones it was possible to receive up to 50 points, but for one unsolved problem to lose more than 10 points. The deformed though at a less degree, the system of estimation of the examination results has also remained within the frame of the USE: 12 initial points for group B elementary problems at recalculation in 100 – point scale turn in to 50 points, and 18 points for group C problems corresponds less than 50 points. It is difficult to name such a "thermometer" an ideal measuring instrument of higher school graduates training quality in mathematics.

The basic, system drawback both of the centralized testing, and the USE consists ***in substitution of systematic, consecutive studying of a course of mathematics in 11 form*** of secondary school by cramming pupils with the solution of accurately outlined groups of problems. Within a year, white getting ready for the USE 4 times in the whole co problems country there are held trial examinations on control-measuring materials (CMM), the structure and contents of while completely correspond to the CMM of a real examination. The deviation from the announced before examinations the list and contents of problems at once leads to negative results.

Another drawback of the USE *has clearly enough revealed during the last 2-3 years.* Its essence is that many teachers of mathematics, not being able to solve problems of group C, train their pupils only on solving elementary problems of group B. No wonder that up to 50% of the participants of the uniform graduation examination in mathematics leave the examination room in an hour after the examination beginning.

Some remarks on the structure of the USE task. The refusal from group A problems with the choice of an answer variant is correct step. However, the structure of the USE tasks now does not have any problems of the average level of complexity. In group B it is enough to have 5-6 elementary problems, allowing to get a score, sufficient for overcoming the minimum threshold. Other problems (8-10 problems) should have the complexity level, allowing to check up the pupil's readiness for training in the higher school with the standard requirements to their knowledge in mathematics. In particular, it is possible to include in this group some standard problems from the theory of numbers and some elementary problems with inverse functions. For example, the problem on finding a set of values of the natural argument, where the fractional-rational function is an integer, allows to check up both the level of theoretical training, and the degree if mastering practical skills simultaneously on several themes (the decomposition of the fractional-rational function to the sum of the integer part and a proper fraction, the criteria for divisibility the, concept of prime and composite number). At such a change of the structure of tasks in group C it is enough to have 4 problems of the higher level of complexity.

It is necessary to recognize that the hopes set on the USE that with its help it will be possible to check up not only the basic level of applicants' training but also to provide an adequate evaluation of creative and cognitive potential of gifted children, have not equaled. It is eloquently justified by the results of the additional examination in mathematics on mechanics-mathematical and other departments of the Moscow State University.

According to the data resulted in the official analytical report on USE-2011, 735 450 persons took part in the examination in mathematics [3]. According to the results of work all participants are divided by the authors of the report by their training level into 4 groups: I – low, with the test points from 0 to 30; II – basis, with a test points from 34 to 56; III – higher, with the test points from 60 to 82; IV – high, with the test points more than 84. Group I included 15,6% participants of the USE, group II - 57,9%, group III – 25,3%, group IV – 1,2% [3]. The authors of the report actually recognize that the training level of the majority of the school leavers is insufficient for training in technical colleges. In their opinion, the representatives of group IV (8825 applicants) – "are a contingent of physical and mathematical specialties of the leading classical universities and technical colleges, and also prestigious economic higher schools", groups III (186069 persons) are "... basically technical college applicants". At once the authors of the report notice that this number of applicant from the III-d group is not enough to meet the requirements of technical colleges and consequently in them, some applicants from the second group were enlisted there and also to the speciality' teacher of mathematics'. We will underline: really, they were enlisted and in a considerable number.

One of the best groups of the first-year students at Ryazan state radio-engineering university (the specialty "Radio engineering ", 2011) 72,7% of its make-up have got at the USE in mathematics from 30 to 56 points and only 27,3% - from 56 to 66 points. We will notice that 50% of the students in this group have in mathematics from 30 to 49 points. The authors of the report fairly consider that the applicants from group II "... have no sufficient training for their successful further education at specialities demanding a higher and high level of mathematical competence" [3]. In practice, the representatives of exactly this group, rather than group III as the authors of the report consider, make up the greatest part of students at some technical specialities in provincial higher schools though, probably not only in them. It is necessary to note that it occurs contrary to a great demand for experts of a corresponding profile on a labor market.

With such audience it is difficult, and more likely impossible at all, to provide a high level of mathematical training. It', a pity that in this case *there's no possibility to receive a quality education* for that part of the student's audience the representatives of which have a good mathematical training and a high level of motivation.

As a matter of fact, there's broken the *constitutional right* of this part of students for getting is high-quality education.

The mistrust of the public and experts to the USE results has generated a new form of examination of yesterday evening school leavers. It is so-called ***the entrance control of permanent knowledge*** in a given subject. Such a control is done annually, at the Ryazan state radio-engineering university as well. A short analysis of the entrance control results in 2009/2010 academic year is given below. The task consisted of 17 problems broken into three groups according to the level of complexity. The tasks were taken mainly from the problems of the centralized testing and the USE tasks of the previous years. The task doing was given 90 minutes. Each of 10 tasks in the first part was given 1 point, each of 6 tasks in the second part –2 points, the last – 17th problem with the parameter in the third part –3 points. Total 25 points.

575 first-year students of three technical departments of the university took part in the test. The main results of the entrance control are as follows:

— **there is no problem which was solved by all the students**; the maximum result is received by the students while solving an elementary exponential equation: it was solved by 3/4 students from the total number;
— **only 7 problems from 15** were solved by more than half of the students from the sample (finding the greatest common devisor of three numbers (63%). The calculation of numerical expression with degrees (62%), the simplification of the algebraic expression (59%), the elementary in equation with fractional-rational function (61%), the calculation of value of one trigonometrically function on the value of another (65%), the exponential equation (73%), finding set of values of the simplest elementary function (58%); in brackets there is the percent of students done a corresponding problem);
— **unsatisfactory results** are received in the problems of average complexity **in trigonometry** (the calculation of the value of trigonometrically expression with the use of trigonometry formulas (28%), the solution of the elementary trigonometrically equation with the selection of roots (20%)), in the problem on the properties of roots of the quadratic equation (20%), derivative application (22%). There are even worse results with the text problem (10%) and with a simple problem on geometry (4%);
— **low results** of the solution of problems in trigonometry, the ignorance of properties of the derivative and inability to apply them in researches, poor knowledge of the properties of quadratic function (formula VIETA, the discrimination of the perfect square from the quadratic trinomial) **cause a special anxiety** as they concern mathematics sections, most demanded in a mathematics course in technical college.

More detailed results of the research of permanent knowledge of the first-year students are given in [4].

The Search of the education purposes adequate to the occurring changes in the country and in the world, in present-day Russia has undoubtedly run over time [5]. The Dim character of these purposes, and especially the criteria of quality in education, supports a long-term crisis of the education system. The system of examinations in mathematical education should carry out simultaneously both an important training function, and the function of the adequate quality assurance of education results.

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INTERDISCIPLINARY COMMUNICATION IN TEACHING PARTIAL DIFFERENTIAL EQUATIONS

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Key words: interdisciplinary communication, professionally-oriented tasks, mathematical modeling

AMS Mathematics Subject Classification: 97I70

Abstract. This article discusses the need to identify the relationship of learning content of differential equations, partial differential equations (PDE) and the disciplines of the professional cycle, which is the basis for the consideration of mathematical models of heat and power processes in the course of PDE and ensuring the implementation of interdisciplinary connections of PDE with the professional disciplines by means of solutions professional-oriented tasks

1 Introduction

Interdisciplinary communication is an important condition and result of a comprehensive approach to training and educating students of the heat and power engineering, helping to form a integrated system of mathematical methods and their application in specific disciplines in solving professional problems. The most efficient show the interdisciplinary communication and natural science research method used at the intersection of science allows the integrated education.

Integration of training implies a transfer from the coordination of teaching different subjects to a deep interaction between them and above all a significant development and deepening of interdisciplinary relations, which are analogous to interscientific communications [1]. With interdisciplinary relationships not only at a qualitatively new level to solve the problem of training, development and education of students, but also lay the foundation for an integrated vision, approach and solve complex problems of reality [2].

It is not so much the authors of methodological literature trying to identify ways to realization interdisciplinary connections, due to the lack of uniform theoretical basis, which is manifested at the level of the definition of interdisciplinary connections (as a principle of learning, didactic condition, part of the principle of systematicity, one of the necessary conditions for successful learning)[3].

Thus, it is necessary to identify opportunities of realization interdisciplinary connections of PDE and professional disciplines means of modeling heat and power processes. By considering the notion of a mathematical model of a rod connecting the PDE with other academic disciplines and practices.

Chapter contents PDE	The content of professional disciplines cycle
Partial differential equations of first order; linear wave equation of first order; nonlinear wave equation of first order	Process of propagation of linear and nonlinear waves in one-dimensional case; linear one-dimensional convection equation, linear equation Burgers; inviscid Burgers equation
System of partial differential equations of first order	Dimensional supersonic stream of inviscid liquid;nonlinear convection equation
Decision problems for hyperbolic equations: a) for the wave equation; b) for the hyperbolic heat conduction equation	Propagation of sound waves in a homogeneous environment; distribution of heat with powerful pulse thermal influence
Mixed problems for the heat conductivity equation: a) with the boundary conditions in the form of Dirichlet, Neumann, Robin; b) in cylindrical and spherical domains; c) with nonlinear boundary conditions	Determination of nonstationary temperature field in a solid: a) the temperature field during the cooling process (heating) of the plate; b) the temperature field during cooling process (heating) of an infinitely long cylinder, sphere, and some bodies of finite dimensions; c) cooling plate with allowance for radiation
Boundary problems for elliptic equations: a) Dirichlet and Neumann problems for Laplace's equation; b) the Dirichlet and Neumann problems for the Poisson equation	Determination of of a stationary temperature field: a) two-dimensional thermal field and heat flow in a flat edge, the calculation of subsonic nonrational flow of gas; b) the temperature field of a plane with nonuniform supply and removal of heat at the boundary
System of partial differential equations of mixed types	Stationary and not the stationary Navier-Stokes equations for compressible and incompressible fluid
Integro-differential equations	The integral equation for heat flow the thermal boundary layer; integral equation of impulse for the thermal boundary layer

Established by us the relationship content learning Partial Differential Equations (PDE) and disciplines of the professional cycle (heat and mass transfer, fluid dynamics, thermodynamics) provides an opportunity consideration of mathematical models of heat and power processes to solve professionally oriented tasks within the framework of education the PDE; it can be an important didactic condition further improve the efficiency of cognitive activity of students the heat - and - power engineering.

We offer as an example the problem of aligning of rapidly changing temperature field, a mathematical model which is a mixed problem for hyperbolic equation of heat conduction.

Problem. Let the initial time is set to a sinusoidal temperature distribution along the thickness of the plate, with a maximum in the middle. We shall assume that the initial state was maintained for some time, the initial temperature change speed over time is zero. The temperature at the surface is supported at a constant zero level. Is required, using numerical methods for solving, to calculate - on the hyperbolic model - the process of alignment of the temperature field. The coefficient of thermal diffusivity $0,361m^2/s$ relaxation time $10^{-3}s$ thickness of the plate $0,06m$

Professionally-oriented tasks are at the crossroads of different character knowledge systems and means of activity. The solution of such problems allows students to move to the level of interdisciplinary synthesis, provides them with a holistic understanding, that allows to activate the motivation to study and implement the development of professionally significant qualities of the individual.

The integration of training content PDE and disciplines of professional cycle promotes to:

- versatile disclosing of the maintenance educational disciplines in the relationship and interdependence, which helps more coherent and systematic mastering of educational information;

- development of cognitive activity of students in the process of learning new knowledge in classes and independent work by means of attraction knowledge from other subjects;

- to formation at students of abilities to use knowledge of various disciplines in mastering of new knowledge and in practical activities more operatively.

Thus, an integrative approach for teaching PDE can be represented as a complete system applied mathematical methods and their application in specific disciplines that promotes the development of skills of mathematical modeling and the formation of professional competencies of future professionals of heat and power direction..

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THE MODELLING IN THE PROCESS OF INTENSIVE HIGHER MATHEMATICS TEACHING IN MODERN UNIVERSITIES

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Key words: intensive teaching, differential teaching, modeling information pedagogical environment

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Abstract. In the paper some aspects of modern pedagogical problems of the modeling in the process of intensive higher mathematics teaching in technical, classical and pedagogical universities are discussed. The synthetic approach to the modeling of the training process which consists of application of certain mathematical theories is discussed. This approach leads to the formation of new mathematical tools which can be applied to pedagogical environment. It is represented to us that the concepts of topological model and topological relational system can be quite successfully used. The state graph, received at such approach, is exact enough analogue of a pedagogical situation.

The analysis of the modern situation of social development leads to the conclusion that the scientific technical progress and the increasing flow of information in all areas of human activity are crucial components in the social and economic life of society and significantly increase the contradictions between traditional and innovative aspects of social development.

The end of the pervious and the beginning of our century shows that modern society is in the way of a fast-development, which is not only a powerful acceleration in the production and spiritual spheres, but also a great dynamism of changes in social relations. The change in social relations entails the need for rapid positive action in the management of education.

Increase of the role of quality of knowledge, the need for and the importance of rapid learning (including, in new specialties) and retraining of specialists in the various fields of activity, and therefore, the rapid development of the information educational environment establish the relevance of the development of new models of intensive training in different fields, and first of all in the system of higher and secondary education. Improving the quality of knowledge and authority of graduates, the overcoming of technologies in education - challenges also in the modern higher school. In addition, the discrepancy between the increasing flow of new information, which has great importance for the future successful professional activity of students, and the need for exploration of the deep fundamental knowledge, without which in the current difficult to navigate, may be permitted only intense, methodically grounded methods. The reform of education, which takes place in recent years in our country, led to the formation of a new educational paradigm. To achieve real success of the necessary reforms, in virtue of the peculiarities of culture of the modern society and the rapid growth of the information flow, we can only teach the students of the higher educational institutions of intensive methods of development and processing of information. Reform of the system of teaching mathematics at the high school first of all requires a clear definition of objectives, tactics and strategy of the training subjects of mathematical cycles. It should be noted that the transition to a multi-level system of training (bachelor - master - doctorate) already practically in all higher educational institutions of Russia makes topical and important, not only the definition of goals, objectives, content and volume of mathematical courses for each of these levels, but also methods, methodologies and models training. This is especially important in the training of students of mathematics at the specialties for which this discipline is the main subject. Therefore, the problem of intensification of training of students of mathematical disciplines and its special tasks, resulting profile specialty and training system, are of great importance and relevance [7].

Principle of reasonable severity allows you to establish the criteria of expediency of requirements to knowledge of students in the diversification of their training in intensive methods. Methodically competent application of this principle can contribute to and humanization in teaching mathematics, that is the decision of one of the most important tasks of modern educational paradigm. According to the principle of reasonable severity of presentation of educational mathematical material in higher educational institutions it is important to adapt the presentation to the real level of pre-university preparation of students in mathematics, so that it was possible to develop a well-trained students, and, at the same time, help the alignment of knowledge of those of them who do not have enough high-quality basic training. And, thus, contribute to the development of as weak, so strong and students.

Academician of the European Academy of sciences, corresponding member of RAS, professor Lev Kudryavtsev in his article "On the mathematics" noted that "*in the teaching of mathematics should pay special attention to the development of the students a clear logical thinking, what for it is necessary that the presentation of mathematics was strictly logical, clear, understandable and as brief as possible*" [4, 22].

Therefore, according to the principle of rational severity, the presentation of the educational material as at any stage of the training and in the training on specialties with different relevance of mathematical training, should be clear and quite strict, but can also have a different depth analysis of the mathematical concepts, which depends on the university course, the nature of the contingent and the level of preparation of students.

So, for example, with the current academic year at the Moscow physical technical institute (the state university) at the department of higher mathematics students were offered the curricula of all learning math courses of two levels - base and so-called "advanced". Moreover, although the level examination program it is recommended that students individually teachers, leading seminars, the student has the right to make a choice of the level of the exam, provided that he was recommended for the examination in "advanced" level. Note that the transition to the two and even more multi-layered system of education requires serious research and literate approach to solving the problem. So the experience of the first session of the "multilevel education" in the Moscow physic-technical institute led to the fact that the vast majority of students, even the strong, select the level of basic, a program which was considerably shorter and information weaker, because, according to this program it has been possible to get a rating of "excellent". Thus, methodically poorly reasoned and well thought-experiment, without serious incentives and priorities, giving students the opportunity to more easily obtain high scores, can lead to an apparent reduction in the quality of mathematical knowledge. And it happened when training on the specialties of the Moscow physical technique institute, where the role of mathematical knowledge has always been very important and in demand. In the experiment, showing clearly negative result for the quality of knowledge, was violated one of the fundamental principles of intensification of teaching mathematics, which was grounded in the study . It is clear that the differentiation of education should encourage the student to deepen the level of mastering the educational material, and not to provoke it in a more simple way to get the satisfying of his assessment.

The above example shows how important methodologically sound approach to innovations in education, as well as the role of modeling educational process of teaching (in particular, higher mathematics), including intensive training in universities today. However, the principles proposed as a basis for constructing a model of learning a particular mathematical discipline, should be general enough and have the opportunity to be common for training and other mathematical and non mathematical disciplines.

According to the developed by the authors of the principles of intensive learning of mathematics disciplines, simulation of a pedagogical process should meet the following requirements:

1. *the teaching of mathematics should be parallel-multi-level of complexity and depth study of the training material and are characterized by the expediency of the requirements of level of preparation of students;*
2. *the teaching of mathematics should be based on "the principle of reasonable rigor" in a summary of its basic provisions;*
3. *the training of mathematics must be built on a differentiated basis, and differentiation of education should encourage the student to deepen the level of mastering the educational material;*
4. *the training of mathematics should be problem-developing nature;*
5. *the teaching mathematics should stimulate and strengthen an independent cognitive activity and to educate the students' abilities, skills and inclination to continuous self-education, self-development, analysis and selection of new useful information;*
6. *the teaching mathematics should make fuller use of psychological and pedagogical approaches to teaching;*
7. *the education of mathematics should be characterized by scientism, which is being implemented through the content and the logic of building a school mathematics course and be based on the fundamental teaching knowledge;*
8. *axiomatic, inductive and deductive methods and principles of building courses should be widely used in the teaching of mathematics;*
9. *the training of mathematics must develop the mathematical intuition students, it is advisable to use in appropriate situations heuristic techniques;*
10. *the control of the typical knowledge of students it is expedient to carry out the systematic use of interactive tools;*
11. *during training in mathematics it is important to proceed from the interpretation of it not only as an academic and scientific discipline, but also as an element of human culture* [8].

Naturally suggesting that intensive methods of teaching a separate mathematical discipline will be much more effective if they are applied in the study of the totality of the university of educational subjects, reasonableness and consistency of the training programmers of various disciplines and integrated intensification of the organization of the entire process of training of students in the higher educational institution.

In the concept of an intensification of teaching mathematics in higher school outlines the main directions of the didactic impacts, among them are parallel-level, psychological-pedagogical, specially substantive, software and controls and humanitarian.

Note that modern information and pedagogical environment, due to the bilateral process of learning and cognition (in a simplified scheme of teacher - student), especially when developing disciplines mathematical cycle in higher educational institutions, is a complex, hierarchical system. This phenomenon is expressed variety of objects, their different properties and multi relations between them, as well as the necessity of taking into account a large number of factors of the learning process for the identification, analysis and synthesis of all possible options, significantly influencing the decision making process, as students (each student), and also from the educational structure of the department, the faculty, the teaching staff. At the present time, there are a number of approaches to the development of methods of formalization and simulation, applied in pedagogy. It is "hard" and "soft" model of V.I. Arnold [1], generalization methods in teaching of prof. V.V. Davydov [2], the analysis of multi-level differentiated learning mathematical disciplines prof. V.T. Petrova [8], the differentiation of the information educational environment in dissertation research of V.K. Zharov [3].

Effective means of achieving the intensification of teaching mathematical disciplines in the modern school, in particular higher, can serve as a correct choice of didactic bases and schemes for the creation and effective use of educational-methodical complexes which would include (in order to optimize the impact on the quality of education and relevant professional activities of a future specialist's, bachelor's degree, master's degree) methodically grounded

- *The selection of the material for lectures and practical sessions;*
- *The application of interdisciplinary relations in the training courses;*
- *Professional orientation of educational material;*
- *Preparation of the competent educational-methodological manuals;*
- *Differentiated multi-level tasks for the students;*
- *Handout (on electronic and paper carriers);*
- *Control and measuring materials;*
- *Technology of tests, examinations, systematic testing.*

Note that a certain intensification of the educational process, in particular, can be achieved by application of the provisions of fusionism and use of interdisciplinary relations. For example, in the “Geometry 1”, “the Geometry 2”, “the Geometry 3” in the Moscow state regional university in the preparation of teachers, as well as bachelors in the specialties of mathematics, physics and informatics is the conceptual construction requirements for modeling of the teaching methods. The issues of higher geometry in two- and three-dimensional spaces set out in parallel, there is a systematic comparative analysis of the Euclidean and affine geometry, affine and projective geometry, projective and non Euclidean geometries. And in the course of differential geometry, theory of curves in the plane and three-dimensional space logically summarized in the theory of smooth surfaces in three-dimensional space.

Further studied by the students of the material is combined in the optional course “Manifolds of affine connection”.

Such an approach is not only important for future teachers, but may be useful for the future of physicists-theorists, since the geometric ideas are used, first of all, in theoretical mechanics, the special and general theories of relativity, in mathematical modeling of various physical theories. An example of realization of ideas of intensive teaching mathematical disciplines through the use of interdisciplinary communication and vocational guidance could serve as a training course in the Moscow state regional university "Computational geometry" - subject, relatively recently emerged at the junction of geometry and computer science. This course allows you to orientate bachelors - information on parenting skills of competent construction of mathematical models of various natural and social phenomena.

The synthetic approach to modeling the training process, consists of an application various powerful mathematical theories, and leads to the formation of new mathematical tools which can be applied to the teaching practice. As it is represented to us, the notion of topological models and topological relation systems may be successfully used for the modeling of intensive training the mathematical disciplines at the higher school. It appears that semantic bases (linguistic thesauruses) and hierarchical process of the representation of basic objects by sets of the basic signs with the internal associative and non- associative communications have a good enough description in topology language. Thus, the indication space is allocated with the topological structure and the set of relations. The received status graph quite precisely can characterize a pedagogical situation.

For an example we will consider a sketch of a skeleton of the language training model. Let $\boldsymbol{D}=\boldsymbol{Q}\ \boldsymbol{P}\ \boldsymbol{S}$, where $\boldsymbol{Q}$ is a set of terms of the thesaurus, $\boldsymbol{P}$ is a set of the characteristics on the indication space, $\boldsymbol{S}$ is a set of indistinct operations and the relations, describing the given object in the language of science in the certain area of knowledge (a training subject). The thesaurus is a multilevel system. By means of unary operation of joining a word of one level is translated in the following level. Many-placed relations characterize dictionary nests which carry out associative and not associative communications between various levels of the thesaurus. The indication space is allocated with the topology caused by semantic laws of the given thesaurus, focused on a certain part of the subject domain. Construction of similar models of thesauruses of the given class can be directed on gradual expansion of a lexicon of students, on mastering them the key concepts of studied mathematical discipline. So the students have the deeper understanding of the mathematical facts and the theorems, which proof shouldn’t be the jagged chain of some logic tautology.

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MODERN DIRECTIONS OF INFORMATION OF MATHEMATICAL FORMATION

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Key words: mathematical formation, information, computer tutorials

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Abstract. In article the basic directions of modernization of modern mathematical formation are considered. One of the most perspective is introduction of modern information and communication technologies in practice of training to the mathematician.

Modern education modernization is directed on a priority of the human person which development should become the main value and the major result of formation. These new reference points of an education system are shown in various directions of its development: in construction of system of the continuous formation, the lichnostno-focused training, occurrence of forms of alternative training, working out of new approaches to formation of the maintenance of formation, creation of the new information-educational environment etc.

Traditional process of training has a number of classical contradictions which can be formulated as follows:

— Activity of the teacher and passivity of the trained,
— The curriculum is calculated on an average trained,
— An individual approach lack to the person of the trained,
— The information is presented in the is abstract-logic form;
— Limitation in time, etc.

According to a number of researchers in structure of modern educational process one of leading components become the tutorials focused on an intensification of teaching and educational process, increase of its efficiency and quality, preparation trained to work and life in the conditions of the information society, capable appreciably to reduce terms of training and to raise its quality.

In a pedagogical science the concept of "tutorial" has till now no unequivocal interpretation. Many researchers use the various definitions at times contradicting each other. While the importance of tutorials in educational process is marked by many scientists, a number of researches in the field of effective application of tutorials is spent.

The problem of use of tutorials in educational process is considered by many authors. Various classifications of tutorials (V.V.Voronov, V.Okon, Pidkasistyj P. I, Smrnov S.D., Studenikin M. S, etc.) are developed.

At various stages of teaching and educational process such scientists as Arhangelskij S.I., Shahmaev N.M., Belkin E.L., Kyverjalga A.A., Kuzmin N.V., Tihonov I.I., Shapovalenko, etc. showed Of this year Efficiency of application of separate kinds of tutorials Boltjansky Century G., Chashko L.V., Serdjuk A.V., Gelmont A.I., Libenson E.V., Hozjainov G. I were engaged in a substantiation of necessity of application of means of training, etc.

Development of modern information and communication technologies, was naturally reflected and in an education system. Introduction in educational process of means of the information and communication technologies providing conditions for formation of formation of new type, development meeting requirements and self-development of the person in a modern society became one of the major strategic directions of modernization of Russian education.

Has historically developed so what exactly formation became one of the first areas of information of the society, called to form new information culture of the person – persons, able to work in the conditions of introduction of information technology, information of all fields of activity of the person.

The named tendencies and principles are characteristic for all formation and are inherent in mathematical formation, both the general, and the higher. Mathematical formation – an invariable component of formation at school and in high school in Russia and in the world.

The mathematics in a modern Russian society and the world appears in two forms: as a science and as a subject. Mathematical sciences is not only object for employment of science officers, they have practical reflection, application, introduction and use in all spheres of life, especially in the technician and technology, economy and the finance, manufacture and a life, social sphere, family and private life. But it is necessary to notice that decrease in interest of a society to this science, scientific and educational disciplines is now observed.

The mathematics throughout all history of mankind was an integral part, a key to knowledge, base of scientific and technical progress, and important компонентой developments of the person. Under mathematical formation understand the teaching and educational process which is carried out during studying mathematicians at all steps of continuous formation at which occurs not only mastering of certain set of mathematical knowledge, skills, but also development of thinking of pupils, formation of their moral and spiritual culture.

Thus it is necessary to mean that “ideals of mathematical formation on a history extent didn’t remain invariable, and changed from one historical period to another depending on those problems which were put forward by a society to mathematical knowledge and abilities of the citizens". Importance of mathematical formation at the present stage is underlined by many researchers, and according to the various bases.

Mathematical formation is important from the various points of view:

— Logic - mathematics studying is a source and means of active intellectual development of the person, its mental faculties;
— Informative - with the help of mathematics the world around, its spatial and quantitative relations »is learned;
— Applied - the mathematics is that base which provides readiness of the person as to mastering by related subjects, and many trades, does it accessible continuous formation and self-education;
— Historical - on examples from history of development of mathematics development not only her, but also human culture as a whole is traced;
— Philosophical - the mathematics helps to comprehend the world in which we live, to generate at the person developing scientific representations about real physical space.

The modern mathematics in a combination to computer science and the COMPUTER becomes interdisciplinary toolkit which carries out two basic functions: trains the expert-professional to formulate the purpose of this or that process, to define conditions of achievement of this purpose; allows to analyze, i.e. to "lose" possible situations and to receive optimum decisions by means of model. Mathematical modeling should become the obligatory stage previous acceptance of any decision.

The general computerization not only hasn't reduced importance of mathematical formation, but also, on the contrary, has put before it new problems.

Researches in the field of formation information occupy a huge layer in a pedagogical science. (S.A. Beshenkova, Ja.A. Vagramenko, I.E. Vostroknutova, A.G. Gejna, A.P. Ershov, O.A. Kozlova, A.A. Kuznetsova, M.P. Lapchika, V.M. Monahova, E.A. Rakitinoj, I.V. Robert, G. Semakina, N.D. Ugrinovicha, etc.)

According to the concept of information of formation the maintenance of the present stage of information of formation can be formulated as follows:

— Active development and integration of information technology into traditional subject matters and development on this basis of new methods and organizational forms of work;

- Revision of the maintenance and traditional forms and methods of educational process;
- Working out and introduction of the uchebno-methodical maintenance based on application of information technology - training software, systems of the testing, training systems, subject-oriented environments etc.

According to some researchers, it is impossible will divide new pedagogical and information-communication technologies as only wide introduction of new pedagogical technologies will allow to change a paradigm of formation, and only new information technology will allow to realize most effectively the possibilities put in new pedagogical technologies.

Concepts of use of information and communication technologies in formation, as training effective remedies are developed. (N.S. Anisimov, I.G. Zaharova, O.A. Kozlov, V.V. Laptev, M.P. Lapchik, N.V. Makarova, E.S. Polat, I.V. Robert, N.I. Ryzhov, I.V. Simonov, etc.).

Doesn't raise the doubts that fact that methodically competent use of modern information technology is an unlimited reserve of increase of efficiency and quality of training. With introduction of information and communication technologies formation process is carried out in essentially other environment. It means that technology addition now isn't simple, it will transform formation according to requirements of an information society with considerable consequences for an education system in the organization of educational process, methods of teaching and the training maintenance.

Development of information technology goes so promptly that the pedagogical science isn't in time behind technical progress. The majority of modern tutorials on the basis of information technology take root into educational process, only after they "have technically become outdated".

The analysis of the most perspective of a direction of development of modern means of information technology has shown, increase in a role of electronic tutorials in teaching and educational process at all steps of formation.

All great popularity is received by electronic tutorials, on the basis of computer technologies. But it is how much effective, this or that electronic tutorial, and what its didactic functions, remains a basis of problems of various pedagogical researches. There is a number of researches when electronic means training, is duplicated traditional, for example by numbering of the printing version of the textbook that leads to decrease or total absence of efficiency of application of the given tutorials. Essentially new tutorial as much as possible using all functionality of modern information technology, realized by means of active "dialogue" with trained is necessary.

In formation information the new direction of modern tutorials – interactive tutorials is allocated.

Occurrence of interactive tutorials provides such new kinds of educational activity as registration, gathering, accumulation, storage, processing about studied objects, the phenomena, processes, transfer is enough great volumes of the information presented in the various form, management of display on the screen models of various objects, the phenomena of processes, and the main feature presence of "interactive dialogue» immediate feedback between trained and a tutorial.

Wide circulation interactive boards within the limits of the National project have got Education which are successfully applied in educational process, but it is not all spectrum of the interactive equipment which can and be used in training practice.

For today only separate components of interactive tutorials are applied, and there are few complete research works in which problems of use of interactive tutorials would be considered in a complex, as a unit. Absence of researches in the field of technical features (program and hardware) interactive tutorials, the analysis of existing pedagogical and methodical workings out in this area makes the extremely negative impact on introduction and active use of interactive technologies at all steps of domestic education.

Let's list prominent features of training to the mathematics with application of modern information technology:

- An active position of the trained;
- Transition of process of knowledge from a category to "learn" to study in a category any subject осознанно and it is independent;
- Interactive communication with various educational resources (libraries, dictionaries, encyclopedias) and educational communities (colleagues, advisers, partners);
- An information saturation of training;
- "Immersings" trained in the special information environment which in the best way motivates and are stimulated with training process;
- A self-appraisal of results of training.

Complex use of possibilities of modern tutorials promotes dynamism, an intensification of process of training, its novelty, большей to an individualization and differentiation, variability of educational activity and, in particular, purposeful integration of various kinds of activity that allows to organize interaction of all subjects of training in a new fashion.

Now it is possible to say with confidence that interactive tutorials is that area in which the technology and a training technique develop as though synchronously.

The technique of training to the mathematician conducts to occurrence of more perfect interactive tutorials, in turn, development of interactive technologies stimulates development of more perfect technique of training.

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APPROXIMATION OF GEOMETRIC SHAPES IN A SPECIAL CLASS THE FOCAL CURVE

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Key words: curves, focuses, lemniscate, ellipse, shape, invariant basis, the metric, approximation, the degrees of freedom, the generation of forms.

AMS Mathematics Subject Classification: 00A71

Abstract. Developed focal method analytic approximation of geometric forms, represented a smooth closed curves. The analytical basis of the method are multifocal lemniscates, the parameters of the method are a finite number of foci within the lemniscate and its radius. Analysis of the more general class of quasi-lemniscates highlights lemniscates family as satisfying the most general requirements for the distance function and description of geometric forms and their invariants.

1 Introduction

Analytical description of the empirical geometric forms presented by a smooth closed curve, as a traditional problem of applied mathematics, is in constant progress. Numerous methods use different classes of approximating functions, depending on application area requirements and curves relevant properties. For example, a widely used class of trigonometric polynomials is relevant to such properties of signals as frequency and smoothness.

Numerous sources define a geometric form as a spatially organized structure, which geometric properties are independent from coordinate system and are described by its symmetries invariants. Most often the shape of the curve is determined by proportional ratios of its constituent points coordinates.

What should an adequate analytical representation of an empirical closed curve for its shape description be?

Continuous approximation of discrete description of what is the point-to-point description of empirical geometrical form, is obviously a composite of continuous and discrete. Approximate methods differ in their components. Thus, the harmonic approximation represents a curve with a discrete system of coefficients in harmonics continuous basis.

In this paper we discuss the class of approximating functions for the analytic approximation of the shape of smooth closed curves called multifocal lemniscates. Focus approximation represents the curve with a discrete system of points-foci within the curve.

2 Multifocal lemniscates

Lemniscate are the focal curve is completely determined by the system of a finite number k of points-foci in it and a numeric parameter R – radius. Invariant of the lemniscate is constant along the curve product the distances to the foci of all k:

$$\prod_{j=1}^{k} r_j = R^k, \tag{1}$$

where r_j – Euclidean distance from any point of the lemniscate (x, y) to the j–th focus f_j with the coordinates of (a_j, b_j). Lemniscate with k foci will be called kf-lemniscate, and the focal system of k foci $f = \{f_1, ..., f_k\}$ – kf-system.

Lemniscates are smooth closed plane curves without self-intersections, not necessarily simply connected. For a fixed set of k foci lemniscates with different radii form a family of embedded curves from k-connected, for small values of radius R, to simply connected, for large values. The curves with a larger radius cover the curves with a smaller radius without crossing (Fig. 1a). In a certain range of values of the radius the lemniscate have a large variation in form, that may influence their use as approximating functions for a wide range of curve forms [1-3].

The family of multifocal lemniscates forms class of approximating functions for smooth closed curves approximation (continuous basis). Foci of the lemniscate form $f_j = \{a_j, b_j\}, j = 1, ..., k$ a discrete focal system.

The fact that the approximating class of the lemniscates is a family of smooth functions, brings the focal method of approximation with such classical methods as an approximation of power or trigonometric polynomials. However, there is a fundamental difference of focal approximation from these methods. The power and trigonometric approximation, based on a curves parametric description, are relevant to analysis of signals one-dimensional curves. Their degrees of freedom are objects of others in relation to curve spaces and in general not invariant in relation to coordinate transformations, that preserve the curves shape, such as shift, rotation, scaling. The focal method is more appropriate for tasks related to the curve representation form. Freedom degrees of lemniscates are invariant in relation to these transformations. This is easily seen, since the object of approximation (the curve) and freedom degrees of approach (focuses of the lemniscate), belong to the same space.

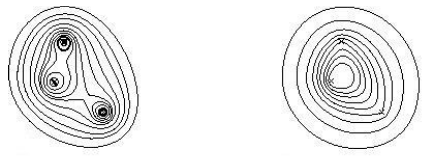

Figure 1. The families of the curves with the same focal system: a) $3f$-lemniscates and b) $3f$-ellipses

3 Approximation by the multifocal lemniscates

The task of closed curves approximation by the multifocal lemniscates can be formulated as follows:

Let n *values* $\{x_i, y_i\}, i = 1, ..., n,$ *of a smooth closed curve are given in the plane. It's required to find such a foci system* $\{f_j\}, j = 1, ..., k,$ *that with a certain radius* R*, a lemniscate corresponding to these foci will be close to the curve in terms of selected criterion of curves closeness.*

The analysis of multifocal lemniscates approximation options has showed that by choosing an appropriate number of foci and their location on the plane, we can get the lemniscate, which is close to any predetermined smooth closed curve. However, the question of how to determine a radius and corresponding system of focal points for a particular empirical curve, given by its points coordinates, remained open. Appropriate methods for finding of arbitrary curve focal approximation, determining focal points number, their location and radius of the approximating lemniscate, are developed for both real and complex [3] descriptions. As a criterion of proximity the minimax criterion of Euclidean distances between curves points is used.

The form focal representation does not impose on the curve data any requirements to parameterization and ordering the points can be given in any order (as opposed to harmonic representation). The result of the focal approach is also parameterization free and is determined only by the curve form.

Examples of empirical curve focal approximation are shown in Fig. $2a$, $5d$. For comparison the result of approximating of the same curve in the trigonometric basis is shown (to achieve equal precision approximately the same number of parameters are required: 19 parameters for the focal and 22 parameters for the harmonic method).

A natural question arises about the possibility of focal approximations in other classes of the focal curves close to lemniscates. It's interesting, in particular, to consider a more general class of such functions in terms of the approaching problem of curves shape and determine thus lemniscates place in a series of similar families of approximating functions. In other words:

"What are unique features of the class of lemniscates within the class of other focal curves in terms of describing curve shape?"

Figure 2. Approximation of the empirical curve: a) foci (9), b)harmonics (5)

4 Additive invariant

Invariant of the multifocal lemniscate in the form of (1) allows identifying it as a multiplicative invariant of Euclidean distances between every point and all foci of the lemniscate. Let's consider the wider class of functions, including the multifocal lemniscates class as a special case.

The equation of the lemniscate (1) can be converted to an additive invariant of logarithmic functions of distances as follows:

$$\sum_{j=1}^{k} \ln r_j(x, y) = S, \tag{2}$$

where $S = k \ln R$. Similarly, an additive invariant can be represented by the class of curves, which are multifocal generalization of ellipses for which the invariant is a sum of the distances to its two foci within the curve:

$$\sum_{j=1}^{k} r_j(x, y) = S \tag{3}$$

Multifocal curves with additive invariant (3) will be called multifocal ellipses with a radius of S by analogy with the lemniscates. Parameterized family of the isofocal ellipses is shown in Fig. 1*b* for the same focal system, as the lemniscates family in Fig. 1*a*. The limiting case of large values of S is also a circle, for small values of S the curves connection, in contrast to the lemniscates, is not broken the curves, remaining convex, are contracted to the point (in this case to a single focus at the apex of the obtuse angle) with the smallest total distance to the foci. A comparison of the two families in Fig.1 shows that the multifocal lemniscates demonstrate much greater variety of forms than multifocal ellipses with the same foci. In particular, in the case of k foci, forming a regular system of located in the vertices of a regular k-gon, kf-ellipses are shrunk to the center of the circumscribed circle. This is well illustrated by the family of $6f$-ellipses (Fig. 3c) and by similar family $7f$-ellipses (Fig. $4a$, *bottom*) in contrast to the lemniscates (Fig. $3b$ and Fig.$4a$, *top*).

5 Quasi-lemniscates

Bearing in mind the common representations of both families: multifocal lemniscates (2) and ellipses (3), it is interesting to consider the equation:

$$\sum_{j=1}^{k} \varphi(r_j(x,y)) = S, \qquad (4)$$

where an arbitrary function of φ is their generalization. Defined by equation (4) curves are called quasi-lemniscates. Ordinary multifocal lemniscates are obtained by setting $\varphi(r) = \ln r$, and multifocal ellipse, if we take as φ identity mapping [4].

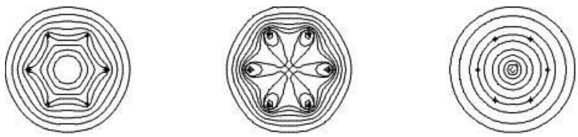

Figure 3. Families of isofocal $6f$-quasi-lemniscates with hexagonal focal system: a)$\varphi(r) = r^{1/3}$; b)$\varphi(r) = \ln(r)$ (lemniscates); c)$\varphi(r) = r$ (ellipses)

This paper devoted to the analysis of possible usage quasi-lemniscates as a class of approximating curves. The main conclusion is that the lemniscate in some sense are the most appropriate for the approximation of curves arbitrary form among all quasi-lemniscates under certain requirements to the function φ, that is to say:

- function φ should be monotonically increasing, because it is in some sense an analogue of the distance;,
- function φ should provide an approximation complexity of the method, sufficient for approximation of smooth curves of arbitrary shape;

— approximation method must be invariant in relation to transformations that preserve the curve shape: shift, rotation and scaling.

The last requirement is divided into two parts: the invariance under movement of rigid body: shifts and rotations, and scaling invariance.

It is easily to see that lemniscates satisfy the formulated requirements to function φ.

Lemniscate function $\varphi = \ln r$ satisfies obviously demands of monotone increase and complexity.

Euclidean distance r is invariant under transformations of movement, so that all the functionals of type (4) are invariant under transformations of shift and rotation.

The scale transformation, as it is not difficult to show, converts the kf-lemniscate in kf-lemniscate but with a different radius. Therefore, up to the radius values, family of the kf-lemniscates is invariant under scale transformations, which corresponds to the definition of the form.

Thus, the realistic requirements for the class of functions that are useful for approximating the curves shapes are to be fulfilled for the multifocal lemniscates class. And how are these requirements satisfied by quasi-lemniscates?

6 Convex functions

The first result is that, if the function φ is convex (in this case the convexity is downwards) then the corresponding quasi-lemniscates given by equation (4) are convex curves. This refers to convexity of its limited set. This fact is proved as follows:

1. If φ is a convex monotonically increasing function, then function $\varphi(r(x, y))$ of points (x, y) is also a convex. Here $r(x, y)$ is equal – as before to the Euclidean distance from point (x, y) to some fixed points of the plane.
2. The sum of convex functions is a convex function.
3. Level line of a convex function is a convex curve.

Let us present the proof of these simple statements.

We use the following definition of convexity of functions:

$$F(\alpha p + \beta q) \leqslant \alpha F(p) + \beta F(q),$$

for any points p and q of the plane and any numbers α and β ($\alpha, \beta \geqslant 0$) such that $\alpha + \beta = 1$. (Points p and q are regarded as vectors with the usual addition and multiplication operations by a scalar). It's required therefore to prove the convexity of $F(p) = \varphi(r(p))$. You can assume without loss of generality that $r(p)$

is the distance from p to the origin, i.e. $r(p) = |p|$. According to the triangle inequality we have for any $p,\ q,\ \beta,\ \alpha$:

$$|\alpha p + \beta q| \leqslant |\alpha p| + |\beta q| = \alpha |p| + \beta |q|,$$

hereof due to monotony and convexity of φ, we conclude that

$$\varphi(r(\alpha p + \beta q)) = \varphi(|\alpha p + \beta q|) \leqslant \varphi(\alpha |p| + \beta |q|) \leqslant \alpha\varphi(|p|) + \beta\varphi(|q|) =$$

$$= \alpha\varphi(r(p)) + \beta\varphi(r(q)).$$

The second assertion for the function:

$$F(p) = \sum F_j(p)$$

obviously follows from inequality:

$$F(\alpha p + \beta q) = \sum F_j(\alpha p + \beta q) \leqslant \sum(\alpha F_j(p) + \beta F_j(q)) = \alpha F(p) + \beta F(q)$$

Finally, to prove the last assertion, let's consider the set $P : P = \{p : F(p) \leqslant S\}$ – inside the level curve $F(p) = S$ and prove its convexity. Let $p, q \in P$, i.e. $F(p) \leqslant S$ and $F(q) \leqslant S$. Then every point $\alpha p + \beta q$ from $[p, q]$ lies in P, as:

$$F(\alpha p + \beta q) \leqslant \alpha F(p) + \beta F(q) \leqslant \alpha S + \beta S = S.$$

This is the condition of the convexity.

Thus, if the distance is given by convex functions, then functional (4) generates only convex (downward) families of functions that can not be generally used as a class of approximating functions when approximation method is developed. The convex functions of distance determine multifocal ellipses (3) as well, so their use is only possible if it's a priori known that approximated curve itself can be a convex function.

7 Scale invariance

The second result concerns invariance in relation to movement transformations. It is provided by the invariance in relation to these transformations of the Euclidean distance that is in the functional argument (4). As far as scale transformation is concerned, the lemniscates, as shown above, satisfy this requirement up to the radius R.

Let us consider the quasi-lemniscates. The assertion is the following:

$$if \ \sum_{j=1}^{k} \varphi(r_j(x,y)) = const, \quad then \ \ \sum_{j=1}^{k} \varphi(\alpha(r_j(x,y))) = const,$$

which may, generally speaking, have a different meaning (α – scale coefficient).

Let us assume that the function $\varphi(r)$ defines the class of quasi-lemniscates invariant under scale transformation. To obtain necessary conditions, we can consider two foci. The assertion is that:

$$if \ \varphi(r_1) + \varphi(r_2) = const, \quad then \ \ \varphi(\alpha r_1) + \varphi(\alpha r_2) = const,$$

To prove it, we fix the foci and define the function $F(\alpha, S)$, as follows: this function value will be assumed as a radius of the quasi-lemniscate appeared from the quasi-lemniscate with radius S as a result of stretching with a coefficient α. Thus, for any point on the quasi-lemniscates with radius S, the following equality is true:

$$\varphi(\alpha r_1) + \varphi(\alpha r_2) = F(\alpha S) = F(\alpha, \varphi(r_1) + \varphi(r_2))$$

Differentiating of r_1 and r_2 the extremes of this equality, we obtain:

$$\alpha\varphi'(\alpha r_1) = \frac{\partial F}{\partial S}(\alpha, \varphi(r_1) + \varphi(r_2))\varphi'(r_1), \ \ \alpha\varphi'(\alpha r_2) = \frac{\partial F}{\partial S}(\alpha, \varphi(r_1) + \varphi(r_2))\varphi'(r_2)$$

The equations can be rewritten as:

$$\frac{\varphi'(\alpha r_1)}{\varphi'(r_1)} = \frac{1}{\alpha}\frac{\partial F}{\partial S}, \qquad \frac{\varphi'(\alpha r_2)}{\varphi'(r_2)} = \frac{1}{\alpha}\frac{\partial F}{\partial S}$$

Equating the left-hand sides, we find that the ratio of the derivatives on the new curve and on the original at the same point does not depend on this point, but only on α. Denote this ratio by $g(\alpha)$:

$$\frac{\varphi'(\alpha r_1)}{\varphi'(r_1)} = \frac{\varphi'(\alpha r_1)}{\varphi'(r_2)} = g(\alpha)$$

Differentiating further on α we get:

$$\frac{r\varphi''(\alpha r)}{\varphi'(r)} = g'(\alpha)$$

For $\alpha = 1$ the right side of the equation becomes a constant $G = g(1)$. As a result, we obtain the following equation:

$$r\varphi''(r) = G\varphi'(r)$$

By adopting the notation $\psi = \varphi'$, we obtain the first order differential equation:

$$r\psi'(r) = G\psi(r)$$

solution of which is known to be:

$$\psi = C_1 r^G$$

Returning to the function ψ , we finally obtain:

$$\varphi(r) = C_1 \int r^G dr + C_2 = \begin{cases} \dfrac{C}{G=1} r^{G=1} + C_2, & G \neq -1 \\ C_1 \ln r + C_2, & G = -1 \end{cases}$$

Thus, we have the class of functions for which specified above functional is invariant under a scale transformation:

$$\varphi(r) = c \ln r \tag{5}$$

or

$$\varphi(r) = cr^{\alpha} \tag{6}$$

the first of which is the lemniscates (2).

Direct substitution verifies that the found functions really define classes of quasi-lemniscates invariant under scaling, and also already for any number of foci.

Thus, apart from the usual lemniscate given by the solution of (5), quasi-lemniscates can be regarded interesting for the purposes of approximation – but quasi-lemniscates that are defined by power functions of the form (6), where only the values of α in the range $0 < \alpha < 1$. Indeed, at $\alpha \geqslant 1$ the function $\varphi(r)$ turns out to be convex that, which is not difficult to show, results in convexity of the entire class of approximating functions, severely restricting the class of the approximated functions. When $\alpha < 0$ the function $\varphi(r)$ is decreasing, that contradicts the requirement of monotonic increasing of the function with meaning of the distance.

8 Families of quasi-lemniscates

On materials of this work a comparative computer experiment was carried out. For the same foci system the families of quasi-lemniscates were built for different φ-distance functions and for a wide range of radius S (fig. 6).

Figure 4a (top) shows a family of curves with a logarithmic function of distance $\varphi(r) = \ln r$- the family of lemniscates (5). Here, as in fig. 1a the considered above properties has been also revealed, including the property of holding all foci inside lemniscates, in particular. fig. 4a (bottom) shows another family of a quasi-lemniscates given by the solution of (6) for the $\alpha = 1$: $\varphi(r) = r$ - family of multifocal ellipses. All curves in this family, as in fig. 1b, are convex, their form is poor and with decreasing radius they "lose" their foci, leaving them outside the curve (and the more $\alpha > 1$, the stronger these properties are manifested).

The other two pictures (fig. 4b) are another two families of quasi-lemniscates generated by the solution of (6), with the functions of the distance $\varphi(r) = \sqrt{r}$ (fig. 4b, bottom) and $\varphi(r) = \sqrt[4]{r}$ (fig. 4b, top). The curves of these families are not all convex, in contrast to ellipses, but their forms are poorer than the lemniscates ones, and they, like ellipses, "lose" their foci, though not with the same readiness. Comparison of the same families of curves with power functions of distance (fig. 4b) shows that degree of such characteristics as foci holding and forms variability in a family with lower degree value (fig. 4b, top), is higher than in the family with the higher value (fig. 4b, bottom).

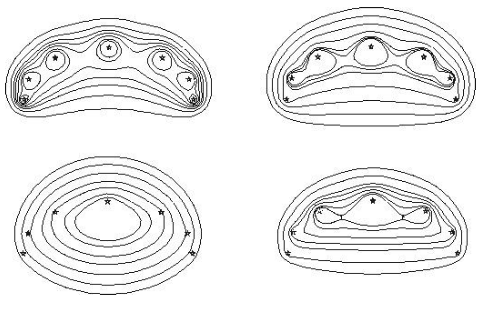

Figure 4. The families of a multifocal quasi-lemniscates with the different φ-function of distance: a) the lemniscates (top), ellipses (bottom) and b) $\varphi(r) = r^{1/4}$ (top), $\varphi(r) = r^{1/2}$ (bottom)

It can be concluded that with increasing degree of root n in a function of distance $f(r) = \sqrt[n]{r}$ corresponding family of quasi-lemniscates will move away in its properties from the family of ellipses, and come close to the family of lemniscates.

It is natural, having in mind a relation between the logarithm and the roots:

$$n(\sqrt[n]{r}-1) \to \ln r, \quad n \to \infty \tag{7}$$

The observed regularity is vividly expressed in the task of approximating the same empirical curve in the four discussed bases (fig. 5).

The above properties of quasi-lemniscates given by different functions of distance are reflected in their approximation properties. As an illustration, fig. 5 shows examples of approximations of the same empirical curve by the six-foci quasi-lemniscates: ellipses with $f(r) = r$ (a), power functions with: $f(r) = \sqrt{r}$ (b), $f(r) = \sqrt[4]{r}$ (c) and lemniscates with $f(r) = \ln r$ (d).

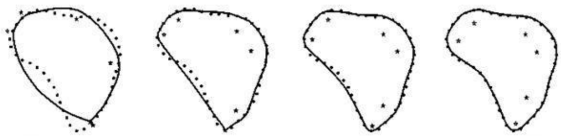

Figure 5. Approximation of the empirical curve by quasi-lemniscates: $\alpha = 1,\ 1/2,\ 1/4$ and lemniscates

As it can be clearly seen by comparing the figures, quasi-lemniscates are not able to approximate a given curve (the result of approximation by ellipses remains convex), while the approach, carried out with the same number of foci lemniscates can be considered quite satisfactory [3]. It should be noted that the higher the root of α is, the closer is this approximation to the approximation by the lemniscates (and to the required one). Thus, analysis of kf-family of quasi-lemniscates identifies family of lemniscates as the most satisfying the general requirements to the distance function and description of geometric forms and their invariants.

9 Conclusion

Focal system has a representation in the same coordinate system as a curve, and is subject to transformations, preserving curve form. Preserving compressed information about the curve form, the focal system inherits also the symmetries of the curve form, unlike the harmonic system. It is shown that the lemniscate, and hence the basis of lemniscates, have the same symmetries group as its focal system.

Focal representation of the form of the curves enables you to enter the plane poly-polar coordinate system. Lemniscates family and the family of gradient curves form two mutually orthogonal families of coordinate curves $\rho(x, y) = const$ and $\varphi(x, y) = const$, where the metric component can be of any shape, for example, the shape of a given curve [5]. fig. 6 shows the grid poly-polar lemniscate coordinate system generated by the focal approximated empirical curve. Such coordinate system can be called its own for the given object.

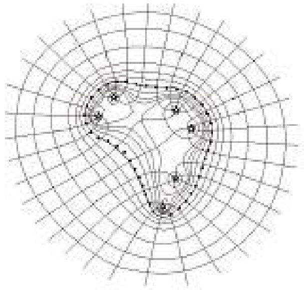

Figure 6. Poly-polar own system coordinates of the empirical form

Thus, the shape of the object image can be related to its own coordinate system by implementing a focal approach. Metric component can be in this case rather complex, manually or automatically adjusted, and for any form of the metric component the angular component is obtained as orthogonal. In other words, the focal representation of arbitrary shape generates its own poly-polar coordinate system with orthogonal system of coordinate curves.

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TEACHING ANALYSIS IN THE HUMANITIES: MATHEMATICAL ESSENTIALS, COMPUTATION AND APPLICATIONS FOR OPTIMIZATION

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Key words: teaching analysis, the humanities, levels of education

AMS Mathematics Subject Classification: 97A40, 97A80

Abstract. Mathematical component in multileveled system of higher education for the humanities is considered. The main investigated aspects herein are the pure mathematical essences inside applied tasks and problems, the selection of appropriate mathematical models and methods, the methodical support of the teaching process itself. The educational paradigm consisting in the stream from the simple one-dimensional models towards more complicated multi-dimensional ones, from the simple mathematical tasks towards applied situated problems (so-called "case-studies") is implemented. The real teaching experience of the methodical package usage is shown.

1 Introduction

The problem of teaching analysis in the humanities is multi-dimensional one [1, 2]. The dimensions herein are as follows: the difference between humanity and engineering, the difference among the concrete humanities, the variety of mathematical essentials involved, the teaching and cognitive style, the variety of educational levels, the analytical aspect in comparison with the computational one, the optimization concept in ideal mathematical models and in real processes of social behavior, and so on. It is important to consider and to investigate the difference among economics, sociology and philosophy being the "typical" humanities from the "mathematical" point of view. Quite different fields for teaching analysis are herein due to different scientific and cognitive essence of these three humanities.

It is generally accepted point of view that economics is the most mathematical one among the humanities [3, 4]. The variety of mathematical disciplines in economical education is even more than in engineering one. These disciplines are calculus, linear algebra, differential equations, discrete mathematics, optimization, game theory, probability, mathematical statistics, econometric, etc. But this variety leads to mathematical education "in width" rather than "in depth". The root here is both the dispersion of real economical problems and the lack of academic hours ("the credits") for mathematics in economical education. So, the teaching analysis for economists covers the set of "points of growth for future" to supply "the package of purchases stored".

The philosophy is quite different from economics in the mathematical dimension. General cultural and mathematical aspects play the leading part here. It is important for philosophers to know the history of mathematics and the branches, the genesis of mathematical ideas, concepts and objects, the "meta-mathematics" – set theory, general algebra, mathematical logic. It gives the conceptual and operational tools for philosophy.

Sociology takes special place in "mathematical continuum". Sociology demands from mathematics the systematic concrete knowledge to maintain fuzzy humanitarian categories ("the qualities") on the base of measurements ("the quantities"). So, the teaching analysis here involves both the general mathematical disciplines and the operational ones (conjoined in the paradigm of cognitive qualimetry) connected with the behavior – probability, computational statistics and applied optimization first of all. Real applied optimization ("sub-optimization") can be implemented here through the multidimensional paradigm "Data Mining" which has essential computational content based upon applied computer packages.

2 The variety of humanities and their essences

Theory of data and information systems having deals with the human essence quite differ from the natural sciences [5, 6]. The presence of behavioral aspects leads to rather specific theoretical and applied properties both of data itself and of it's collection, analysis and optimal decisions making. On the other hand the behavioral aspects have revealed their importance in the wide range of economical applications such as finance, marketing, etc. So, the problems of theory of data in social sciences are both of mathematical specificity and of applied width and importance.

What are the essentials herein? First of all we cannot measure straight the object of our interest (so-called latent parameter, or construction). But we can measure its obvious characteristics. For example, we cannot measure straight the latent parameter "the loyality of the consumer to the trade mark". But we can measure such characteristics as "the period of time during which the consumer buys the products of this trade mark only", "the part of this trade mark in the consumer's budget", "the frequency of buys", etc. So, the mathematical modelling of the dependence of the latent parameter from the set of its characteristics is of obvious vital importance.

Some approaches are known here based on special scales and specific scaling procedures. Nevertheless, we have no any reliable mathematical model here except of so-called “Data Mining” researching paradigm. This is only one example of specific essentials herein. The list can be continued. What are the mathematical problems herein [7, 8]? The latent parameter statistical estimator must be (i) unbiased one and (ii) consistent one first of all. The social sciences analogues are so-called (i) validity and (ii) reliability. So, the mathematical methods to prove the properties of the estimation used are of vital necessity here. Some approaches based upon the correlation analysis are fruitful ones here. We have a wide range of different fine delicate types in (i) unbiased and (ii) consistent estimators such as so-called internal consistency reliability, split-half reliability, Cronbach’s alpha, etc. Nevertheless, all mathematical restrictions of the correlation analysis must be taken into account from the applied point of view. This is only one example of mathematical problems here. The list can be continued too.

Some mathematical approaches and real cases in the area of behavioral economics researches are considered including the consumer behavior at so-called high competitive market and so-called subjective discretionary income construction.

3 The mathematical essentials, applied problems and multileveled teaching system

There are two main approaches to the original problem investigation: to find out the real mechanism (model) or to elaborate the relations among the parameters without trying any rather simple mechanism (model) finding out. The first researching approach is adequate to the situation in which the intrinsic mechanism exists. The second one is much more wary based on the absence of any simple mechanism. The first approach is typical for so-called “natural” sciences (engineering, industry, etc), the second one is typical for so-called “behavioral” sciences (sociology, psychology, politics, etc). It is important that economics and business are between here depending of the prevailing look: on the base of some objective “laws” (the “hard look”) or on the base of subjective “behaviors” (the “soft look”). Both of them are in progress [3, 6]. We had focused in practice on some soft tasks typical for behavioral “soft look”. Those tasks are in the areas of marketing (so-called “marketing-mix”) and the political elections (“political marketing”) [5, 8]. The methodological generality here is the inadequacy of some standard econometric methods aimed at the simple models finding out (for example, time-series analysis). There are some reasons of the inadequacy: the absence of consumer (elector) stable traditions, “short” samples, the processes non-stationarity, etc. The most important actual reason herein is the absence of the stable model itself. The “soft” paradigm “Data Mining” has become much more effective than the “hard” time-series one. That is using a variety of different structural/statistical methods is the real way for searching any stable tendencies.

It is typical for complicated situations to originate the different conclusions (statistical inferences) given by different methods/approaches. It is the evidence of complexity of the situation itself, of the presence the noises, the variety of uncontrollable factors, the multidimensionality, etc. The reasons here is the searching just the tendencies only, not the models, is the constructing the suitable data mining techniques (based, for example, upon so-called the "Attributive Perceptual Mapping" and the "Cognitive Mapping", not upon the regression analysis and the time-series techniques).

What does it mean for the pedagogic and learning aspects? There are following different situations:

(i) Teaching junior students in management, marketing, sociology, political science, etc.
(ii) Teaching junior students in mathematical modelling (in applied mathematics).
(iii) Re-teaching for adults – both with their initial "natural" education and with their initial "humanitarian" one.

There are three different syllabi and educational (pedagogic and learning) paradigms.

(i) "Humanitarian" syllabus accompanied by some mathematical and modelling disciplines.
(ii) "Natural" syllabus accompanied by some humanitarian disciplines.
(iii) Some kind of coaching by means the variety of special short courses in very applied topics in the area of socially-behavior data collecting, processing, interpretation and decision making.

There is the whole educational system in the Russia of today aimed at the (i), (ii), (iii) implementation. The first one is realized by humanitarian institutions/universities, the second one is realized by mathematical/computer institutions/universities, the third one is realized by business schools. The Russian labour-market evaluates the quality here as follows: (ii) is preferable for initial (introductory) job-positions, (iii) is preferable for the advanced ones, (i) is losing to both (ii) and (iii) because of lack of the mathematical/computing training in comparison with (ii) and because of lack of the practical experience in comparison with (iii).

So, it is the problem for educational system to adjust the humanitarian education to the current necessities of humanitarian investigations and practice. The creative and fruitful implementation of the "Data Mining" paradigm must be based on the fundamental mathematical disciplines. But standard humanitarian syllabi in sociology, political science, etc have no the sufficient credits in mathematics. And the “natural” syllabi do! So, the current situation is evidential preference of the “natural” graduates in comparison with the humanitarian ones in mathematical modelling for the actual areas of humanities. The typical standard syllabi of Russian institutions in mathematical modelling is considered to illustrate the problems have been outlined above. The author is involved deeply into the construction, elaboration and actual implementation of those syllabi being the professor at the variety of the universities (in math, modelling, computer science, economics and humanities), being the member of the councils under the Ministry of Education and under the Russian business bodies. The experimental material shows the full picture of the current Russian actual activities in mathematical modelling in the social areas – from scientific investigations, through business practice, to education for the students and for the adults.

4 The educational paradigm and teaching experience

The prevailing look at the mathematical education for economists and managers both in Russia and in the West nowadays is as strictly reduced one for physicists and engineers. The result of the reduction herein is so-called “Calculus” in the West and so-called “Mathematics for economists” in Russia. It leads actually to so-called “coaching” for the “multiple choice” exams, not to real teaching and training in mathematics itself. But Russian students armed with such mathematical tools had been coached (not trained) are out of real capability of dealing and operating with current economical problems. Why does this phenomenon take place just in Russia, not in the West, taking into account the equality of the mathematical syllabi at the typical Russian and Western economical Institutions? The answer here is the deep difference between the economical syllabi at the Institutions. The economical syllabi at Russian institutions are of the humanities (out of vital connections with the previous mathematics). The ones at the Western institutions are on the base of the whole previous mathematics, with appropriate continuations and implementations.

Those items are the reason of impossibility for the mathematical component (of Russian economical education) to be the copy of the Western one, as the economical component itself is not the copy of the Western one. The great difference between the economical components exists and the strategic way is to eliminate it in future. But current Russian mathematical component ought to be much more essential than the Western one. It ought to be aimed just for teaching and training both in mathematics itself and in mathematical models in economics. It is so-called “Mathematics of economy”, not the reduced “Mathematics for economists”. Mathematical component of economical education in Russia must resolve the economical problems, must build the modern mathematical paradigm of the economical education itself. So, the coaching paradigm here is quite insufficient. The creative paradigm is of vital necessity to train in mathematics and in mathematical models in economics. This paradigm is not traditional one for the West.

The inherent benchmarking, the educational models and the technologies are in the scope of professional interest for the paradigm‘s implementation. The paradigm is so-called “phys-tech system” (in accordance with the title of the most prominent Russian University - Moscow Institute of Physics and Technology). The inherent educational essence herein is three-staged education: institutional (general, first-year – third-year), faculty (problem-oriented, third-year – fourth-year), specialized (professionally applied, fourth-year – sixth-year). There are one institutional syllabus, about ten faculty syllabi and about hundred specialized syllabi at the original prototype. The most vital part of the paradigm is the second stage. The first and the third stages are two separate elements, not the system. Just the second stage addition – the interface between the first and the third ones – transforms the whole educational process (of six years duration) into the system. The most important systematic effect herein is the actual implementation of the general mathematical education (the first stage) inside the real professional problems (the third stage) – through the interface (the second stage). Thus, we have the system of three elements (at the “phys-tech paradigm”) instead of two standard elements (at the majority of the real institutions) due to the interface, which duration (only two years) is less than of the other two stages. Therefore, the way to the effective training for economists and managers in current Russia is quite evident. What is the explanation of the full absence of real examples where the paradigm has been implemented in economics and management? There are two main reasons in the Russia of today: (i) the top-managers at Russian economical and managerial institutions are out of any previous mathematical education themselves and (ii) the traditional “Russian” gap between the mathematics itself and it’s real applications exists inside the heads of mathematicians themselves. The reasons show the tendency and the methodological path to overcome it.

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POSSIBILITIES OF INTERACTIVE MEANS IN THE TRAINING OF HIGHER MATHEMATICS

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Key words: interactive training, interactive means, interactive electronic boards

AMS Mathematics Subject Classification: 97B40, 97D40

Abstract. Interactive means – means, which provides dialogue occurrence, that is an exchange of messages between the user and information system in a mode of real time. Occurrence of interactive means provides such new kinds of educational activity as registration, gathering, accumulation, storage, processing of the information on studied objects, the phenomena, processes, transfer is enough great volumes of the information presented in the various form, management of display on the screen models of various objects, the phenomena, processes.

1 Introduction

The modern pedagogical technologies applied in educational process, should solve a number of problems connected with formation reforming. Preparation trained in such conditions should pass to qualitatively new level meeting modern requirements which can be reached at the expense of an intensification of educational process, an optimum combination of traditional and innovative forms, methods and tutorials.

Many methodical innovations are connected today with application of interactive tutorials. Therefore, introduction and active application of interactive tutorials is one of perspective directions of perfection of the organization of educational process. Use of interactive tutorials in educational process allows to realize innovative ideas and directions of an individualization and formation information in practice.

The word “interactiv” occurs from an English word “interact” where “inter” — mutual, “act” — to operate. Interactive training is, first of all, dialogue training in which course development of experience by the trainee (spontaneous or specially organized) on the basis of interaction with something (computer) or someone (person) is carried out. We will allocate two approaches to definition of interactive training: biological and technological.

With positions of the biological approach, under an interactive method of training, understand the method of the training constructed on strengthened intersubject interaction of all participants of process of training. In this case interactive training

— training in a mode of the strengthened interaction and dialogue of subjects of process of training.

The technological approach means an interactive method of training, as the method of training constructed on use of possibilities of a two-way communication of tutorials. Then interactive training is a training, in a mode of interaction of the person and a tutorial (as a kind of electronic training — training in a mode of interaction of the person and the computer).

In a context of use of information technology as interactivity understand possibility of the user actively to cooperate with a data carrier, the term «interactive dialogue» is more often used.

I.V. Robert defines interactive dialogue as interaction of the user with the program system, characterized (unlike dialogue, assuming an exchange of text commands, inquiries, answers, invitations) realization of more developed means of conducting dialogue (for example, possibility to ask questions in the involuntary form with "keyword" use, in shape with the limited character set), is thus provided possibility of a choice of variants of the maintenance of a teaching material, an operating mode.

The interactive mode of interaction of the user from the computer is characteristic that its each inquiry is caused by reciprocal action of the program and, on the contrary, a remark of last demands reaction of the user [1].

At the same time distinguish three interactive forms of interaction:

— Interpersonal interactivity which means two-forked correspondence between people in which course the addressee and the sender of messages can exchange in places and creates original interaction if stay an active position and mutual interest (e-mail or debatable lists on certain subjects),

— Information interactivity which is limited by the possibilities in advance provided at designing of electronic editions. They are aimed at information representation, navigation under the maintenance and placing of any data, include use of hyperlinks, filling of forms, search of the data in keywords and other forms of interaction,

— Human-computer interaction or interaction of the person with the computer - the area concerning interaction between the user also is computer firmware, for example, through such devices and a handshaking, as the mouse, the keyboard, the graphic interface, recognition of vocal commands [2].

There are various opinions on ways of realization of interactivity. If process of interactive dialogue to consider between the computer and the person it is possible to present it as set of the elementary elements "inquiry-reaction" where "inquiry" — a certain signal from the user to the training program (or back), and "reaction" — the answer to it, and the answer influences the further actions of the requesting party. For example, at training to the problem decision in an interactive mode under the first answer given to the pupil the error message is formed, helps are given; and after the second attempt of input of the answer other set of messages is deduced [3].

Interactive training allows to pass from passive training to an active way of realization of educational activity, to generate an individual trajectory of training for each pupil who becomes in this case the main participant of educational process [4].

Interaction possibilities considerably increase efficiency of mastering of knowledge pupils. Now the interactivity principle is realized at creation of various elements of electronic editions.

A number of authors in the researches is considered by questions of a combination of demonstration and interactive models with traditional methods of training and possibility of their integration into educational process. They notice that application of interactive models promotes formation at pupils of a fuller appreciation about studied objects. Demonstration models with a soundtrack acquaint pupils with a structure and properties of studied objects. The subsequent use of similar interactive models at the decision of problems and at the permission of problem situations gives the chance to pupils to put the theoretical knowledge into practice. The combination of application of interactive computer models to demonstration of real objects, their photos and video images allows pupils to receive adequate representation about a structure and properties of various substances, promotes increase of interest to studied questions. Application of interactive animation models is well combined with a technique of problem training, promotes mastering of scientific concepts and laws on the basis of the personal experience got by pupils at manipulations with models. The bright impressions resulting interaction with them, promote strong storing of data by the subject and raise quality of mastering [5].

There exists two basic approaches in training the mathematics to the students of humanitarian specialties:

- `The acquaintance of students to the basic directions of mathematics in the form to a survey of the general educational course,`
- `The adaptation of a course of mathematics taking into account prospective level to the preparation of students and a professional orientation of training.`

Despite distinction of specialties, for all students it is necessary to be able to analyze the information, to allocate a question essence, to own the logic of reasoning (accurately and logically to state the thoughts to deny or prove judgments, is given reason to argue), to generalize a statistical material and correctly to interpret it. All these qualities develop in training to the mathematicis. Considering the growth of volume of the new information, it is necessary to form the independent informative activity of students. There is the professional-focused problem to create the catalyst of development of self-education of students in the course of training the mathematics. Such problems reflect the basic maintenance of the mathematical discipline, the model informative, the research and creative problems of professional work of the future expert.

Thus, the use of the professional-focused problems in the course of training promotes the mathematics of students of humanitarian specialties:

— `To formation of positive motivation of trainees to studying in the bases of the mathematical science`,
— `To training improvement of quality, at the expense of use of mathematical methods in the decision of professional problems`,
— `To development of scientific outlook and research competence of the expert of the future professional work`.

Let's notice that a priority direction in system of the higher vocational training, is the second approach which leans against basic principles of training: the scientific character, the fundamental nature, the variability, the variability, the individualization and professional additions.

Interactive means – means, which provides dialogue occurrence, that is an exchange of messages between the user and information system in a mode of real time. Occurrence of interactive means provides such new kinds of educational activity as registration, gathering, accumulation, storage, processing of the information on studied objects, the phenomena, processes, transfer is enough great volumes of the information presented in the various form, management of display on the screen models of various objects, the phenomena, processes.

Interactivity degree is defined proceeding from the actions, which students can make with the information: to choose a movement way on a subject information field, to add or withdraw the information put in a tutorial at its creation.

In the scientifically-methodical literature distinguish, following interactive means of educational appointment:

— `Interactive textbooks (the programming-methodical complex providing possibility independently to master a training course or its big section`,

- — Subject-oriented environments (the program, the software package, allowing to operate with objects, operations over objects and the relations, corresponding to their definition, and also provides evident representation of objects and their properties),
- — A laboratory practical work (serve for carrying out of supervision over objects or their properties, for processing of results of supervision, for their numerical and graphic representation, for research of various aspects of use of these objects in practice),
- — Training apparatus (serve for working off and fastening of technical skills of the decision of problems, provide information reception under the theory and receptions of the decision of problems, training at various levels of independence, control),
- — Supervising programs (the software intended for check (an estimation) qualities of knowledge),
- — Directories, databases of educational appointment (programs of this class are intended for storage and a presentation to the pupil of the various educational information of help character. For them the hierarchical organization of a material and means of fast information search in various signs or on a context) are characteristic.

Let's list the basic interactive means which can be applied successfully at studying of the mathematical analysis in the university: interactive projectors; interactive boards; interactive panels; interactive displays; interactive tablets; interactive computers; interactive systems of voting; interactive systems of testing; interactive presentation tribunes; interactive projective complete sets; interactive projective prefixes; interactive classrooms; interactive manuals.

Let's stop on a technological component of the modern interactive boards applied in educational process. Interactive electronic boards are made with application of various technologies of definition of position of a marker or a finger on a surface. Distinguish: touch resistive, optical, infra-red, ultrasonic, electromagnetic technologies. Let's allocate key possibilities of the interactive boards qualitatively distinguishing them from a projector and the screen. It is necessary to carry introduction of records to the basic functions over the entered image, and as preservation of the received image, concealment of objects, a conformity designation between objects or concepts, visual effects and animated objects. Interactive boards staffed voting systems or tablets allow to organize feedback with students.

Application of interactive boards in training to the mathematical analysis allows to solve a number of pedagogical problems: qualitatively to change material giving;

to raise interest to a subject; to expand kinds of educational activity; economy of school hours.

Interactive electronic boards are made with application of various technologies of definition of position of a marker or a finger on a surface. Now exist: touch resistive, optical, infra-red, ultrasonic, electromagnetic technologies. Key possibilities of the interactive boards qualitatively distinguishing them from a projector and the screen, carry the following: introduction of records over the deduced image, preservation of the received image, concealment of objects, a conformity designation between objects or concepts, visual effects and animated objects. The interactive boards completed with systems of voting or tablets, allow to organize feedback with pupils.

Let's notice as that one of perspective directions of application of interactive technologies, the interactive textbooks representing a programming-methodical complex are, providing possibility independently to master a training course or its big section. In a basis of such textbooks interactive interaction trained with the information training environment is necessary. Interactive textbooks unite properties of the usual textbook, a directory, the book of problems and a laboratory practical work and a training apparatus.

Let's allocate a number of didactic advantages of interactive textbooks over traditional printing: use of several channels of perception in educational process; an individual trajectory of training; material representation by various ways; representation of processes in the dynamic form; modeling of difficult real experiments; visualization of the abstract maintenance; management of process of training by creation of problem situations; motivation to training. Now textbooks on the basis of interactive environments are created.

The most popular interactive means in studying of the mathematical analysis are: Mathcad, Advanced Grapher, Plotter.

Application of the given interactive means will allow to open such sections, as «Functions, sequences, limits», «Differentiation of functions of one variable», «Research of function by means of a derivative», «Integral calculus of functions of one variable», «Differential and integral calculus of functions of several variables», «the Differential equations». Use of interactive means in training to the mathematical analysis will allow to build various schedules on a plane, to conduct researches of functions, to receive analytical expression for a derivative, to carry out numerical integration, graphically to solve inequalities, perform regression analysis etc.

We develop a course, on training to higher mathematics with application of modern interactive technologies. Feature of the given course consists that the developed interactive fragments of employment are realized on the interactive equipment. The developed interactive uchebno-methodical complex includes a number of multimedia presentations for carrying out of lecture employment on higher mathematics and analytical geometry, methodical materials for carrying out of a practical training under the mathematical analysis on the basis of interactive Mathcad environments, Advanced Grapher, interactive control and measuring materials on all sections of higher mathematics.

Using of interactive means in system with classical training in pedagogical high schools will allow to construct a modern course of the mathematical analysis.

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RESPONSE SYSTEMS IN HIGHER EDUCATION

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Key words: teaching technologies, computer technologies, teaching process support, higher education

AMS Mathematics Subject Classification: 40-03, 01A50

Abstract. The article analyses the potential of the response system SRS (clickers). SRS is a technology that enables every student to answer immediately to the teacher's questions with the help of the keyboard (device) that transfers the information into the teachers' computer. SRS technology has bee implemented into the system of Higher education and is successfully used in the process of different subjects teaching.

Nowadays lectures are conducted in big classes, containing hundreds of students and though during the lecture (90 min) the teacher from time to time gives a chance to the students to ask him questions only 1-2 students can get the answers.

In the result od such teaching conditions many students consider themselves as passive participants of the lecture, inactive part of teacher-and-student interaction. Such interaction is very important because if the students don't feel involved in the teaching process, they don't make much effort to take in the material.

Development of students' ability of logical thinking and creative usage of their knowledge while solving the problem is one of the main goals of the University education.

Any part of the lecture can be activated by introducing new technical appliances. New technical devices usage makes it possible to improve the quality of the teaching material and intensify the teaching effects as it gives the teacher an additional opportunity to work out an individual teaching line and enables the teacher to put into practice the differentiative approach to the students with different levels of readiness to the education. Systematic usage of informational technologies in combination with the traditional methods can significantly increase the teaching effectiveness.

Interrogation is an excellent way to attract the students' attention. Interrogation system is used to get a quick students' response and gives an immediate result showing how successfully the teacher has got his words over to the students. In other words the teacher gets the diagram that shows how many students worked during the lecture, how many of them understood the material, how many were not attentive. Making such interrogations every 15-30 minutes the teacher can improve his lecture taking into consideration the data received in order to make it more interesting. There is one more opportunity – to make interrogation at the beginning of the lecture to get the information about the students' knowledge level.

The SRS enables to speed up and simplify the process of testing, and also makes the process of feedback from a student to the teacher effective. The system consists of the set containing the clickers, the infra-red receiver and the software. The clickers are little devices that are easy to keep in hand and that enable the students to answer the questions. The clicker is a plastic case with monochrome liquid-crystal display and a keyboard/ students can type in the answers. If necessary the can also show that they are not sure about the answer. The software enables to type in both digital and symbolic information. The Main Clicker manages the test programme startup, the choice of the type of the test and the results, changes of the image size. The clickers of the respondents are the set of the digital keys and have the individual numbers.

The participants of the interrogation choose the number of the answer they consider correct on the clicker. You can immediately get several types of the test results (general of the group with the amount of the answers to every variant and the percentage of the correct answers and individual).

Test can be made with multimedia files, pictures, sound, video. You can make your test time-limited. For every answer a certain amount of points is given and the results for every participant are calculated.

The teacher can use several regimes:

— `Full interrogation (every student should give an answer).`
— `The quickest answer (the interrogation is over as soon as the first correct answer is received).`
— `Attention-getting mechanism (test are not used; the students can show that they want to ask or answer the question)`
— `Show the names (identification numbers) of the respondent on the general scheme`
— `Show the numbers of the answered questions on the general scheme`

A good teaching method provides teacher and students' interest unity on the basis of the necessity of subject learning. Methods of active teaching are one of the most perspective ways of professional training system improvement.

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CLASSROOM PROJECT "MATHEMATICAL ANALISIS IN THE UNITY AND DIVERSITY"

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Key words: Mathematical analysis, Training project

AMS Mathematics Subject Classification: 97D40

Abstract. Classroom project aims at revealing the presence or absence of analogies in one- and multi-dimensional mathematical analysis helps to establish similarities and differences, to systematize material. It is necessary to discuss mathematical analysis as a whole and in details, in its interconnection and interrelationship from different points of view and with different attitude.

1 Main

Realization of the classroom project is a complicated process therefore to understand the diversity of the interconnections within the subject the technique for comparing, which allows you to structure the investigated material, was developed. The technique of comparative analysis fulfillment.

1. To expose possible cases of a uniform record of the material being studied within mathematical analysis (concerning formulas).

2. To identify aspects in the case of the lack of analogy between mathematical quantities depending on spatial dimension.

3. To sort out similar definitions, notions, statements regardless of spatial dimension.

4. To discover the invariance (non-invariance) of some properties of objects as a result of various algebraic and nonalgebraic operations, relationships between various properties of functions, inverse functions.

5. To consider geometric interpretation of mathematical notions. To illustrate the unity of analytical and geometric approaches.

6. To establish the presence or absence of certain information in the recommended literature followed by the complete presentation of the studied material, issue, unit.

7. To state issues, subjects for self-research.

Let us consider the content of each point in details.

1. Carrying out a comparative analysis it is necessary to state what is given and what we should get, to consider a subject in terms of availability of the uniformity of some notions. A comparison of univariate and multivariate analysis allows us to find out that the most important notions such as a point, a function, a function limit, function continuity, a function differential, a derivative, an integral in the vectorial form look the same and are recorded in the same way.

2. Analyzing the material we study not only common features but also differences. So another constituent part of the project is to find information that does not allow not only the same way of recording but also so it is necessary to find the information that does not allow the same representation. Besides there are some facts which exist in space if one dimension and does not exist in other.

2.1. In the case $f\colon\ R^m \longrightarrow R^n$ the following does not happen: the property of the conservation of the function limit value sign, which is not zero, at a point; the property of the conservation of the value sign, which is not zero, of a function continuous at a point; the existence of a product limit, a quotient of two functions having a limit at a point; the continuity of a product, a quotient of two functions continuous at a point; the issue of taking by functions the minimum and maximum of its values on a compact; the issue of taking intermediate values by the functions which are continuous on a compact. But all these properties happen in the case of real value function $f\colon\ R^m \longrightarrow R$.

2.2. In the case $f\colon\ \mathcal{R}^m \longrightarrow \mathcal{R}$, $m>1$ the definition of iterated limits is introduced. Such a notion does not exist in the case of one-dimensional case $f\colon\ R \longrightarrow R$.

2.3. In the case $f\colon\ R \longrightarrow R$ the presence of a differential, the differentiability of a function at a point and the presence of a finite derived function at a point are equivalent conditions. The existence of partial derivative of each variable at the point in a function follows from the differentiability of the function in the interior points of applicable domain for the functions with many variables. But the differentiability of a function at a point does not follow from the fact of the existence of a partial derivative of a function at the point. However, if partial derivatives of a function exist and are continuous at a point the function is differentiable at this point.

2.4. Some properties of multiple integral related by specific features differ from the one-dimensional integral; the integration of a multiple integral can be reduced to an iterated one.

2.5. In considering the problem with the smallest and largest values of continuous functions on a compact in the multivariate mathematical analysis the notion of conditional extremum, which does not exist in the case of $f\colon\ R \longrightarrow R$, is introduced.

3. In mathematical analysis, some definitions, notions, statements, theorems do not depend on spatial dimension, i.e. they are similar. Let us point out some of them: a bijection; the definition of a sequence of points; the limit of a sequence of points in terms of distance if we consider that in a one-dimentional case $\rho(x_1, x_2) =$

$|x_1 - x_2|$; the Cauchy's criterion on the convergence of a sequence of points; the definition of a function limit; equivalent definitions of a function limit according to Cauchy and Heine, the Cauchy's criterion of the existence of a function limit; local properties of functions having a limit; the definition of a composition and its limit; continuity and uniform continuity of a function; the composition of continuous functions; the local properties of continuous functions; differentiability of a function; extremum points; the necessary and sufficient conditions of a function extremum in terms of the first and second differentials. Regardless of the dimension, the above mentioned is justified and proved in much the same way as in one-dimensional case.

4. The following notions have the invariance property, regardless of a spatial dimension: the existence of the limit of a sum, the product and the quotient of converging sequences, Cauchy's criterion of the convergence of a sequence; the existence of the limit of a sum, a product and a quotient of a real-valued functions; the existence of a function composition limit; the continuity of a sum, a product, a quotient of continuous real-valued functions; the continuity of a composition of continuous functions; Lagrange's formula in vector notation; the formula for determining the differentiability of a function in vector form; the equations of a tangent line, a plane, a hyperplane in a vector form; the relationship between continuity and differentiability. Inverse functions are invariant under monotonicity, continuity and existence of a finite or infinite derivative with a definite sign, but are not invariant under convexity, concavity. In addition, it appears that there is no invariance of functional properties concerning non-algebraic operations of differentiation and integration.

5. To make studied notions more understandable and visible is one of the objectives of the classroom project. Geometric interpretation favours better mastering of studied material by a student during his independent research. In the study of mathematical analysis it is appropriate, where possible, to follow the principle of unity of analytical and geometrical approaches. For example, when considering a derivative it is necessary to combine analytical and geometric representation of the issues of existence (nonexistence) of finite (infinite) derivatives with the existence (nonexistence) of nonvertical and vertical tangents; to point out the dynamics of the transition from the tangent line and its equation to the tangent plane and its equation to the hyperplane and its equation. From the point of view of geometry, we have a transition from a one-dimensional object, straight line, to two-dimensional object, a tangent plane, and multi-dimensional one, a hyperplane. The system simultaneous consideration of equations allows us to establish their generality; all the equations are expressed in terms of derivatives.

6. The student is invited to study educational literature [1-3] in order to identify structural elements which are constituent parts of the content of different areas of mathematical analysis (the theory of limits of sequences, functions and etc.) contained in each textbook. As a result of analysis, identification, comparison, synthesis of information from the recommended textbooks, the use of personal experience, self-maintained enlargement of the main or related material the student selects the main structural elements in a specific unit aiming the creation of relative completeness, depth and breadth in the material coverage, the possibility of simplifying the statements and proofs of some theorems, obtaining his own new or updated results, formulation of new questions, problems, etc. Let us describe various situations which the student can face in studying the material of the textbooks [1-3].

6.1. The textbook [1] contains a complete list of properties of a certain integral expressed by the equalities and inequalities, but does not contain the second mean value theorem. In [2] there is no mean value theorems for integrable and for continuous on the interval [a, b] functions, it does not contain the integrability property of the quotient of two integrable functions, but in contrast to [1] it has the justification of the integrability of the integrable function square as well as the theorem on the integrability of a complex function. In [3] there exists a complete list of properties of a definite integral, but it does not specify the properties of integrability of the product and the quotient of two integrable functions. The project will contain joint information about the properties of one-dimentional definite integral given in [1-3].

6.2. In [1] it is emphasized that the properties of a multiple integral are similar to those of the integral of a one-variable function on the interval and it supports a complete list of these properties except for the second mean value theorem. In [2] the first mean value theorem for continuous functions is not considered, but the mean value theorem for an arbitrary function is indicated, the property of integrability of the quotient of two integrable functions is not pointed out. [3] also gives a complete list of the properties of a multiple definite integral, but it does not specify the properties of the integrability of a product and a quotient of two integrable functions. Similarly to the one-dimensional case of a definite integral the combined information given in [1-3] will be included into the project on the properties of the multiple integral in the multidimensional case. Within the project working with various sources allows to find out which material is presented in one group of textbooks and is absent in the other, in what way the same facts are introduced, maybe some items are not presented at all, and whether it is possible to complete the information during the self-study work. We will show this on the basis of the issues "The Limit of a Sequence" and "The Limit of a Function." Classroom project is carried out in the form of a table that reflects the various approaches to the proof of the theory of sequence limits and function limits by the authors of textbooks [1-3]. We shall give a comparative analysis of the table content without presenting the table itself. In some textbooks as the sources of information some issues are not covered completely or partially and other issues are examined but do not include all possible situations, there is no complete proof and therefore they are proposed for research by the students themselves. Mentioned above will be taken into account at the final stage of the project, in the course of which the student will propose, consider, justify, prove all these issues. The project shows that there are two definitions of the limit of a sequence and four definitions of a function limit. Subject to which of these definitions are taken as the basis, in what order infinitesimal sequences α_n, the sequence $\{x_n\}$ and their limits, infinitesimal functions $\alpha(\tilde{0})$, functions *f(x)* and their limits, as well as the relations $x_n = a + \alpha_n, x - a = \alpha_n, f(x) = A + \alpha(x), f(x) - A = \alpha(x)$ are studied, different approaches of the authors of the textbooks [1-3] to the proof of the theory of limits of sequences and limits of functions are found. In particular, the initial starting points are: a sequence and its limit in [1], the infinitely small and infinitely large sequences in [2], error estimates, the limit of a sequence in [3]. In some textbooks some questions are not considered at all, for example, the infinite function; the connection between infinite sequence (a function) and an infinitesimal sequence (function) are not considered in [2], [3]; the relationship between the sequence having a limit and an infinitesimal sequence in [3]. However, they are important issues of mathematical analysis which should be studied. If the study of the material is made with one textbook, some issues will not be covered, will drop out of the consideration and will not be mastered. The order of the material presentation in [1] - [3] is as follows: In [1]: the limit of a sequence, infinite limits, infinitesimal sequences, the relationship between the sequence $\{x_n\}$ having a finite limit a and an infinitesimal sequence $\{x_n\}$: $x_n = a + \alpha_n$. The limit of a function, infinitesimal and infinite functions and interrelation between them. In [2]: infinitesimal and infinite sequences, the limit of a sequence. A function limit, infinitesimal function. In [3]: the limit of a sequence, infinitesimal sequences are not introduced, but the properties of the limit of a sequence are justified in terms of errors. A function limit, infinitesimal functions. The properties of function limits are shown in terms of features of infinitesimal functions.

7. Recently mathematical analysis is considered as an established academic discipline, but this does not mean that in it there are not enough issues for research and profound discoveries. In the course of the carrying out the project it was found that the scientific literature does not specially reflect the fact that one-to-one (bijective) functions do not reach an extremum at interior points of the domain, they are not invariant concerning convexity, concavity. This fact suggests the idea that in the theory of bijective functions there is uninvestigated material, in particular, in the point of the integral representation of special subclasses of bijective functions. The solution of this problem becomes easier in the classes of functions having integral representation, such as the Stieltjes's integral. In this connection let us state and prove the theorem. We shall consider the set of bijective functions as a class. Let us denote the set of bijective on the interval *(a,b)* functions by BO*(a,b)*, subclasses of bijective function classes BO(h_i), i=1,2,3 defined on the intervals $h_1 = (-\infty; 1), h_2 = (-1; 1), h_3 = (1; +\infty)$. Let M$[-\pi, \pi]$ represents the set of non-decreasing on the interval $[-\pi, \pi]$ functions $\mu(\vartheta)$ which satisfy the condition $\int\limits_{-\pi}^{\pi} d\mu(\vartheta) = 1$.

Theorem 1. *Let* $\mu(\vartheta) \in M[-\pi, \pi]$. *Then the function is*

$$\gamma_i(x) = \int (\int\limits_{-\pi}^{\pi} \frac{1 - x^2}{1 - 2x\cos\vartheta + x^2} d\mu(\vartheta))dx, x \in h_i, i = \overline{1,3}. \qquad (1)$$

bijective on the intervals $h_i, i = 1, 2, 3$.

Proof. Differentiating functions $\gamma_i(x)$ for x, we obtain

$$\gamma_i'(x) = \int\limits_{-\pi}^{\pi} \frac{1 - x^2}{1 - 2x\cos\vartheta + x^2} d\mu(\vartheta) \qquad (2)$$

Since $\mu(\vartheta) \in$ M$[-\pi, \pi]$ and the integrands in (2) are strictly constant in the sign in $h_i, i = 1, 2, 3$, then the expressions on the right side (2) are also strictly sign-constant. On the basis of the applicable theorems of mathematical analysis, strictly defined sign of derivatives is a sufficient condition of the bijectivity of functions $\gamma_i(x)$ in (1) on proper intervals. *Theorem is proved.* □

This part of the project involves further development and study of special subclasses of the class of bijective functions with their integral representation according to the information given above.

As a result of the project fulfillment the technique for comparative analysis of univariate and multivariate mathematical analysis has been developed due to which: a mathematical analysis was examined from different perspectives, from different points of view; the number of examples for the existence of invariance of some properties of mathematical objects was increased, the fact that differentiable bijective functions do not take extreme values at interior points of the applicable domain was elicited; the range of issues, problems for student's self-study was outlined. It is noticed that: bijective functions do not possess the invariance property in the cases of convexity and concavity; the inverse function can possess finite derivatives or infinite derivative with a definite sign on applicable intervals, if the original function is differentiable on some interval or equals to zero at this point of the interval.

Prospects for the project. In connection with the turn to multilevel system of education according to which the number of classes was reduced and the number of hours for self-study work was increased, a student should work a lot with education literature, where different approaches to the studied material are possibly used, and it is difficult to work with a variety of alternative variants of the mathematical analysis presentation.

General outlook. Due to classroom projects we have the opportunity: to unify some of the definitions, notions, theorems and their proofs regardless of spatial dimension; to issue the textbook focused on standardization, uniform presentation of separate items of mathematical analysis regardless of an author's approach to the issues covered and with the emphasis not on the form of presentation but the content of the material.

Particular perspectives. 1. Introduction of special classes of bijective functions with their integral representations since the presence of integral representations facilitates the study of functions. 2. The study of functionals, operators defined on functions of classes introduced before. 3. The stating and proof of the theorem on the differentiability of an inverse function provided that the original function is differentiable everywhere on an interval except for the point at which there is an infinite derivative with a definite sign. 4. The performance of the comparative analysis for in other areas of mathematical analysis.

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VI.2 Teaching analysis at schools

(Sessions organizers: M. Klyaklya, A.A. Rusakov, P.V. Semenov)

COMPLEX NUMBERS METHOD IN PLANE GEOMETRY TRAINING TECHNIQUES AT SECONDARY SCHOOL

T. A. Chernetskaya

Key words: intra-subject connections in mathematics, complex numbers, plane geometry

AMS Mathematics Subject Classification: 97G40

Abstract. Some plane geometry tasks which solutions are based on non-traditional for school geometry course geometric figure conception and special training technique, involving not only analytic and coordinate methods but also complex number method were considered in the article.

The mathematical analysis methods in plane geometry training techniques at secondary school are based on the well known mathematical education universality principle. This principle is connected with mathematical science universality generality and interpenetration of its methods. The connection between mathematical analysis and geometry is the example of such interpenetration, which is based on many facts from antique method of geometric exhaustion to modern analytic and differential geometry.

In present time there are several problems of mathematical education content in secondary school. Acknowledgement of intra-subject connections in mathematics by secondary school graduates is one of these problems. In fact, algebra and mathematical analysis methods are widely used in geometry and vice versa. That is the example of such intra-subject connections that our school graduates must realize.

A very important part of school mathematics – complex numbers – was excepted from the school mathematics course content several years ago and it disadvantaged this course in perspective of forming school graduates' knowledge of origin and development such term in mathematic as "number" and created some difficulties for complex numbers usage in plane geometry. It's a well-known fact, that complex numbers are widely used in mathematics ant its applications, especially in complex number's analytical functions. These functions are attractive themselves, in addition they are used in aero- and hydrodynamics, in non-Euclidean geometries, in number theory. And with them, complex numbers can be used in such simpler mathematic and physics applications as elementary geometry, trigonometry, similarity and movement theories, electrotechnology etc.

It is a well known fact that in secondary school plane geometry training techniques use analytic, vector and coordinate methods. We also suggest using complex numbers method in plane geometry for school graduates who are interested in mathematic and including this method in mathematical education content in specialized mathematical schools and classes. Complex numbers are the one of the traditional contents for supplemental and elective courses in 9-11th grades (basic level of Russian secondary education mathematic standard) and it is obligatory for studying in specialized mathematical schools and classes (core level of Russian secondary education mathematic standard), but only complex number geometric illustration on a plane is considered as a rule without future studying complex numbers method in plane geometry. We can mark one important advantage of complex numbers method in plane geometry: this method allows to solve geometric tasks using standard formulas, that are unambiguously defined by the task type. That's why this method is incredibly simple in comparison with vector and coordinate methods, analytical method, etc., whose usage demands enough quick wits and time.

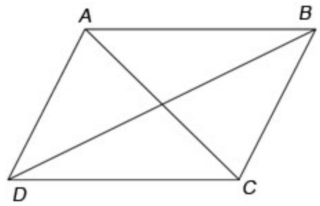
Is it possible to use complex numbers method in plane geometry in secondary school even for supplemental and elective courses in the 9−11th grades? We suppose that the answer is "yes". In fact, in the 9th grade our students already have some knowledge of vectors, their coordinates and operating rules. On the other hand the following terms of complex number theory are enough to set using complex numbers method in plane geometry:

— complex numbers plane;
— complex number's trigonometric form;
— complex-conjugate numbers, sum, product and quotient of complex numbers;
— complex number's vector interpretation and geometric meaning of sum, product and quotient of complex numbers (homothetic tern);
— formula for line segment division in given relationship and condition of three points lie on one line;
— line segment length formula and vector's scalar product formula in complex number's form.

These terms are not difficult for secondary school students and it's enough to discuss several interesting geometric tasks. We can note, that using school geometry ordinary methods for these tasks are more difficult.

Our first task deals with the well-known metric correlation in parallelogram: the sum of square length of its diagonals is equal to the sum of square length of all its sides. This correlation is elementary proved using the cosine theorem.

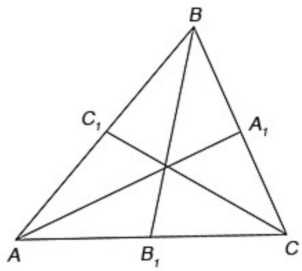
Task 1. Prove that for $ABCD$ parallelogram is true that $AD^2 + BC^2 = 2(AB^2 + BC^2)$. Let $AD = d1$, $BC = d_2$, $AC = a$, $AB = b$, angle $ACD = \alpha$, then $d_{12} = a^2 + b^2 - 2ab\cos\alpha$, $d_{22} = a^2 + b^2 + 2ab\cos(\pi - \alpha)$. To prove our correlation it's necessary to sum up these equalities: $d_{12} + d_{22} = 2(a^2 + b^2)$.



But in connection with this task it becomes interesting: is there a similar correlation for arbitrary quadrilateral? The answer is "yes", it's a well-known fact, that for arbitrary quadrilateral the sum of square length of all its sides is equal to the sum of square length of its diagonals and quadruple square length between bisecting points of diagonals. This correlation can be proved by complex numbers method, that is easier than by other methods.

Task 2. Points M and N are the bisecting points of diagonals AC and BD of arbitrary quadrilateral $ABCD$. Prove that $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2$. Let's to designate point's complex coordinates as small letters. As $m = \frac{1}{2}(a+c)$, $n = \frac{1}{2}(b+d)$, than: $AB^2+BC^2+CD^2+DA^2 = (a-b)(\bar{a}-\bar{b})+(b-c)(\bar{b}-\bar{c})+(c-d)(\bar{c}-\bar{d})+(d-a)(\bar{d}-\bar{a}) = 2(a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d})-(a\bar{b}+\bar{a}b+b\bar{c}+c\bar{d}+\bar{c}d+d\bar{a}+\bar{d}a)$, $AC^2 + BD^2 + 4MN^2 = (a - c)(\bar{a} - \bar{c}) + (b - d)(\bar{b} - \bar{d}) + 4(m - n)(\bar{m} - \bar{n}) = (a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d}) - (a\bar{c} + \bar{a}c + b\bar{d} + \bar{b}d) + (a + c - b - d)(\bar{a} + \bar{c} - \bar{b} - \bar{d}) = 2(a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d}) - (a\bar{b} + \bar{a}b + b\bar{c} + \bar{b}c + c\bar{d} + \bar{c}d + d\bar{a} + \bar{d}a)$.

Task 3. Prove that for an arbitrary triangle the sum of its medians square length is equal to $\frac{3}{4}$ of the sum of all its sides square length. Use the median's length formula for our first proving way: $4m_a^2 = 2(b^2 + c^2) - a^2$, than $4AA_1^2 = 2(AB^2 + AC^2) - BC^2$, $4BB_1^2 = 2(AB^2 + BC^2) - AC^2$, $4CC_1^2 = 2(AC^2 + BC^2) - AB^2$. To prove our correlation it's necessary to sum up these equalities: $AA_1^2+BB_1^2+CC_1^2 = \frac{3}{4}AC^2 + BC^2 + AB^2)$.



And there is another proving way based on complex numbers method.

Let's to designate point's complex coordinates as small letters. As $c_1 = \frac{1}{2}(a+b)$, $a_1 = \frac{1}{2}(b+c)$, $b_1 = \frac{1}{2}(a+c)$, than: $AB^2 = (a-b)(\bar{a}-\bar{b})$, $BC^2 = (b-c)(\bar{b}-\bar{c})$, $AC^2 = (a-c)(\bar{a}-\bar{c})$, $AA_a^2 = (a-a_1)(\bar{a}-\bar{a}_1)$, $BB_1^2 = (b-b_1)(\bar{b}-\bar{b}_1)$, $CC_1^2 = (c-c_1)(\bar{c}-\bar{c}_1)$,

We've used the formula of distance between two points on complex plane. And now it is enough to place these equalities under the proving correlation.

In our report we've considered some plane geometry tasks which solutions are based on non-traditional for school geometry course geometric figure conception and special training technique, involving not only analytic and coordinate methods but also complex numbers method. These tasks are simple, but we are of opinion, that complex numbers method has a significant possibilities while using at secondary school geometric courses. In fact, using such terms as vector's collinearity and perpendicularity makes it possible to prove such classical elementary geometry theorems as Newton's theorem, Gauss's theorem, Pascal's theorem, Ptolemaic theorem etc., moreover, it becomes possible to get triangle's equality and similarity terms, than to take on to plane transformations ˘ movements and homothetic transformations ˘ and their compositions and to solve some tasks which are, in our opinion, important to form school graduates' mathematical culture.

Thereby, complex numbers method in plane geometry is another effective way to solve geometric tasks, which extends our possibilities in studying geometry and realize intra-subject connections in school mathematic. We believe, that involving this method in secondary school mathematics elective courses content for school graduates who are interested in mathematics and including this method in mathematical education content in core and specialized mathematical schools and classes.

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DIFFERENTIATION OF LEARNING AS THE MEAN OF ITS HUMANIZATION

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Key words: differentiation of learning; forms of the organization of educational process

AMS Mathematics Subject Classification: 97G40

Abstract. Forms of the organization of the educational process are considered, allowing the teacher to work individually, on groups and with collective of a class and to realize in practice the acceptable models of forming research skills of students (including gifted), choosing certain strategy of their training and the developments exercised through administration by the teaching and research activities of schoolboys.

One of the modern trends in education is its humanization. It is intended to increase attention to the personality of each student, systematically connected to the educational process as a subject. Turning to the student, his individual personal development leads to a change in the style of education towards full compliance with the needs of the individual.

When we say that the student has become the subject of the educational process, we sometimes sly a bit, because, traditionally in accordance with methodology of teaching, during the real learning process he has often no opportunity to participate in the development of the content of educational program material or to choose an appropriate topic for his interests, etc. Therefore, it remains only to use the principle of perspectives in the development of basic mathematical concepts, ideas of teaching of mathematics courses at high school and direct the learning process to increase the level of students' subjectivity. It can be realized if the student clearly knows when he begins to learn the new material and what issues previously covered are connected with this material. So he must be psychologically prepared for the development of the course and in some cases the student can be a collaborator of process.

Thus, this eternal problem has led to the development of appropriate forms of organization of educational activity. It's relatively independent didactic task that needs a special consideration.

Most experts in the field of didactics agree that the form of training activities includes the following features: the nature of the relationship between teacher and students in class, the grouping of students, the nature of their activities, etc. Forms of training (according to I. P. Podlasiy) are the outward expression of a coherent work of the teacher and students. This work is carried out in a certain order and mode.

The well-known teaching methods specialist, academician RAE Y. M. Kolyagin determines the forms of learning as a way of organizing the learning process. Besides the common forms of learning (the appointed class, class-group, laboratory and practicum), he also highlights problem and differentiated forms of learning. However, we can say that the same form of education can hide different ways of organizing the cognitive activity of students using both teaching methods and the methods of self-study of the subject.

Using different methods and forms of learning activities can gradually and systematically develop students certain skills, which help them to carry out independent search and research activities.

The methodology of teaching mathematics of our country pays attention to the importance of formation following skills:

- The ability to identify and formulate educational problems;
- The ability to select, evaluate and use information related to specific learning problem;
- The ability to formulate and propose a plausible hypothesis;
- The ability to evaluate and validate the hypothesis put forward by them and their conclusions or generalizations;
- The ability to plan the activity and realize it according to plan, etc.

It is obvious that all of these skills are the structural elements of learning and researching skills that are generated in the learning process. Analysis of the structure of the educational process has shown that in practice, even during the ordinary lesson we can notice clearly defined individual, group and collective ways of organizing of the teaching of mathematics. But learning is more effective if we use the differentiation of students in compliance with their abilities. It's effective to offer them appropriate differentiation of levels of the tasks.

As the problem of development of all children is seen by us in the context of educational activities in the village school, where classes are rather small, and the differentiation of students into groups of three levels: sufficient, medium and high. In accordance with the levels of students, it is advisable to give the appropriate level of tasks. For example, a class VIII, after studying the properties of the true number of inequalities, it is important to give the task (for proving of the truth of inequality) in accordance with the level of groups of students:

- Without calculating the values of expressions, where $\sqrt{a}$ any non-negative number prove that $\frac{3+7}{2} \geq \sqrt{3 \cdot 7}$.

— Prove that TODO $\frac{a+b}{2} \geq \sqrt{ab}$ with $a \geq 0$ and $b \geq 0$ using one of the characteristics of the true number of inequalities (exponentiation).
— Prove that TODO $\frac{a+b}{2} \geq \sqrt{ab}$ with $a \geq 0$ and $b \geq 0$ without squaring both sides of the inequality.

However, this task should be completed by the collective compilation, which done by the group of "high level".

We believe that, in the appointed class the organization of learning math for teacher is complicated, and therefore the work with three groups separately and then summing up the results done with all the students can lead to a sense of personal involvement in the proof of the inequality of Cauchy by each student. Then, this conclusion—property about the squaring both sides of the inequality into natural power formulated verbally and it becomes easy to remember. This way of training allows you to get round the "method of adjustment to the average student" and does not hinder the development of the most talented group of "high level" and pull up the weakest students from the "sufficient" group.

However, social demand is to develop creative persons without changing the form of education. This situation forces researchers to look for ways to provide additional educational support for children of all levels and to realize their creative potential in the classroom.

Thus we achieve our main goal of education and development of the creative person. It's individualization of learning. But it becomes possible only when the organizational form of the class system is changed. That is we should organize the problem approach of learning in conjunction with conducting educational research. Unfortunately, the lack of training time makes even the teachers of high professional level of training practically put individualization of learning into practice in extracurricular activities. It can also be organized during elective and in the centers of pre-university education at the universities).

The appointed class-system is in the tradition of the national school; so we should create the conditions for organizing advanced communication of students allowing them to develop leadership skills (at least among the gifted). It's effective way of personal development of each student.

A collective form of education and upbringing (which uses four methods of training offered by V. K. Dyachenko: individual, pair, group and collective) is among the most promising models of training activities. It's used in order to develop gifted children in comprehensive school. The main method of organizing the collective learning of mathematics is the method "all teach each and everyone teaches all." During the experiment we noticed that in a collective way of learning, gifted children get a special impetus to their development, as they have to teach others more often, passing their own experience to the "weak." And the "weak" students are also developing rapidly, as they report to their group about their educational achievements, justifying their actions and thereby acquiring the skills of self-improvement, as well as having targets to achieve the high level of development. More significant achievement of this lesson is mutual learning and efficient management of learning (teacher support).

The experimental work with this model in the system "gifted child in comprehensive school with a small contingent of students" shows that each pupil can show himself as a leader in the group. And it is a good incentive for further self-development. It allows you to take an important step away from class to class–laboratory with the organization of students' researching. So, the experience of student, his studies associated with the observation, experimentation and research become relevant. Lessons become serious and enjoyable, individual and collective, free and planned.

Researches of the development of individual forms of training are held by many specialists and creative teachers. However, many researchers are inclined to believe that we must develop individual training plans and programs. There are also some proposals for making the plan of action for teachers:

— Determine the level of the child;
— Outline short-and long-term goals and the ways of achieving them;
— Determine the time to learn each theme and individual program in whole;
— Determine the involvement of parents in monitoring of the implementation of individual tasks;
— Identify methods and criteria for evaluating the success of the student.

The work of a tutor-professional is an example of the organization of individualization of learning. However, as we noted earlier, this personalization has a serious drawback: it does not include the work of a student in the group and his responsibility to classmates, as well as mutual learning students (sometimes the students explain to each other more accessible, paying attention to the mistakes which the teacher does not notice). These disadvantages of tutoring method may adversely affect the results of students.

There are some different points of view on the definition of a "form of organization of learning activities of students." But it should be noted that the main feature of this definition is the outward expression of a coherent work of the teacher and students. It's important to develop gradually and systematically the necessary studying skills of students when we use different methods and forms of organization of training activities. All of these forms of educational process allow the teacher to work individually, in groups and with the whole class. It is rather typical for the village schools with a small contingent of students in the classroom. In fact, they are three levels of process which manage the formation of learning skills. In this case it is always important to use positive communication with fellow students and a teacher, which contributes to the development of individual leadership skills, which are not adequately developed at the individual forms of learning. However, each student receives homework which requires the solution of some educational problems. So students often need personal assistance with the creation of conditions for self-initiative and creativity.

In conclusion, we note that in the experiential learning through various forms of training activities of students we were able to create conditions for using different forms of learning and enhance learning activities for each category of students by providing the support for their educational activities. An important criterion for the quality of education is the formation of specific skills especially research skills. The significance of the formation of research skills is that their formation is accompanied by such qualities as intellectual independence, boldness of thought, willingness to volitional tension and focus on creativity.

In studying of the creative act, psychologists distinguish phases of creative mathematics problem solving. For example, researchers identify the following phases:

— The phase of collecting the material that could form the basis for the decisions or the reformulation of the problem;
— The phase of incubation, when the subconscious works basically but on conscious level a person may engage in quite different activities;
— The phase of inspiration, insight, or when a decision is often a sudden and it appears in consciousness;
— The phase of control or test, which requires the full involvement of consciousness.

In the creative act of learning issues the phase of collecting material also includes the necessary updating of knowledge that could form the basis for the solution to the problem. Without the implementation of Phase I and updating of existing knowledge we can say neither about the phase of "incubation" nor the phase of "insight." The phase of control is implemented with the full inclusion of consciousness.

Consider these phases in the solving of the following problem: "Prove that among all the triangles having a common side and equal angles, which are opposite this side, an isosceles triangle has the greatest perimeter."

Students think about the drawing (the model of the problem) at the phase of the analysis the conditions of the problem. In order to match the content of the task and the drawing, you need to remember when we meet the equal angles in studying. The most commonly used sentence is "angles, based on the diameter of a circle are equal." There are no right triangles in the task, therefore, the students should think of more general mathematical sentence—"all the angles inscribed in a circle are equal, if they base on the same arc of a circle." In this phase, the subconscious has already connected and the students realize that we are talking about inscribed angles, based on the same arc, which is contracted by the chord of a circle, which is also the basis of triangles (included in the perimeter of the compared triangle). Figure 1 Appears

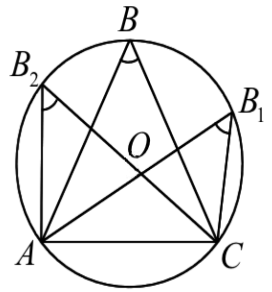

Figure 1

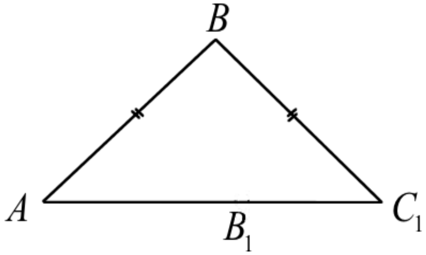

Figure 2

Next, students continue to gather facts. The most active students start to compare the perimeter of an isosceles triangle $\triangle ABC$ with perimeter $\triangle AB_1C$ and $\triangle AB_2C$ and notice that the task must have a solution because the measurements of the perimeters of triangles confirm that the requirement is satisfied if one triangle from a pair of triangles is isosceles. Other students don't measure the length of the perimeter as the sum of its three sides; they draw two sides of $\triangle ABC$ and $\triangle AB_1C$ drawing an arbitrary straight line from one point. As a result they come to the conclusion that it is sufficient to consider only the sum of the sides to compare the perimeter, as the base of the triangles is common and includes in the perimeters of the compared triangles and the base does not affect the values of their perimeters. This conjecture is promoting the appearing of only 2 triangles $\triangle ABC$ and $\triangle AB_1C$.

The second conjecture is closer to the idea, which is the base of plan of solving the problem. The teacher may recall the attenuation of search activity saying that the sum of the two sides can be compared with the help of proper correlation between the sides of the triangle, where the smaller sum is the side, and the large sum is equal to the sum of two other sides. It's useful to help students get Figure 2, in order to understand the way of the comparison of the perimeters, where it is obvious that $AB + BC > AB_1 + B_1C$. But that's not the proof, but only a confirmation of approval. It's only the checking of the lengths, as we don't use the information that the angles are equal at the vertices B and B_1. After this, the most capable students offer to move Figure 1 into Figure 2 as an additional drawing. This proposal is encouraged by the teachers and the students draw Figure 3.

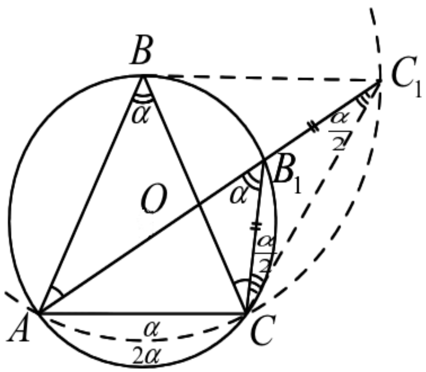

Figure 3

However, it's difficult for students to prove that $AB + BC > AB_1 + B_1C$. Returning to Figure 2, they realize that the point C_1 can be obtained as the intersection of the beam AB_1 and a circle with center B and radius AB. Then, denoting $\angle B = \angle B_1\alpha$, we see that AC arc of a circle with the center at O is equal to 2α, and AC arc of a circle with the center at B is equal to α (because ABC becomes a central angle), and AC_1C becomes inscribed in a circle centered at point B, so $\angle AC_1C = \frac{\alpha}{2}$. Having considered the $\triangle B_1C_1C$ and, using the property of the outer corner of the angle AB_1C which is equal to α, we find that $\angle B_1CC_1 = \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}$. Consequently, $\triangle CB_1C_1$—isosceles and $B_1C_1 = B_1C$, so $AC_1 = AB_1 + B_1C$. So $AB + BC_1 > AC_1$ (by the property of the sides of $\triangle ABC_1$), and therefore $AB + BC > AB_1 + B_1C$, so $P_{ABC} > P_{AB_1C}$.

If you plan to solve the track again, the students come to the conclusion of other methods of proof. It happened in practice, one of the students began to tell about his path: he proposed to draw two concentric circles centered at a radius $r_1 = AB + BC$ and $r_2 = AB_1 + B_1C$. However, in discussing this way, the students did not agree with the proposed solution because it can not be considered the proof in general, as it cannot use the equality of the angles at the vertices B and B_1.

This example of solution shows that the debate of a teacher and students is similar with scientific research, when each new statement must be logically justified. Therefore, in the psychological and educational literature it's widely known the attempts to teach students to research.

Implementing in practice the acceptable models of forming research skills of students (including gifted) and selecting certain learning strategies and the development it's necessary to manage the teaching and research activities, following the phases of general scientific research. This control is much easier if the class is divided into groups according to the levels of their abilities.

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ELEMENTARY METHODS OF PROFOUND STUDY OF MATHEMATICS BY S. M. NIKOLSKY COURSE "ALGEBRA AND THE BEGINNINGS OF THE MATHEMATICAL ANALYSIS. 10–11 CLASSES"

S. M. Nikolsky, A. A. Rusakov, V. N. Rusakova

Key words: profound study, mathematical analysis, elective courses.

AMS Mathematics Subject Classification: 97U20

Abstract. In performance some methodical features of teaching of mathematics at the textbook written by S. M. Nikolsky "Algebra and the beginnings of mathematical analysis. 10–11 classes" in profile mathematical classes and classes with profound studying of a subject.

Figure 1. Discussion: A. A. Rusakov, S. M. Nikolsky

20 years ago our school mathematics was on one of the first places in the world. As the reason for it the successful domestic organization of training serves the mathematics, based on a logic principle, and maintenance of necessary time for training to the mathematics under school curricula. Long-term practice has shown that mastering of a school course by the pupil with average abilities demands dialogue

of the pupil with the teacher, at least, 1 hour per day per each class, i.e. 6 class periods a week. Recently the Ministry of Education of the Russian Federation is occupied by an infinite train of reforms of school education. Unfortunately, it is necessary to ascertain that these "reforms" objectively reduce value of mathematics in school education of our country. Already reached level of mathematical formation obviously decreases.

Nowadays we have to state the reduction of mathematical education level. To give a schoolboy the sufficient mathematical base to continue learning can help elective courses. Especially if it is a question of specialized mathematical classes, teaching mathematically gifted students (as the last we will understand the pupils, capable to make mathematical calculations, i.e. skillfully to transform difficult alphabetic expressions, to find successful ways for the decision of the equations which are not approaching under standard receptions and rules; possessing good geometrical imagination or geometrical intuition; owning art of the consecutive correctly dismembered logic reasoning; a success in the mathematician).

Figure 2. S. M. Nikolsky "Algebra and the beginnings of mathematical analysis. 10–11 classes"

Elective courses are obligatory for attendance courses at the election of pupils as a part of the profile of education in senior high school. The feature of elective courses is that a pupil can choose from the proposed set of courses those ones he is interested in or need. Once a course is chosen it becomes normative, that is a pupil is obliged to attend it and perform the proper term paper (pass a test, to defend the project, etc.). At the same time, they suggest training of senior high school students in small groups of interest, aspiration, potential and they are directed to meet their individual needs, to develop their abilities. In 2010 the publishing house "Enlightenment" in series of "Elective courses" published a textbook by S. M. Nikolsky "Algebra and the beginnings of mathematical analysis. 10–11 classes", the work on which have been tirelessly and successfully continued for the last few years.

"The condition of my sight any more hasn't allowed me all work on preparation of the book for the edition to spend most. Therefore Alexander Aleksandrovich and Vera Nikolaevna Rusakovs helped me with a writing and typing, selection of exercises, etc.

Despite the extreme old age, I all the same am engaged in sciences today. And in the morning I was engaged in sciences. In practice, even if there will be with me something, all the same I will already leave such records which can be published which represent, from my point of view, essential interest. I have ceased to teach, but also now I am engaged in school affairs, school textbooks. Both now prepares and already for certain we will prepare the second book of algebra and the analysis beginnings. Though to teach I has ceased, but me recently school affairs very strongly interest. . . "

Nowadays S. M. Nikolsky coauthored with A. A. Rusakov and V. N. Rusakova continues to work on the second part of the book—a task book with the solutions and guidelines.

Despite the absence in the already published book "gimmicked" (as defined Sergei Mikhailovich) tasks of increased complexity this textbook, no doubt, can be used not only for studying the program material for School Mathematics, but also for a deeper acquaintance with individual themes that fall beyond the scope of it. Complex numbers, approximate calculations, differential equations, Taylor's formula, Chebyshev's polynomials—all these and other issues of higher mathematics are stated in the manual at a level accessible for a pupil, with usage of simple, understandable language, having kept sufficient mathematical severity. Thus from the pupil free possession of a native language—only is required then the accurate analysis of the resulted data and problems, clearness of understanding of a stated material is possible.

Distinctive feature of this course is the unique style of Sergei Mikhailovich in the presentation of teaching material and also the system of tasks matched to each

Figure 3. Group of authors: S. M. Nikolsky, A. A. Rusakov, V. N. Rusakova

of the theoretical issue. Pupils are not given only bare facts—every new concept, a new formula a pupil is forced to "find" (in the text) himself. Only having read all the material of the given section and thus having leant the logic of the above proof (a kind of mini-research) we can allocate necessary idea for the solution of the above tasks. Such a methodical "zest" obviously is the style of S. M. Nikolsky's teacher—the famous mathematician A. N. Kolmogorov, according to whom methodical and mathematical situation, naturally assigns a schoolboy in a position where he is mastering the new training material, not only collects the amount of knowledge, but in the process of successful solution of some system of tasks, it is formed an impression that he solved mathematical task by himself, the task that according to its complexity is available to students-mathematicians. The most important thing in this case, as noted by Andrey Nikolaevich, is that a pupil can carry this feeling of success and necessity throughout his life.

Thus, the manual promotes an intensification of independent work of schoolboys on mathematics studying in wider, than general educational, volume. The problem of the teacher at a management of independent work of pupils—to help them with the rational organization of the work, to impart skill of deep considering of tasks at which are combined persistence at movement to the purpose,—to a right decision finding,—and the flexibility necessary for a choice of several possible ways of performance of the task. In connection with the developing potential of the book, such

Figure 4. S. M. Nikolsky in the study room of his teacher A. N. Kolmogorov

approach provides deep mastering of various methods of the decision of mathematical problems at increase of level of independence of the schoolboy in acquisition of these skills.

Figure 5. S. M. Nikolsky discusses with teachers about problems of the Russian mathematical education

Deeper studying of a subject is promoted also by presence enough the developed, full proofs, allowing to achieve more realized relation of pupils of mathematical classes to carrying out of proofs and, in wider plan, clearness of understanding them of structure of the mathematical theory, a role of proofs in it.

Figure 6. The best educational edition on the mathematician

The material is presented in such a manner that training begins with the most simple and evident concepts, and gradually it passes to uncommon reasonings,— promotes development of mental faculties of the schoolboy and its logic thinking, display of its interest to a studied subject, intuition strengthening in studying of the various phenomena and to formation of the firm approach to studying of daily practical activities.

The textbook by S. M. Nikolsky "Algebra and the beginnings of the mathematical analysis. 10–11 classes"—the winner of the First All-Russia competition of Scientifically-methodical council about mathematics of the Ministry of Education and Science of the Russian Federation "The best educational edition on the mathematician" 2010 in a nomination: "Mathematics at high school. The additional literature".

Thus, the elective course offered by Sergei Mikhailovich can serve both as a good tool for better mastering the basic skills in mathematics and for the development of natural and scientific endowments of pupils.

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SOLUTION OF DIFFERENTIAL EQUATIONS AS MEANS OF PROFOUND STUDY OF MATHEMATICAL ANALYSIS AND THE VARIETY OF SUBJECTS FOR A PUPIL'S RESEARCH

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Key words: differential equations, in-depth study of mathematical analysis, pupils research

AMS Mathematics Subject Classification: 97I70

Abstract. This article reveals the didactic possibilities of studying the subject "Differential Equations" in-depth study of mathematics, including its potential to develop in pupils the skills of research.

1 Introduction

Historically, in the high school course of the algebra and mathematical analysis includes a subject "Differential Equations", which introduces students to the concept of differential equations, often on an intuitive level, with a few examples, such as harmonic vibrations, the laws of motion of the body, radioactive decay, etc.

We emphasize that the study of this subject in a general education course is more for informational purposes and is a consolidation of the basic skills of the basic elementary functions integration and differentiation.

However, the wide differential equations application, its use to describe a wide variety of real processes requires a more detailed acquaintance with them. Certainly, the objective complexity of most of these problems can not enter them in the general education mathematics course. But at the core classes and classes with intensive study of mathematics can give them a little more attention, and for students, who are interested with this issue, organize elective course, and even offer them a topic for research in this area.

What is the attraction of studying the subject at school?

In addition to the above-mentioned fastening material on the basic elementary functions differential and integral calculus, the use of these functions properties (without the knowledge of the mathematical analysis general course can not learn integrating differential equations technique), we can note the diagnostics and training of the next significant ability of students. Possession of equivalent transformations of the various algebraic and trigonometric equations is very important for the graduate. This is one of the basic skills that determine the quality of student mathematics education. There are often encountered errors associated with the loss and acquisition of extraneous roots. So, just differential equations provide an extensive field for practice in this regard.

Equally important, that other school mathematics topics cannot as widely implemented interdisciplinary connection, to open its applied trend. Differential equations are the most effective means of solving science and technology problems. They are widely used in physics, chemistry, biology, geometry, economics and other scientific and practical fields. Many real-world processes by means of differential equations are described a simply and fully. Many real processes by means of the differential equations are described simply and full.

True, this is connected and the main difficulty of introducing the subject in the school – students theoretical knowledge in these areas is often not sufficient to produce a differential equation, while the simple equations solution are not backed up by practical content, significantly reduces the students' interest in this subject.

But every such problem is the full research task for the schoolboy. To solve it, he requires not only insight into the process described in the problem, but also shows considerable ingenuity. It is equally important mathematical modeling of the real situation skills.

The research activities of students – the first of all the formation of the conditions in which students will receive new impulses:

- for more in-depth mastering of the educational program;

- for the development of advanced training;

- for motivation to develop their own educational product;

- for step by step moving a student from the object through the role of the creative subject and teaching roles for their classmates;

- to identify the subjective novelty of the this activity result and the its execution process (subjectivity is that the results are completely new to the student);

- to conduct their own research, which sometimes end with a new result or discovery in mathematics (with a further publication in a scientific journal); - for understanding unsolved problems and familiarity with the problems in mathematical knowledge.

Certainly, bringig the student to solve such problems should be gradual.

Stage 1. Consideration of the problems that lead to the compilation of differential equations what is motivation to do it. Determination of differential equations and basic concepts associated with it.

Stage 2. This stage consists in an acquaintance with various kinds of the differential equations and methods of their decision. You can give an idea of uniqueness and existence of the solutions.

Here student is receiving a strong core skill to solve differential equations. Do not want to miss this step and immediately offered to students to solve problems with practical orientation, for each task – making equation and its classification and selecting the method of solution – is complex. So it makes sense to train students in the first solution of the equations and then to complicate the problem.

Stage 3. Training in the differential equations development and solution (the theoretical framework must be well-known to student.)

The differential equation development is not a simple task, as a universal method applicable to different situations, there is currently no [2]. So, before you invite a pupil to solve the problem, consider the examples in their preparation, offering solutions for self-similar set. Only acquired some working experience on a template, the student will be able to continue to move away from it and try your hand at solving new problems.

This is the scheme of drawing up and the decision of the differential equation:

1. Carefully read the conditions of the problem. What can be taken as an argument (independent variable)? What can be taken as a required function? If necessary, make a drawing.

2. Find out the meaning of the derivative of the required function (In that case when the derivative has concrete sense). The understanding of it, can a little simplify a stage of search of correlation between function and its derivative.

3. Define the initial conditions (the function value at a fixed argument value).

4. Write down the relation (equation) which describes the relation between the derivative of the required function and an argument between the differentials of the variables or, if the derivative has no specific meaning. The equation relating the differentials dy and dx should be based on the known laws of mathematics, physics, chemistry, etc.

5. Determine the type of the resulting differential equation and select an appropriate solution method.

6. Integrate the differential equation – determine the equation general solution.

7. Find a particular solution of the problem - the required function with the initial conditions.

8. Inspect. Explore the resulting law in limiting cases and study the dependence of solutions on parameters.

Stage 4. Student is invited to research the problem.

At this stage, the student must have their own select and explore the theoretical material in the field of knowledge, defined by the problem. Problems can be allocated in accordance with the inclinations and interests of students. Student may also be invited (unless, of course, there is an agreement) to consult with subject teachers. Next, he will act on the scheme already known to him, but in a qualitatively new situation for themselves.

Here are a few examples of problems that can be offered for school research projects on differential equations.

1. Cylindrical tank with a vertical axis of 6 m of the height and 4 m in diameter has a circular hole at the bottom, radius of which is equal 1/12 m. It is required to establish the dependence of the water level in the tank from time t, as well as to determine the time during which all the water will flow out.

2. Two liquid chemical substances A and B of 10 and 20 liters respectively during the chemical reaction form a new liquid chemical substance C. Assuming that the temperature during the reaction does not change, as well as that of every two volumes of A and one volume of the substance B formed in three volumes of the substance C, to determine the amount of C at any given time t, if for 20 minutes to form a 6 liter of the substance C.

3. The destroyer is hunting for a submarine in a thick fog. At some point in time mist rises and the submarine is detected on water surface at a distance of 3 miles from the destroyer. The destroyer speed is twice the submarine velocity. Required to determine the trajectory (chase curve), to be followed by the destroyer, so it went right under the submarine if the submarine sank immediately after its detection and went at full speed on a straight course in an unknown direction.

These and other interesting tasks can be found in [1].

It should be understood that offering a student working on a particular research problem, the teacher takes on some responsibility. If the wrong approach to teaching pupils in addition they may lose interest in the project. The teacher must accurately determine the level of mathematical training of students, the degree of the necessary mathematical material to solve the problem. In addition, it is necessary consider student personality: the speed and completeness of the learning of new material, its degree of complexity that is accessible to the student, the presence of the student researcher traits - curiosity, patience, ingenuity, etc. [3]-[4].

Question 1. **How to choose a problem feasible to the pupils?** As more often sources for the pupil at work on a problem scientific articles and highly specialized books serve. The pupil is compelled to master a material new to it. Quite often the whole years leave on it. Happens so that, even independently having solved this or that problem, the schoolboy not up to the end understands value of some used terms or concepts.

Difficulties of this stage can negatively affect the decision of the pupil to continue the begun research. The problem of the teacher consists in seeing this moment. In time to prompt, direct, explain, probably, reformulate a problem (temporary or definitive, depending on possibilities of the schoolboy), to create a success situation. How ...?

Question 2. **How to make so that the pupil hasn't stopped to be engaged in a research theme (hasn't lost interest to mathematics)?** The first experience of research work of the schoolboy cardinally differs from those kinds of activity which were offered it earlier at the lessons. Difficulty and narrow prevalence, statement the problems offered for the schoolboys for the abstract work can create for the pupil illusion of reception during research of objectively new result. Happens difficultly enough to convince the schoolboy what that it has made – yet opening that it only approaches to the present research. Many pupils can't overcome the given stage, stopping on the reached subjectively new result.

Question 3. **How to convince the pupil not to stop on the reached?** Being fond of work on own research, participating in various conferences of schoolboys, occupying on them prize-winning places, hearing positive responses for the work from leading scientists-mathematicians, pupils overestimate the abilities, the knowledge in a subject. Far having promoted in the decision of the problem, they start to look at the school course of the mathematics seeming to them simple in comparison with done research haughtily. The failure at lessons and even at examinations because of elementary errors can become result of such relation.

Question 4. **How to motivate studying of a general educational course of mathematics for mathematically creatively gifted schoolboys successfully working over a difficult mathematical research problem?** Work on the project, reading of the serious scientific literature; attempt to understand difficult concepts – not always successful – lead the schoolboy to a condition when he suddenly understands: «I know nothing, I am able nothing». It appears that everything that it has learned for school days – insignificant remains in comparison with huge volume of the knowledge which has been saved up by mankind. And with this conclusion it leaves in adult life. And this too affects results of examinations. It is frightened by an element of competitions, present on introductory tests after all he «knows so a little».

Question 5. **How to overcome fears (complex) of the schoolboy who has seen immensity of Knowledge?**

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PROBLEMS OF TEACHING MATHEMATICAL ANALYSIS IN CONTEMPORARY SCHOOLS

T. Yu. Ryabova

Key words: classes with specialization, mathematical analysis, the competence approach, modular teaching

AMS Mathematics Subject Classification: 97B20

Abstract. In this article the author speaks about fundamental problems in the study of mathematical analysis in contemporary schools and suggests a competence-oriented technology of teaching that provides applicative and practice-oriented system of mathematical analysis course.

Contemporary school education in Russia is going through a difficult phase in its evolution. Implementation of the second generation educational standards, intensification and technologization of educational process, usage of modern means of communication by students in the learning process and outside it — all of it requires of changes and improvements in teaching techniques that are being used in schools nowadays.

Mathematics has traditionally played an important role in school education, being one of the main disciplines studied from the first till eleventh grade. Intensive personality formation process of school children, formation of their general learning skills and ways of their intellectual activity, inculcation of logical thinking and system approach to solving problems at various levels — this is exactly what happens during math classes.

Number of hours to study mathematics has been reducing in the curriculum almost constantly. It has to be compensated by changes in methods of teaching and learning aimed at the intensification of educational process, as well as an increase of independence and responsibility of students.

Mathematical analysis is one of the parts of school mathematics course, which provides a link between school and university mathematics and forms a communication between disciplines, which introduces school children — future university or college students — with methods of data and information analysis generated in real life.

The structure of schooling in contemporary Russia involves a mandatory specialization of education in upper classes as a primary method of education. In this context, the mathematical education of students who, in the nearest future, aim to become proficient in mathematics, physics or natural science, must comply with current requirements and provide an opportunity for effective learning at leading universities in Russia and globally. It also must provide students with a feeling of professional fulfillment in the sector of economics of their choice.

The change in the technology of teaching mathematics is based, primarily, on changes in learning objectives, which in this context can be represented as follows:

- Teaching mathematical language to students, allowing to form their skills of data mathematization and applying mathematical instruments as a rational method of solving real problems
- During the process of learning the principles of mathematical analysis, provide a development of learning competences at the level of functional literacy, which is characterized by the mastery of cognitive tools necessary for further activity in various fields, as well as the implementation of connections between disciplines
- Provide a mathematical competence on research level, which is characterized by achievements in students' practical ability to participate in creative activities in mathematics
- Provide the formation of students' readiness for professional development in the field of mathematical activities, characterized by achievement of a certain mathematical level which is necessary to continue their education in the chosen research area in high school
- Provide the formation of students' information analysis skills and culture of communication, which is necessary for the implementation of mathematical activity in contemporary interactive media tools, such as computer technology, Internet and other

Changing the goals of mathematics education inevitably leads to changes in the student's results of mathematics learning. In the same way that everyone wants to be treated by a competent doctor, get advice from a competent lawyer or live in a building built by a competent architect, the society wants to see competent graduates who have received a specialized education and must meet certain criteria proposed by the society.

In our study we developed a structure of the student's mathematical competence, identified its levels and diagnostic methods for an evaluation of this competence. The following structural components of mathematical competence have been identified:

- fluency with the mathematical language;
- functional mathematical literacy;
- research competence;

— willingness and ability to use mathematical knowledge in future professional activities;
— information and communicative culture.

Each structural unit is represented in this study by three levels of description: elementary, functional and creative.

Currently there are rather traditional schemes for creating content of the math courses which were defined during the formation and development of the Soviet economic system. The content of mathematics education met those requirements which were brought by the society. Due to changes occurring around us, it would be logical to make some changes to the content of mathematics courses. First of all, the content of math education should vary depending on the goals set by students. If a student does not plan to educate himself further and does not require an advanced level of mathematics for that, then his educational content should be limited by the level of general cultural competence. If the student expects to receive education which requires a higher level of background, then the content of his mathematical education should correspond to the levels that are determined by either specialized or advanced levels of mathematics.

In terms of a competence-based approach, in classes where mathematics is studied at a specialized level, it is necessary to preserve the traditional set of topics, but to change the structure of hourly split-up of educations process, giving preference to those meaningful moments that allow to form the named above competencies. For example, when studying the topic of "The derivative and its applications", more attention should be paid not to technical aspects of teaching methods and techniques of differentiation, but to applying derivative instruments to approximate calculations, to explaining the geometric and mechanical meanings of the derivative, to reading and making diagrams, to solving optimization problems of various areas of human activity, to using derivatives in non-standard solutions of equations and equations with parameters. It is necessary to ensure the formation of a logical understanding of the concept of derivative, to ensure incorporation of the concept into a common conceptual system of mathematics.

As for the structure of the content in this section, we should agree with a certaing viewpoint, that the student who has chosed an advanced level of mathematics for his specialization, should learn how to use fluently the contemporary math methods which are available at the present stage of knowledge. Since the mathematical profile of learnign is most often chosen by those who plan to use the results of the General State Exam (GSE) in mathematics as a competitive advantage when applying for a higher education (which suggests, this way or another, to link their professional lives with math), it is non-rational to reduce the number of mathematical subjects that are traditionally studied in that classes. In our view, a set of topics of mathematical analysis in this case should be as follows:

— real numbers;
— limit of the sequence and its properties;
— functions and their properties;
— derivative and its application;
— integral;
— differential equations.

This consequence of narration allows for a systematic and logical approach to the reception of this course.

In order to achieve the purpose and goals we have stated, the following educational method targeted at formation of the mathematical competence structure within the students' mindset has been proposed.

Projection of the course of mathematical analysis principles is being designed in accordance to action approach, combined with a modular education methodology, computer, project-oriented and research technologies. Educational process organization is based on a variative scheme, which implies an opportunity to form and realize an individual educational route.

Method of formation of all 5 elements of mathematical competence during the study the principles of mathematical analysis described in our research.

We propose to use modular education as a basic structure of educational process, which allows the students to think and act more independently.

In the beginning of each module (such as "The study of functions using derivatives", for example) each student of a profile class receives topics for projects and research of his choice, as well as some problems to be solved (up to 15). At the end of the module he must present the results of his work on them on a given day. All the tasks are divided into three subgroups, according to the level of mathematical competence, which are: 4-5 tasks targeted at estimating an elementary mathematical competence, as well as 5-6 tasks that require some creative thinking. Among the tasks of the third kind, there must be not only non-standard ones for a particular module, but also tasks from the next module.

Thus, not only are we checking how well the skill to solve standard tasks is being formed, but also provide an opportunity for displaying mathematical talent to those who posess it. Apart from technical part of solving a problem, students are asked to develop a classification of these problems according to various features. For example, 1) by the inherent methodical quality of the solution they use, 2) by the main solution approach (analytical, graphic, synthetic), 3) by creativity. The student may also suggest his own classification system and provide a reasoning for that, which will be taken into account for an evaluation of his work.

Table 1

Topics and competences

Name of the topic	**Formation of mathematical competence as a result of the study of the topic**
Real numbers	Provides the basis for learning professional language of math, forms major functional concepts of mathematical analysis
Limit of the sequence and its properties	Helps to understand the connection between several topics, demonstrates the universal relevance of mathematical approaches, provides a minimal level of mathematical literacy for future learning of mathematical analysis methods
Functions and their properties	Provides the knowledge of basic functional grammar, brings up the realization of elementary practical skills, such as research and construction of functional graphics, allows to develop the culture of dealing with information
Derivative and its application	Provides a basic level of mathematical knowledge and skills of learning the principles of mathematical analysis, provides a propedeutical effect on research skills formation, enables the future application of mathematical knowledge in the area of future professional expertise
Integral	Helps to define the dual nature and mutual irreversibility of mathematical dependencies, increase interdisciplinary connections, develop creative mathematical skills
Differential equations	Provides an interdisciplinary integration, formation of a logical and informational culture, learning the language of mathematics

Our next step is the definition of terminology and concepts that are used to solve a particular mathematical problem and description of genetic connections for one of them. It is suggested that students put all the data in one chart, for their convenience. While doing that, they are allowed to use various levels of help, which is to be registered by an expert.

Since among the proposed tasks there are some involving the terms that are yet unfamiliar for students, but are tightly connected to the ones that have already been learnt, each student has a chance to show their creative capabilities by forming a new concept for themselves, create a new algorithm based on the skills and tools they have already been familiar with so far. By that, we can tell whether the level of mathematical competence of students has changed.

Table 2

Scoring the level of mathematical competence of students

№of stage	**Name of the diagnostic stage**	**Maximum number of points**
1	The solution of 15 exercises, ranging from level 1 to 3	40 points
2	Exercises classification	10 points
3	Working with the concepts, construction of the “tree” concept	15 points
4	The manifestation of creativity	15 points
5	Presentation and defense of the work	20 points
	Total	100 points

The total score is 100 points. If required, points can be translated to a school system of grades. To do so, it is recommended to use the following scale of correspondence: over 30 points — “3”, over 50 points — “4”, over 80 points — “5”. From our viewpoint, a student with 50 points and more can be considered mathematically competent.

Thus, if the proposed method is used to in school education, it will allow schoolboy studying in a specialized class to become mathematically competent, helping him to improve his knowledge and skills and increase his adaptivity at next levels of education.

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A SHORT WAY TO LEARN ABOUT FUNCTION FROM ITS DERIVATIVE

P. V. Semenov

Key words: Lagrange (Mean Value) Theorem, derivative, monotone function, Cantor intersection theorem on nested sequence of segments

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Abstract. We show that the main Rules of Mathematical Analysis for learning about $f(x)$ from $f'(x)$ are direct corollaries of the Cantor intersection theorem. This fact gives a short and correct way for applications which avoids the well-known and standard sequence of classical theorems of Calculus.

1 Introduction

The classical approach for obtaining the basic rules in Calculus:

Rule 1: "If $f' > 0$ over an interval I then f is increasing on I", and
Rule 2: "If $f' \equiv 0$ over an interval I then f is constant on I",

is to derive these Rules from a series of such deep facts in Calculus as Weierstrass, Fermat, Rolle and Lagrange (Mean Value) Theorems.

Unfortunately the new Federal Educational Standards (No.3), as a rule, give no real chance for passing all details in these Theorems on a satisfactory provable level, due to the short of auditorium time. On the other hand, both Rules 1 and 2 are strongly needed for such principal facts of Calculus as finding of bending points, extremal values, checking on convexity, properties of indefinite integrals, Newton - Leibnitz formula, various applications, etc.

To resolve such a contradiction below one can find a reduction of both Rules to the more elementary Cantor intersection theorem on nested sequence of segments.

2 Preliminary

Let f be a function over a segment $\Delta = [a; b]$, l be the line drawing through the points $(a; f(a))$ and $(b; f(b))$ of the graph of f. Denote

$$T(f, \Delta) = \frac{f(b) - f(a)}{b - a}$$

the slope ratio of l . Shortly, $T(f,\Delta)$ is the "tangent of f over Δ".

Let us show that absolute value of the tangent $T(f,\Delta)$ of a function f is nondecreasing with respect to subdivision of the segment. So, pick $a<c<b$ and denote $[a;c]=\Delta_1,\ [c;b]=\Delta_2$. Making the following (well-known in from convexity theory) algebraic trick

$$\frac{f(b)-f(a)}{b-a}=\frac{c-a}{b-a}\cdot\frac{f(c)-f(a)}{c-a}+\frac{b-c}{b-a}\cdot\frac{f(b)-f(c)}{b-c}$$

we see that

$$T(f,\Delta)=(1-t)\cdot T(f,\Delta_1)+t\cdot T(f,\Delta_2),\quad t=\frac{b-c}{b-a}\in(0;1).$$

In other words, $T(f,\Delta)$ is displaced between $T(f,\Delta_1)$ and $T(f,\Delta_2)$. So, if $T(f,\Delta)$ is positive then $T(f,\Delta_1)\geqslant T(f,\Delta)$, or $T(f,\Delta_2)\geqslant T(f,\Delta)$ and $T(f,\Delta_1)\leqslant T(f,\Delta)$, or $T(f,\Delta_2)\leqslant T(f,\Delta)$ for the case of negative $T(f,\Delta)$.

3 The main proof

Now we are ready to obtain both Rules 1 and 2 from the Cantor intersection theorem on nested sequence of segments.

So, suppose that a continuous on interval I function f has the nonpositive derivative f' inside the interval. We want to see that such a function is nondecreasing over the interval.

To the contrary, let $f(a)<f(b)$ for some $a<b$ from the interval and that is why $T(f,\Delta)=\frac{f(b)-f(a)}{b-a}>0$. Due to the continuity of f one can assume that both a and b are inner points of I.

Divide the segment Δ into two equal subsegments and denote Δ_1 a segment for which $T(f,\Delta_1)\geqslant T(f,\Delta)$, see previous section. Divide the segment Δ_1 into two equal parts and denote Δ_2 a segment for which

$$T(f,\Delta_2)\geqslant T(f,\Delta_1)\geqslant T(f,\Delta)>0,$$

etc.

Applying Cantor theorem for the nested sequence $\Delta\supset\Delta_1\supset\Delta_2\supset\ldots$ one can find a point $x_0=\bigcap_n\Delta_n$. This point x_0 is inner point of I and divides each segment Δ_n into two subsegments. Denote ∇_n a segment for which

$$T(f,\nabla_n)\geqslant T(f,\Delta_n)\geqslant\ldots\geqslant T(f,\Delta)>0,$$

and denote x_n the endpoint of ∇_n which differs from x_0. Then x_n converges to x_0, when $n \to \infty$ and

$$f'(x_0) = \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = \lim_{n\to\infty} T(f; \nabla_n) \geqslant T(f, \Delta) > 0.$$

Contradiction with the assumption that $f' \leqslant 0$ inside I.

The proofs for the cases $f' < 0$, $f' > 0$, $f' \geqslant 0$ are quite similar. Rule 1 is proved.

As for Rule 2, working once again on contrary, suppose that $f' \equiv 0$, but f is not a constant function. It means that $T(f; \Delta) > 0$ or, $T(f; \Delta) < 0$ for an appropriate subsegment Δ inside I. Repeating the above construction one can easily find a point x_0 with $f'(x_0) > 0$, or with $f'(x_0) < 0$. Contradiction with $f' \equiv 0$. Rule 2 is proved.

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VII. History of analysis

(Sessions organizers: S.S. Demidov, L. Pepe)

GIUSEPPE VITALI: RESEARCH ON REAL ANALYSIS AND RELATIONSHIP WITH POLISH AND RUSSIAN MATHEMATICIANS

M. T. Borgato

Key words: Giuseppe Vitali, Measure and integration, BV functions

AMS Mathematics Subject Classification: 01A60

Abstract. Giuseppe Vitali (1875-1932) was one of the most interesting protagonists of Italian and European mathematics in the early twentieth century. His most significant production took place in the field of real and complex analysis over a short period: the first decade of the century. It was then that he achieved success in demonstrating some remarkable results, such as the necessary and sufficient condition for Riemann integrability, the first example of non–Lebesgue measurable sets, the theorem of compacity of a family of holomorphic functions, the covering theorem, which are still of great relevance in the literature of mathematics today. His research was linked to the new field opened in France by Lebesgue and Borel, and which in Italy had originated from the work of Ulisse Dini and his school. After a long period of teaching in secondary schools, owing to the difficulty in obtaining an academic position, Vitali returned to university in 1922 and he had to catch up with the latest research: in the meanwhile the new analysis was being developed by mathematicians in Russia such as Egorov and Luzin or in Poland like Sierpiński, Mazurkiewicz, Nikodym and Banach.

1 Research on measure and integration

The name of Giuseppe Vitali is mainly linked to his first research works on real analysis of the period from 1903 to 1908. The main sources for Vitali's life and work are the edition of his papers on real and complex analysis, which contains his biography, the complete list of publications and his scientific correspondence edited by M.T. Borgato and L. Pepe [1], and specific studies on different aspects of his research: on real analysis, complex analysis, differential geometry and didactics of mathematics: [2, 2, 3, 3, 6].

The research carried out in Italy by Ulisse Dini and his school had formed the basis of the successive development in France by Borel, Baire, Lebesgue and Fréchet. Baire was in Italy in 1898 when he worked with Volterra, and his thesis ("Sur les fonctions de variables réelles") was published in the *Annali di matematica pura* *ed applicata* of which Dini was the editor [11]. Lebesgue's thesis, too, ("Integral, longueur, aire") was published in the *Annali di matematica* in 1902 [12, vol. I, pp. 20-331].

At the beginning of the twentieth century the legacy left by Dini and Arzelà tied up with the new research that Borel and Lebesgue were carrying out on the theory of measure.

To the question of Riemann's integrability of a function in relation to the set of his points of discontinuity, Vitali devoted three memoirs [1, pp. 125-128, 133-137, 139-149]. In the first one Vitali introduced the concept of *minimal extension* ("estensione minima") a definition which coincides with Lebesgue outer measure, and provided his criterion for *Riemann integrabiliy*, that is that the *minimal extension* ("estensione minima") of the points of discontinuity must be equal to zero. In the second note he reformulated and clarified the concept, he also observed that countable sets had minimal extension zero and that this extension coincides, for closed sets and only on these, with the *Inhalt* of Cantor. In the third note, Vitali, inspired by Borel, defined the measurable sets and studied the properties that make the *minimal extension* a measure for measurable linear sets of points: *finite* and *countable additivity*, *closure* under the operations of countable union, and intersection, comparing it with the measures of Jordan and Borel.

The *minimal extension* of a linear set of points in Vitali's theory is the least upper bound of the series of measures of (finite or countable) families of pairwise disjoint open intervals, which cover the set. Vitali did not introduce an inner measure and defined measurable sets by means of a scalar function ("allacciamento" - *linking*) which link a set E of an interval (a, b) to its complement $E^* : Z(E, E^*) = (\text{mis}_e E + \text{mis}_e E^*) - (b - a)$. Those sets for which the linking is zero are defined *measurable*.

Other problems connected to the theory of measure and the integral of Lebesgue, to which Vitali gave important contributions were: the existence of non-measurable sets, the characterisation of the integrals of summable functions, the extension of the fundamental theorem of calculus to Lebesgue integrals, the integration of series term by term, the integrability on unlimited intervals, the extension to functions of two or more variables.

In particular, in 1905 Vitali gave the first example of non–Lebesgue measurable set, which is constructed using the *axiom of choice* in the form of the well-ordering theorem, which Vitali immediately accepted [1, pp. 231-235]. Several later research works refer to this problem, among which those of F. Hausdorff and W. Sierpinski. In more recent times, to this question is related the famous result by Robert M. Solovay [13].

Vitali's works of this period have many intersections with Lebesgue's. In 1904 Lebesgue had demonstrated that the indefinite integral of a summable function has as a derivative this function except for a zero measure set [12, pp. 124-125]. The following year Vitali introduced the concept of *absolute continuity*, and therefore demonstrated that the absolute continuity is the necessary and sufficient condition so that a function is the indefinite integral of a summable function. In the same memoir Vitali also provided the first example of a continuous function of bounded variation, but not absolutely continuous, known as *Cantor function* or *Devil's staircase* [1, pp. 205-220].

In 1907-08 Vitali gave, for the first time, a definition of bounded variation for functions of two variables, obtained from the variations of a function in four vertices of a rectangle, and extended the concepts of derived numbers and absolutely continuous functions to more variables [1, pp. 257-276].

Legesgue claimed authorship of various theorems in two letters of 16^{th} and 18^{th} February 1807 to Vitali and pointed out his works were not adequately quoted in Vitali's memoirs. On the other hand, he admitted he had not read many of Vitali's works before Vitali sent them to him. In particular, his criticism was directed to the theorem on integral functions. In reality, Lebesgue had only indicated the result which was completely demonstrated by Vitali in his work of 1905, and was then redemonstrated by Lebesgue in 1907 and once again by Vitali in 1908 with a different method, which could also be extended to functions of more than one variable. Vitali's priority and contribution were then recognized by Lebesgue in many points of his *Notice*, in particular, with reference to the fundamental theorem of calculus Lebesgue says [12, vol. I, p. 127]:

"Toute intégrale indéfinie est continue et à variation bornée, mais la réciproque n'est pas vraie. Pour qu'une fonction soit une intégrale indéfinie, il faut que, de plus, la somme des valeurs absolues de ses accroissements dans des intervalles extérieurs les uns aux autres et de mesure ε, tende vers zéro avec ε. On dit alors, avec M. Vitali, qui à été le premier à publier une démonstration de cet énoncé que j'avais formulé, que la fonction est absolument continue... C'est aussi M. Vitali qui a publié, le premier, des résultats sur la dérivation des intégrales indéfinies des plusieurs variables".

Vitali dealt with term-by-term Integration in two memoirs [1, pp. 189-204, 237-255]. In the second more important memoir of 1907, Vitali considered the integration extended to any measureable set and introduced the concept of equi-absolute continuity of a sequence of functions and complete integrability of series. He provided the characterisation of the term by term integrability of series on the basis of the equi-absolute continuity of the integrals of its partial sums.

Vitali's characterisation turned out to be one of the basic results of the measure theory, extended by Hahn (1922), Nykodym (1931), Saks (1933), Dieudonné (1951) and Grothendieck (1953), and today a great deal of reformulations are inserted into general measure theory.

2 Luzin's Theorem. Vitali's covering theorem

In his thesis of 1899 René-Louis Baire had provided the well-known classification of functions, but Borel and Lebesgue had managed to construct functions that had escaped this classification. The problem of classifying Baire's functions arose. In 1905 Vitali and Lebesgue demonstrated that all and only the functions of Baire are Borel measurable [1, pp. 189-192], [12, vol. III, pp. 103-180]. The same year, however, Vitali also proved that every Borel measurable function can be decomposed in the sum of a Baire function of first or second class, and a function equal to zero almost everywhere [1, pp. 183-188].

In the same memoir in 1905 Vitali demostrated the so-called Luzin's theorem, by which: if a function f is finite and measurable on an interval (a, b) of length l, for every ε there exists a closed set in which f is continuous and whose measure is greater than $l - \varepsilon$.

The theorem had only previously been indicated by Borel and Lebesgue in 1903-04 [14], [12, vol. I, p. 336, vol. II, p. 141]; it inspired various analysts in their research into new definitions of integral, in particular Leonida Tonelli, who posed the functions he called "quasi-continuous" at the basis of his definition of integral. Nikolai Luzin discovered the same theorem independently six years later, starting from a theorem by Dmitri Egorov (*Comptes Rendus*, 1911) and published it for the first time in Russian (*Matematicheskii Sbornik* **2**, 1911) and the following year in *Comptes Rendus* [16].

The line of thought arrived from Borel and Lebesgue, but while in Vitali's research the theorem is applied to the classification of Baire functions, in Luzin's work it is applied to the representation of measurable functions as polynomial series (a generalization of the Weierstrass-Picard theorem on continuous functions) and to their integrability almost everywhere.

Vitali's research on real analysis of this period was crowned by the discovery of the so-called covering theorem. The *covering theorem* was demonstrated by Vitali as an intermediate result in a memoir written at the end of 1907 and was enunciated firstly for the points of the real straight line, and later extended to the plane and then to higher dimensions [1, pp. 257-276]:

"Se Σ è un gruppo di segmenti, il cui nucleo abbia misura finita m_1, esiste un gruppo finito o numerabile di segmenti di Σ a due a due distinti, le cui lunghezze hanno una somma non minore di m_1."

This theorem is preceded by another one, known as the *Vitali covering lemma*: the aim was to cover, up to a measure zero set, a given set E by a disjoint sub-collection extracted from a *Vitali covering* for E: a *Vitali covering* V for E is a collection of sets such that, for every $x \in E$ and $\delta > 0$, there is a set U in the collection V such that $x \in U$ and the diameter of U is non-zero and less than δ.

In the second chapter, Vitali himself provided interesting applications of the covering theorem, on derived numbers of functions of bounded variation and on the integrals of summable functions, extending many of the results of his previous memoirs to functions of two variables.

The lemma and theorem of Vitali have been extended to other measures besides that of Lebesgue, and to more general spaces. Mention is to be made of the formulation which Costantin Carathéodory gave a few years later [17], as well as Stefan Banach's extension, in a fundamental memoir in 1924 [14].

The mathematical production of Vitali, which had considerably slowed down owing to his commitments in the secondary school, especially in concomitance with the first world war, was taken up again with renewed energy on the occasion of examinations for chairs and successive academic activity. Owing to the long interruption, he had to to catch up with the latest research. As we can see in his publications of this period on real and complex analysis, instead of anticipating his contemporaries, Vitali now followed and completed their research, with interesting results too.

But above all, his research interests changed, and were directed to differential geometry, a discipline that in Italy had a long tradition. Giuseppe Vitali published about thirty works from 1922 to 1932 and the essential parts of his papers concerning differential geometry and absolute differential calculus were re-exposed in an organic way in the monography: *Geometria nello spazio hilbertiano* (1929).

3 Latest contributions to real analysis: BV functions

Apart from some useful complements to other authors' works (on continuous rectifiable curves related to the research of Tonelli, on the equivalence between the new definition of integral provided by Beppo Levi and that of Lebesgue...) Vitali's most significant results of the second period in real analysis were the properties of the functions of bounded variation and absolutely continuous functions. He showed that a continuous BV function may be divided into the sum of two functions: the absolutely continuous part, which is represented as an integral, and the singular part (the “scarto”) whose derivative vanishes almost everywhere, similarly to that which occurs for the functions of bounded variation that are the sum of a continuous function, and one that absorbs the discontinuities (the “saltus function” or “jump function”) [1, pp. 303-323].

A similar decomposition had already been obtained by De La Vallée Poussin in 1909, and used by Fréchet in his Stiltjes integral [15, 16], but, as recognized by Fréchet himself in a letter to Vitali (4^{th} May 1923), the representation of the “scarto” given by Vitali as the sum of a countable infinity of elementary differences was considered “nouvelle et intéressante”, allowing classification of functions in accordance with the “scarto” (equal to zero, greater than zero, infinite) into: absolutely continuous, bounded variation, and infinite variation.

Another memoir displays an even more incisive result, demonstrating a characteristic property of BV functions, introducing an equivalent definition of total variation which may be extended to functions of more variables, with a view to establishing results for surfaces similar to those already obtained by Tonelli for the rectification of curves. It was demonstrated that [1, pp. 347-360]: if f is a continuous function of (a, b) on (c, d) and Γ_r is the set of points of (c, d) that f assumes at least r times, the condition necessary and sufficient in order that f is of bounded variation, is that the series of measures of Γ_r is convergent and in any case this series coincides with the total variation of f: $V = \sum_{r=1}^{\infty} \mu(\Gamma_r)$

In his own studies, Stefan Banach had obtained similar results with different procedures [17]. Vitali’s memoir is connected to other research works of the Polish School as Waclaw Sierpiński, Stefan Mazurkiewicz and the Russian, Nikolai Luzin, were, at the same time, studying the measurability and cardinality of the set of values that a continuous function assumes a number of times equal to an assigned cardinal [18]:

“Quant aux ensembles des valeurs qu’une fonction continue prend un nombre assigné de fois, nous avons étudié avec M. Mazurkiewicz l’ensemble qu’on pourrait désigner, d’après Votre notation, par G_2 (Fund. Math. T. VI p. 161). Nous avons démontré que cet ensemble est une projection d’un ensemble mesurable (B) (d’où il résulte qu’il est mesurable au sens de Lebesgue)), et qu’il peut être non mesurable (B) pour des fonctions continue convenables. (Cf. Fund. Math. T. VII, p. 198). Il en résulte tout de suite que l’ensemble $G_{\aleph_0}$ est aussi mesurable (L). Or, nous ne savons pas déterminer la puissance de l’ensemble $G_{\aleph_0}$ (nous savons seulement qu’elle ne peut être comprise entre $\aleph_1$ et $2^{\aleph_0}$). M. Lusin est d’avis que ce problème est un de plus difficiles” (Sierpiński to Vitali, 3rd October 1924)

Relationships with the Polish researchers were frequent and imbued with esteem. It is to be remembered that, in 1924, from one of Banach’s articles he had summarised for the *Bollettino UMI* [19], Vitali drew inspiration for an interesting note of measure theory [1, pp. 327-334]. In the same year, Banach had also published a simplified demonstration of Vitali's covering theorem [20].

Vitali corresponded not only with Sierpiński, but also with Otton Nikodym, with whom he discussed questions of projective geometry (collineations of the complex projective plane, subsets of the complex projective plane which intersect each line in a certain number of points) and whom he had occasion to meet at the International Congress of Bologna in 1829:

"Vous avez résolu les deux problèmes, dont j'ai l'honneur de parler avec vous. Le premier problème consiste à chercher des transformations biunivoques et collinéaires du plan projectif complexe. Vous avez raison, qu'on doit changer l'énoncé du problème, étant donné... qu'il existe une telle transformation antiprojective... La deuxième question s'occupe d'ensembles ε du plan projectif complexe sans la supposition que toute droite coupe ε en deux ou un point. J'ai résolu le problème d'une manière analogue que vous... Néanmoins il reste d'examiner si l'ensemble ε doit nécessairement être une conique, si l'on suppose que ε soit fermé, ou mesurable B, où même partout dense. M. Mazurkiewicz s'occupe d'un problème analogue pour le cas du plan euclidien réel..." (Otton Nikodym to Vitali, 11th May 1928)

Vitali became a member (presented by Nikodym and his wife) of the *Société Polonaise de Mathématique* and published some of his memoirs in the *Annales de la Société Polonaise de Math.* as well as in *Fundamenta Mathematicae.*

Esteem and affection are shown in the touching words of condolences sent by Luzin, on the news of Vitali's sudden death, to the members of the *Seminario matematico* of Padua University [6].

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APPLICATION OF MATHEMATICAL ANALYSIS TO GEOMETRY IN RESEARCH OF L. EULER: ISSUE ON SURFACE DEFORMATION

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Key words: differential geometry, Leonhard Euler, the theory of surfaces, history of mathematics

AMS Mathematics Subject Classification: 01A50

Abstract. There is a clever presentation of L. Euler's research on a problem of surface deformation. Considerably Euler has applied the majority of means of the infinitesimal analysis to resolve one of the typical geometrical question. The submitted material is an illustration of process of formation of differential geometry in XVIII century.

In XVIII century the issues connected with application of deferential and integral calculus methods to geometry which had been already developed by Isaak Newton (1643–1727) and Gottfried Wilhelm Leibniz (1646–1716) were of great significance. Exactly these problems encouraged the origin of differential geometry.

Leonhard Euler (1707–1783) played a great role in formation of differential geometry. This fact was noted by many historians of mathematics: H. Wieleitner [1], M.Y. Vygodsky [2], B.N. Delone [3], V.V. Kotek [2], B. Kommerell [3], B.A. Rosenfeld [6], [4], D. J. Struik [8], A.Speiser [9], A. P. Juschkewitsch [10] and others. The analyses we have carried out on the basis of Euler's memoirs being a part of "Opera omnia", published scientist's correspondence [11] as well as unpublished materials taken from his notebooks [12] kept at St. Petersburg branch of the Russian Academy of Sciences have enabled us to prove that he obtained fundamental results in the above mentioned field.

The problems of the cartography, geodesy and mechanics made Euler investigate spatial problems of differential geometry. In the proposed article the issue on deformation of one surface onto the other in Euler's research will be surveyed. We remind that the deformation is called such deformation of the surface when the length of each oriented edge of any line drawn out on the surface remains intact. The issue on surface deformation is important as in case one surface is deformed into the other their intrinsic geometries are equal. The development of surface on the plane is the special case of the deformation.

Euler published the results of investigation on the issue of developable surfaces in his memoir "On bodies whose surface is possible to develop on the plane" (1771, issued in 1772) [13]. For the first time the notion of developable surface as the surface which can be laid on the plane without foldings and ruptures is defined there. Euler based on the fact that infinitely small triangle on such a surface should be congruent to the matched triangle on the plane where it was developed. He describes the coordinates x, y, z of a point (fig. 1) asfunctions of two variates t and u where t and u are the coordinates of the above point on the surface. Thus Euler introduces the so called isothermal (or Gaussian) coordinates.

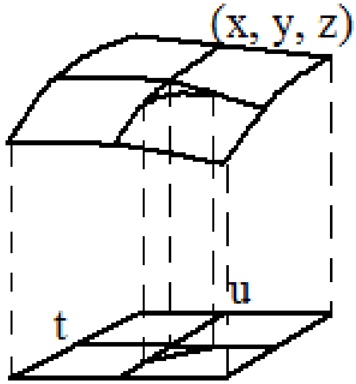

Figure 1

To the points $(t+dt, u)$ and $(t, u+du)$ of the surface the points whose coordinates are as follows correspond: $(x+ldt; y+mdt; z+hdt)$ and $(x+\lambda du;\ y+\mu du; z+\nu du)$, where $l, m, n, \lambda, \mu, \nu$ are the functions of the variates u, t and fulfill conditions:

$$l=\frac{\partial x}{\partial t},\ m=\frac{\partial y}{\partial t},\ h=\frac{\partial z}{\partial t},\ \lambda=\frac{\partial x}{\partial u},\ \mu=\frac{\partial y}{\partial u},\ \nu=\frac{\partial z}{\partial u}.$$

To the point $(t+dt, u+du)$ of the surface the following surface point corresponds $(x+dx, y+dy, z+dz)$, than $dx = ldt + \lambda du$, $dy = mdt + \mu du$, $dz = ndt + \nu du$, besides, in order the formulas of the right side equation being the total differential, the following proportions shall be fulfilled:

$$\frac{\partial l}{\partial u}=\frac{\partial \lambda}{\partial t},\quad \frac{\partial m}{\partial u}=\frac{\partial \mu}{\partial t},\quad \frac{\partial n}{\partial u}=\frac{\partial \nu}{\partial t}. \tag{1}$$

For the first time in this work the ds line element of surface is introduced (i.e. differential of arc length of curve on it) as the method of investigation the surface characteristics which were called internal later on and which can be researched by means of measuring on that surface without taking into consideration the plane containing it. That idea was developed in future only beginning with Gauss (1828).

The condition of development (or covering condition) of one surface onto the other received by Euler can be formulated as the condition of coincidence of a line element of the developing surface $ds^2 = dx^2 + dy^2 + dz^2$ with a line element of the surface $ds^2 = dt^2 + du^2$ i.e. at any du, dt the distance between points $(t + dt, u)$ and $(t, u+du)$ of the surface should be equal to the distance between corresponding points $(x + ldt; y + mdt; z + hdt)$ and $(x + \lambda du; y + \mu du; z + \nu du)$ of the developing surface:

$$dt^2 + du^2 = (ldt - \lambda du)^2 + (mdt - \mu du)^2 + (hdt - \nu du)^2.$$

Having removed parentheses we have:

$$\begin{aligned} dt^2 + du^2 = l^2dt^2 - 2l\lambda dtdu + \lambda^2du^2 + \\ m^2dt^2 - 2m\mu dtdu + \mu^2du^2 + \\ h^2dt^2 - 2h\nu dtdu + \nu^2du^2. \end{aligned}$$

Therefore we get the condition of covering:

$$\begin{cases} l^2 + m^2 + h^2 = 1, \\ \lambda^2 + \mu^2 + \nu^2 = 1, \\ l\lambda + m\mu + h\nu = 0. \end{cases} \tag{2}$$

From the modern point of view this is the condition of singleness and orthogonality of vectors whose coordinates are equal to partial derivatives of radius-vector of the surface point on coordinates u, t.

Thus, the given geometric task is reduced to the solution of the following purely analytical problem: to find such six functions l, m, n, λ, μ, ν of two variates u and t in order the conditions (1) and (2) are fulfilled.

The solution of this problem Euler has found based on the following geometric ideas: "bodies the surface of which can be developed into the plane should be arranged in such a way that from any point of the surface it would be possible to draw a line which would completely lie in that surface and two such lines would lie in one plane close to each other i.e. if they are not parallel than they meet at one point. Cross points of that lines evidently form the line of such double curvature that all its tangents form the surface of the studied body" [13, p.161].

He examined two close points V and v of the space curve (fig. 2), through them draws tangents VS and vs to the curve and marks the crossing points of this tangents with the surface (t, u) named correspondingly S and s. Points U and u

of the surface (t, u) correspond to the points V and v of the space curve. Further Euler marks the tilting angle of the line SU to the axle Ou, i.e. $\angle MUT$ through ς; tilting angle of the line VS to the axle Ov, i.e. $\angle UVS$ through ϑ. Further, having assumed that the space curve is predetermined by the equations: $u = f(t)$, $v = \varphi(t)$, he offers to determine the tangent to it by means of coordinate t of its tangency point V, and arbitrary point Z on this tangent by means of the its distance s from the point V. Thus, t and s are the coordinates of the arbitrary point Z of some tangent to the examined space curve. These coordinates are connected with the coordinates x, y, z of the point Z by the following equations:

$$x = t - s \sin\vartheta \sin\varsigma; y = u - s \sin\vartheta \cos\varsigma; z = v - s\cos\vartheta.$$

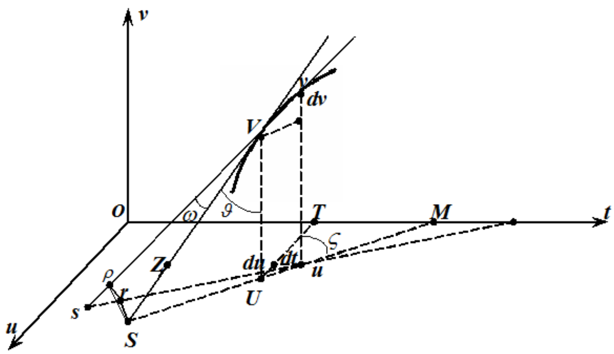

Figure 2

Angles ς and ϑ will be certain functions of the variate t for this curve. On the contrary, if to take any two functions from t for ς and ϑ, than they predetermine the space curve with the accuracy up to its parallel shift.

In this work Euler showed that the searched solution of the system (2) satisfying the condition (1) is expressed by the functions:

$$l = \sin\varsigma \sin\vartheta \sin\omega + \cos\omega \frac{d(\sin\varsigma \sin\vartheta)}{d\omega},$$

$$m = \cos\varsigma \sin\vartheta \sin\omega + \cos\omega \frac{d(\cos\varsigma \sin\vartheta)}{d\omega},$$

$$n = \cos\vartheta \sin\omega + \cos\omega \frac{d\cos\vartheta}{d\omega},$$

$$\lambda = \sin\varsigma \sin\vartheta \cos\omega - \sin\omega \frac{d(\sin\varsigma \sin\vartheta)}{d\omega},$$

$$\mu = \cos\varsigma \sin\vartheta \cos\omega - \sin\omega \frac{d(\cos\varsigma \sin\vartheta)}{d\omega},$$

$$\nu = \cos\vartheta \cos\omega - \sin\omega \frac{d\cos\vartheta}{d\omega},$$

where ω — angle between tangents VS and vs, i.e. $\angle SVs$.

Than he enunciates the converse proposition: if to take such functions $l, m, n, \lambda, \mu, \nu$, than the surface x, y, z, where

$$x = \int (ldt + \lambda du), \quad y = \int (mdt + \mu du), \quad z = \int (ndt + \nu du),$$

all consist of straight lines.

In the same work Euler has enunciated and proved the theorem saying that any developing surface is either a cylinder or a cone or is formed by tangents to certain space curve.

It should be noted that the above viewed results on developing surface were received by Euler in around 1767. That fact is proven by the note made in his notebook No.138 on the sheets 3ob.–5ob. [12].

In the same notebook on the sheets 5ob.–7 for the first time ever Euler has introduced the notion which corresponds now to the term of deformation of surface on the other surface and stated problem on finding the condition of that deformation.

It is required to find two surfaces which deformed into each other in such a way that the length of corresponding lines between corresponding points is preserved.

Let us study the solution of Euler (having changed the designations of corresponding points on the deformed surfaces for the sake of convenience). Let us assume that the surface τ after deformation transforms into surface τ' (fig. 3). The point $A(t, u, v)$ of the surface τ will transform into the point $A'(x, y, z)$ of the surface τ'. Euler notes down that by their nature the coordinates of the surface points are functions of the variates r and s.

If to these variates to set infinitely small increments to dr and ds , than on the surface τ we will get two points $B\left(t + dr\left(\frac{\partial t}{\partial r}\right); u + dr\left(\frac{\partial u}{\partial r}\right); v + dr\left(\frac{\partial v}{\partial r}\right)\right)$ and

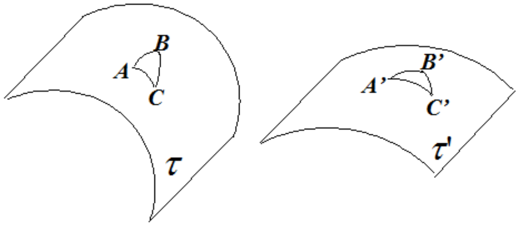

Figure 3

$C\left(t+ds\left(\frac{\partial t}{\partial s}\right);u+ds\left(\frac{\partial u}{\partial s}\right);v+ds\left(\frac{\partial v}{\partial s}\right)\right)$, and on the surface τ' correspondingly the points $B'\left(x+dr\left(\frac{\partial x}{\partial r}\right);y+dr\left(\frac{\partial y}{\partial r}\right);z+dr\left(\frac{\partial z}{\partial r}\right)\right)$ and $C'\left(x+ds\left(\frac{\partial x}{\partial s}\right);y+ds\left(\frac{\partial y}{\partial s}\right);z+ds\left(\frac{\partial z}{\partial s}\right)\right)$.

The points A and B are very close as the increment dr is infinitely small. Therefore the length of the arc AB can be considered equal to the length of the segment AB . At the same time we note, that $AB^2 = dr^2\left(\left(\frac{\partial t}{\partial r}\right)^2+\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial v}{\partial r}\right)^2\right)$.

By analogy we get that the length of the arc AC is equal to the length of the segment AC, where $AC^2 = ds^2\left(\left(\frac{\partial t}{\partial s}\right)^2+\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial v}{\partial s}\right)^2\right)$.

The length of the arc BC is also equal to the length of the segment BC, where:

$$\begin{aligned} BC^2 = {} & \left(dr\left(\frac{\partial t}{\partial r}\right)-ds\left(\frac{\partial t}{\partial s}\right)\right)^2+\left(dr\left(\frac{\partial u}{\partial r}\right)-ds\left(\frac{\partial u}{\partial s}\right)\right)^2+ \\ & +\left(dr\left(\frac{\partial v}{\partial r}\right)-ds\left(\frac{\partial v}{\partial s}\right)\right)^2 = AB^2+AC^2- \\ & -2drds\left(\left(\frac{\partial t}{\partial r}\right)\left(\frac{\partial t}{\partial s}\right)+\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial s}\right)+\left(\frac{\partial v}{\partial r}\right)\left(\frac{\partial v}{\partial s}\right)\right). \end{aligned}$$

We have similar ratio for the points A', B' , C' of the surface τ':

$$A'B'^2 = dr^2\left(\left(\frac{\partial x}{\partial r}\right)^2+\left(\frac{\partial y}{\partial r}\right)^2+\left(\frac{\partial z}{\partial r}\right)^2\right);$$

$$A'C'^2 = ds^2\left(\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 + \left(\frac{\partial z}{\partial s}\right)^2\right);$$

$$B'C'^2 = A'B'^2 + A'C'^2 - 2drds\left(\left(\frac{\partial x}{\partial r}\right)\left(\frac{\partial x}{\partial s}\right) + \left(\frac{\partial y}{\partial r}\right)\left(\frac{\partial y}{\partial s}\right) + \left(\frac{\partial z}{\partial r}\right)\left(\frac{\partial z}{\partial s}\right)\right).$$

Later, as the length of the corresponding arcs are preserved at deformation, we have $AB^2 = A'B'^2$, $AC^2 = A'C'^2$,$BC^2 = B'C'^2$, whence it follows the system of three equations:

$$\begin{cases} \left(\frac{\partial t}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2 = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2, \\ \left(\frac{\partial t}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial v}{\partial s}\right)^2 = \left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 + \left(\frac{\partial z}{\partial s}\right)^2, \\ \left(\frac{\partial t}{\partial r}\right)\left(\frac{\partial t}{\partial s}\right) + \left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial s}\right) + \left(\frac{\partial v}{\partial r}\right)\left(\frac{\partial v}{\partial s}\right) = \left(\frac{\partial x}{\partial r}\right)\left(\frac{\partial x}{\partial s}\right) + \\ + \left(\frac{\partial y}{\partial r}\right)\left(\frac{\partial y}{\partial s}\right) + \left(\frac{\partial z}{\partial r}\right)\left(\frac{\partial z}{\partial s}\right). \end{cases} \tag{3}$$

predefining the conditions of deformation of one surface into the other.

In the modern terminology this system expressed the condition of equality of the corresponding coefficients of first Gausse's quadratic forms for surfaces line elements:

$$\begin{cases} E = E', \\ G = G', \\ F = F'. \end{cases}$$

Unfortunately that remarkable result of Euler remained unknown practically for one hundred years. It was published by Golovyn M.E. only in 1862 in the first chapter L. Euleri Opera posthuma of the section "Continuatio Fragmentorum ex Adversariis mathematicis depromtorum" [14, p. 494–496]. Euler offered to search for the solution of this system (3) in such a way:

$$\begin{aligned} t &= \int Jdr + Js, \quad & x &= \int Ldr + Ls, \\ u &= \int Gdr + Gs, \quad & y &= \int Mdr + Ms, \\ v &= \int Hdr + Hs, \quad & z &= \int Ndr + Ns, \end{aligned} \tag{4}$$

where functions J, G, H and L, M, N depend only on the variate r.

In the comments [9, vol. 29, p. XLI] to this work Speiser A. noted that of to assume that $s = 0$ herein, than we will get the space curve on each of the examined surface depending only on the parameter r. Where (J, G, H) are the coordinates of the tangent vector drawn at the point (t, u, v) to the curve laid on the first surface; (L, M, N) are the coordinates of the tangent vector drawn at the point (x, y, z) to the curve laid on the second surface.

Plugging expressions (4) into the system of equations (3), Euler came to the system of the following kind after several transformations:

$$\begin{cases} J^2 + G^2 + H^2 = L^2 + M^2 + N^2, \\ JdJ + GdG + HdH = LdL + MdM + NdN, \\ (dJ)^2 + (dG)^2 + (dH)^2 = (dL)^2 + (dM)^2 + (dN)^2. \end{cases} \tag{5}$$

Later, having put $J^2+G^2+H^2 = L^2+M^2+N^2 = p^2$, Euler expressed functions J, G, H and L, M, N as follows:

$$\begin{aligned} &J = p\sin m \sin n, \quad G = p\cos m \sin n, \quad H = p\cos n, \\ &L = p\sin \mu \sin \nu, \quad M = p\cos \mu \sin \nu, \quad N = p\cos \nu. \end{aligned} \tag{6}$$

Herein the value p expresses the length of each of the vectors (J, G, H) and (L, M, N); n should be interpreted as the angle of deviation of the vector (J, G, H) from the axle v; m is the angle of deviation of the vector projection (J, G, H) on the plane (t, u) from the axle u; v is the angle of deviation of the vector (L, M, N) from the axle z; μ — is the angle of deviation of the vector projection (L, M, N) on the plane (x, y) from the axle y. At the same time p as well as the angles m, n, μ and ν depend only on the parameter r . In other words, Euler has used herein the so-called spherical coordinates.

Having plugged the expression (6) into the last equality of the system (5) he has developed the ratio: $dm^2 \sin^2 n + dn^2 = d\mu^2 \sin^2 \nu + d\nu^2$, which shows that one of the four angles m, n,μ and ν can be always expressed through the rest three.

Thus, the coordinates of the corresponding points (t, u, v) and (x, y, z) of developed onto each other surfaces are expressed herein by the following ratios:

$$\begin{aligned} t &= \int p\sin m \sin n dr + ps\sin m \sin n, & x &= \int p\sin \mu \sin \nu dr + ps\sin \mu \sin \nu, \\ u &= \int p\cos m \sin n dr + ps\cos m \sin n, & y &= \int p\cos \mu \sin \nu dr + ps\cos \mu \sin \nu, \\ v &= \int p\cos n dr + ps\cos n, & z &= \int p\cos \nu dr + ps\cos \nu. \end{aligned}$$

In epilogue Euler has made important remark: sphere is a surface that can not deform as it is closed [and convex]; if to take only part of the sphere than it can be deformed.

Received results he used in the works on cartography: “On presentation of sphere surface on plane” (1777) [15], “On geographical projection of sphere surface” (1777) [16], “On Delil’s geographical projection used on the general map of the Russian Empire” (1777) [17].

The works of Euler differential geometry of surfaces influenced Gaspard Monge (1746–1818) and Carl Friedrich Gauss (1777–1855) and were the basis for their future research in that field.

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ON THE HISTORY OF DIVERGENT SERIES BY L. EULER

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Abstract. We want to show that the example, demonstrating the idea of the envelopment of the series and disappearing from the researchers' attention, we can find in Euler's Institutiones Calculi Differentialis (1755).

1. It is known that it was L.Euler who was the founder of the theory of divergent series, see [1–3]. He extended the notion of the sum by suggesting several methods of summation of divergent series and successfully applied them for numerical calculations. He showed that many important constants – such as π – as well as integrals, solutions of differential equations, slowly convergent series, etc. may be obtained effectively and rather precisely with the help of these new methods.

While working with divergent series Euler met with totally new notions of asymptotical series and series that enveloping some value. Although he did not formulate any definition – although his experience was based on enormous computational practice – he managed to find some important properties of divergent series. In particular, he noticed that in many cases of a divergent alternate series whose terms decrease and then start to increase fast, the tail does not exceed the first dropped term, similarly to tails of convergent alternate series, such as of Leibnitz type. So, if such divergent series does have a "sum", then the best approximation to this sum provides the stopping rule "until the minimal term"; this is the famous Euler's method of "summation to the minimal term". Euler himself expressed it in the following way [2]: «the more the values of n, the more the series converges, although, it converges until some term, starting from which the terms start increasing. By this reason, it is not right to use the series earlier than the moment where the terms start increasing, but it is better to stop where the maximal convergence holds true». If the series is enveloping, then this Euler's recipe is correct.

There is an opinion in the literature that it was A.-M.Legendre who was the first to suggest explicitly the idea of enveloping series in 1811, on several examples of Euler-Maclaurin series, in particular, for the function $f(x) = \frac{1}{(a+nx)^2}$ [3]. Legendre noticed that partial sums of the corresponding series become, in turn, more or less than the exact value, «and one has to stop at the term where convergence is seized» [3]; this is equivalent to the enveloping around the sum for this series. In this talk we will show one Euler's example, which demonstrates the idea of enveloping series and which, apparently, was not noticed by researchers. Before that, let us remind the modern definition. A series $\sum_{n=0}^{\propto} a_n$ is called enveloping a value A if its partial sums are in turn more and less than A. In other words, the remainder r_n given by the formula

$$A = a_0 + a_1 + ... + a_n + r_n$$

is alternating (see [6]). This definition is naturally extended to function series (see [6]).

2. In the sixth chapter of "Differential Calculus", Euler [4] provides examples of applications of his famous formula — known as Euler-Maclaurin's formula — for asymptotic evaluation of partial sums, for approximate integral computation and for finding sums of slow convergent series. He writes this formula as

$$\sum_{k=1}^{n} f(k) \sim C + \int\limits_{1}^{n} f(x)dx + \frac{1}{2}f(n) + \sum_{k=1}^{\propto} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(n),$$

where B_k are the Bernoulli numbers and C is the Euler-Maclaurin constant. Here the sign of equality is not used because, generally speaking, the series may diverge. It may be shown (see [8]) that under certain conditions, the sum of a convergent series equals C. By using his formula, Euler computes exactly or approximately the sums for slowly convergent series for the power functions $f(x) = frac1_{x^p}$ for $p = 2, 3, \ldots, 16$, i. e., the expressions for zeta-function $\varsigma(p)$. He shows the calculus only for $p = 2$ and $p = 3$ and gives the answers for the other values of p. Notice that in this example, all Euler-Maclaurin's series are enveloping. The feature of enveloping can be seen very clearly in Euler's computations for $p = 2$ where he knows the sum – that is, the constant C, $(\sum_{n=1}^{\propto} \frac{1}{x^2} = \frac{\pi^2}{6})$ – and the terms of Euler-Maclaurin's series are computed in their natural order; for $p = 3$ the calculus does not show the enveloping effect explicitly because Euler sums up positive and negative terms separately.

Euler writes down the partial sum $s = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... + \frac{1}{x^2}$ of this series by his summation formula,

$$\sum_{k=1}^{n} \frac{1}{K^2} \sim C + \int\limits_{0}^{n} \frac{1}{x^2}dx + \frac{1}{2x^2} + \sum_{n=1}^{\propto} (-1)^n \frac{B_{2n}}{(2n)!} f^{(2n-1}(n).$$

So,

$$s = C - \frac{1}{x} + \frac{1}{2x^2} - \frac{B_2}{x^3} + \frac{B_4}{x^5} - \frac{B_6}{x^7} + \frac{B_8}{x^9} - \frac{B_10}{x^{11}} + ...$$

Hence, as $x \to \infty$, he gets $C = \sum_{n=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. In the sequel, Euler uses the value of $C = 1,644934066848226430$, which he found earlier by considering a divergent series with Bernoulli numbers. However, as usual, he provides yet another method to compute this value.

We show the corresponding paragraph from Euler's "Differential Calculus" [4] where we only made some minor changes, namely, we gave modern notations of the Bernoulli numbers and added some little comments in brackets:

"149. Sin autem summa huius seriei cognita non fuisset, valor constantis illius C ex alio quopiam casu, quo summa actu est inventa, determinari deberet. Hunc in finem ponamus $x = 10$ atque decem terminis actu addendis reperietur

$$s = 1,5497677311665406 90$$

tum est

$$add. \frac{1}{x} = 0,1$$

$$subtr. \frac{1}{2x^2} = 0,005$$

1, 644 767 731 166 540 690 [< C = 1, 644 934 066 848 226 430],

$add. \frac{B_2}{x^3} = 0,000166666666666666$

1, 644 934 397 833 207 356 [> C= 1, 644 934 066 848 226 430],

$subtr. \frac{B_4}{x^5} = 0,000000333333333333$

1, 644 934 064 499 874 023 [< C = 1, 644 934 066 848 226 430],

$add. \frac{B_6}{x^7} = 0,000000002380952381$

1, 644 934 066 880 826 404 [> C= 1, 644 934 066 848 226 430],

$subtr. \frac{B_8}{x^9} = 0,000000000033333333$

1, 644 934 066 847 493 071 [< C = 1, 644 934 066 848 226 430],

$add. \frac{B_{10}}{x^{11}} = 0,000000000000757575$

1, 644 934 066 848 250 646 [> C= 1, 644 934 066 848 226 430],

$subtr.\frac{B_{12}}{x^{13}} = 0,000000000000025311$

1, 644 934 066 848 225 335 [$<$ C = 1, 644 934 066 848 226 430],

$add.\frac{B_{14}}{x^{15}} = 0,000000000000001166$

1, 644 934 066 848 226 501 [$>$ C= 1, 644 934 066 848 226 430],

$subtr.\frac{B_{16}}{x^{17}} = 0,000000000000000071$

1, 644 934 066 848 226 430 = C . "

From this example it is clear that the partial sums are, in turn, more or less than $\frac{\pi^2}{6}$, that is, that they envelope the sum of the original series. Later, this property was used as a base of the definition of enveloping, which we provided above.

We are certain that Euler could not have missed this enveloping property while studying series mainly by computing them.

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