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The 8th Congress of the International Society for Analysis, its Applications, and Computation (ISAAC) was organized by Peoples' Friendship University of Russia, Division of Mathematics of the Russian Academy of Sciences, V. A. Steklov Institute of Mathematics of Russian Academy of Sciences and V. M. Lomonosov Moscow State University, and took place at Peoples' Friendship University of Russia, Moscow, through 22-27 August 2011.

The programme of the Congress included most of the topics of contemporary mathematical analysis, in particular, real, functional, complex analysis, operator theory, theory of ordinary differential equations, theory of partial differential equations, nonlinear analysis, optimization theory, variational analysis, approximation theory, applications of analysis (inverse problems, functional and difference equations, mathematics in medicine, stochastic analysis), teaching analysis at universities and schools, history of analysis.

Vol. 1 contains papers by the plenary speakers D.E. Evans, A.T. Fomenko, K. Mochizuki, M. Sugimoto, and by the participants of Session I (Complex and hypercomplex analysis) and of Session II (Real analysis, functional analysis, and operator theory).

This volume would be of interest for mathematicians working in all main branches of contemporary mathematical analysis and its applications.

Prepared by the Organizing Committee of the 8th Congress of the ISAAC.

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Plenary Speakers

THE SEARCH FOR THE EXOTIC – SUBFACTORS AND CONFORMAL FIELD THEORIES

D. E. Evans, Terry Gannon

Key words: subfactors, conformal field theory

AMS Mathematics Subject Classification: 46L37, 81T40

Abstract. We look at the construction of conformal field theories and their modular invariants via tools from subfactor theory.

1 Introduction

There is a hierachy or pyramid of understanding:

— conformal field theory;
— statistical mechanical models;
— subfactors, vertex operator algebras and twisted K-theory;
— modular tensor categories, pre-projective algebras, Calabi-Yau algebras ...

The most basic algebraic structure here, namely that of a modular tensor category, may arise from subfactors, vertex operator algebras or twisted equivariant K-theory which in turn may give rise to statistical mechanical models which at criticality may produce conformal invariant field theories. That is to say, two-dimensional conformal field theories can be understood from the vantage point of conformal nets of subfactors or vertex operator algebras. In this paper we focus on the former setting, using von Neumann algebras of operators to understand modular invariant partition functions in statistical mechanics and conformal field theory. Our primary interest here is the search for integrable models or solvable models beyond what one can construct from loop groups and quantum groups or orbifolds from finite groups and related constructions like coset theories. For this purpose, subfactors are convenient. However, an alternative K-theoretic approach based on the twisted equivariant K-theoretic description of Verlinde algebras by [23] has been proposed by us in [13, 14, 16].

1.1 Operator algebras

Let us start with the basics of analytic and measure theoretic objects of operator algebras, i.e. with some fundamental examples of C^*-algebras and von Neumann algebras. We will begin with a fundamental example of an operator algebra –

tensor powers of 2×2 matrices: $\bigotimes^n M_2 \simeq M_{2^n} \simeq End(\bigotimes^n \mathbb{C}^2)$. We can complete this under the embedding $x \to x \otimes 1$ in the norm topology to get a meaning for the infinite tensor product, called the Pauli algebra: $\bigotimes^\infty M_2 \simeq M_{2^\infty}$.

To get some idea of the algebra, suppose we compute dimensions of projections $e = e^* = e^2$ using the trace: $\dim(\mathrm{e}) = \mathrm{trace(e)}/\mathrm{trace}(1) \in \{0, \frac{1}{2^n}, \frac{2}{2^n}, \frac{3}{2^n} \dots\ , 1\}$. Here we have really normalized the trace to be 1 on the identity operator so that the possible values are these dyadic rationals. They generate the semigroup of positive dyadic rationals $\mathbb{N}[1/2]$ and hence taking the Grothendieck completion the group of dyadic rationals $\mathbb{Z}[1/2]$. This is the K-group of this operator algebra, namely $K_0(\otimes_{\mathbb{N}} M_2)$. If we repeated this exercise with 3×3 matrices we would get the triadic rationals and so the two algebras are very different – they are not isomorphic.

What we are interested in though are von Neumann algebras, which are not only closed in the norm topology but in the weak operator topology. Suppose we complete this infinite tensor product in a different way. First represent the finite tensors on a Hilbert space. This can be done by turning the matrices into a Hilbert space using the trace as an inner product $H_2 = M_2\,, \langle x, y\rangle = \mathrm{tr}\, y^* x / \mathrm{tr}\, 1$ and letting the algebra act on itself by left multiplication so that $\otimes^n M_2 \subset End(\otimes^n H_2)$. Using the normalized trace, this is compatible as we increase n. We can then take the weak completion and get a different algebra $R = \otimes^\infty M_2 \subset End(\otimes^\infty H_2)$. If we compute the K-group using the dimensions of projections, we find that the gaps get filled in in the dyadic rationals picture and we get the real numbers $K_0(R) \simeq \mathbb{R}$. Remarkably, an isomorphic algebra is obtained from 3×3 matrices.

To get our definitions set up – factors are von Neumann algebras which cannot be split as a sum. This is the same as having trivial centre $R' \cap R = \mathbb{C}$, if R' denotes the commutant, or requiring all non-zero representations to be faithful. Factors are of three kinds. First are those of type I, the matrices and their infinite dimensional counterpart of bounded linear operators on a Hilbert space. We are only going to be concerned with hyperfinite factors [11], i.e. ones which can be approximated by matrices – as naturally occurs in statistical mechanical transfer matrix constructions. There is an unique hyperfinite factor which has a finite trace, the hyperfinite type II_1 R constructed above, and if the algebra is not type I and has an infinite trace the algebra is type II_∞ and is isomorphic to $R \otimes B(H)$. If the factor has no trace at all then the algebra is type III. Consequently, we have:

$$\mathrm{I}: \quad M_n\,, \quad B(H)\,; \qquad \mathrm{II}: \quad R, \quad R \otimes B(H)\,; \qquad \mathrm{III}$$

Indeed, all hyperfinite factors have been classified by Connes [11] with the III_1 case completed by Haagerup [27]. One construction of type III hyperfinite is to repeat the above construction of R with $H_2^j = M_2\,, \langle x, y\rangle_j = \mathrm{tr}\,(e^{-H_j} y^* x)/\mathrm{tr}\, e^{-H_j}$ and then complete $\otimes_{j=1}^n H_2^j$ with sufficiently non trivial Hamiltonians H_j. In the conformal field theory picture, type III (nets of) factors naturally arise from loop group representations.

1.2 Subfactors

A subfactor is an inclusion $N \subset M$ of one factor in another. Suppose to begin with that M is the hyperfinite II_1 factor, then by Connes [11] a subfactor is either a matrix algebra or (the case we are interested in) the hyperfinite II_1 and so isomorphic to M by $\rho : N \to M$. The larger algebra is a left module over the smaller one, and if this module is finitely generated and projective this yields an element of $[_N M] \in K_0(N) \simeq \mathbb{R}$. This is precisely when the Jones index $[N, M]$ is finite and equals this K-theoretic element $[_N M]$. The fundamental result of Jones [31] is that this index value is surprisingly constrained to be in $\{4cos^2(\pi/n)\} \cup [4, \infty)$.

We can extend the inclusion $N \subset M$ either upwards or downwards to a tower and a tunnel. There is a conjugate endomorphism $\bar{\rho}$ on M so that $\rho\bar{\rho} \succeq id_M$ just as for group representations or inverses of group elements. That allows us to continue the inclusion downwards. In the opposite direction we can extend upwards using a bi-module description or using a projection e of M onto N and adjoin:

$$\cdots \subset \rho\bar{\rho}M \subset \rho M \subset M \subset \langle M, e\rangle = M \otimes_N M \subset M \otimes_N M \otimes_N M \subset \cdots$$

$$\leftarrow \quad \text{tunnel} \qquad\qquad\qquad\qquad \text{tower} \quad \rightarrow$$

The sequence of projections e_j constructed in this way describe a Temperley-Lieb algebra. We then have a doubly-infinite sequence of inclusions of factors: $M_k \subset M_l \, , k \leqslant l$, and in the finite index case, the relative commutants $(M_k)' \cap M_l$ are all finite dimensional and thus are sums of matrix algebras.

$$\begin{array}{ccccc} N' \cap M_k & \subset & N' \cap M_{k+1} & \rightarrow & A \\ \cup & & \cup & & \cup \\ M' \cap M_k & \subset & M' \cap M_{k+1} & \rightarrow & B \end{array} \tag{1}$$

An embedding between finite dimensional algebras , e.g. $N' \cap M_k \subset N' \cap M_{k+1}$, gives rise to a multiplicity graph. However due to periodicity $M_k \subset M_l \simeq M_{k+2} \subset M_{l+2}$, which is related to Pontryagin duality, only two bi-partite graphs really arise, called the principal and dual principal graphs, which adjoin as in the example of Figure 1 (i). There is however more information in the square by comparing two ways of embedding $M' \cap M_k \subset N' \cap M_{k+1}$. This is given by a connection in the terminology of Ocneanu [36], (see also [18]), an assignment of a complex number to each square whose edges are labelled by those of the two graphs. This is related to Boltzmann weights of statistical mechanical models with local configurations on the diamond of Figure 1 (ii). If we start with arbitrary graphs and try to set up subfactors by using the model of (1) with squares of finite dimensional approximations, we would need some integrability as in the Yang-Baxter equation of Figure 1 (iii) to ensure that the subfactor $B \subset A$ constructed in this way has the original graphs as their principal graphs. For example, E_7 does not appear in this way as a principal graph, and if we try to build up a subfactor from it in the natural way, then the principal graphs will both be D_{10}.

Figure 1. (i) Principal graphs examples (ii) Boltzmann diamond (iii) Yang-Baxter eqn

We can think of these relative commutants via decomposing endomorphisms or bimodules into irreducibles

$$(\rho\bar{\rho}\rho\bar{\rho}\cdots M)' \cap M \simeq End_N(M \otimes_N \cdots \otimes_N M)_{N\, or\, M}\, .$$

Going from one stage to the next is via multiplication by the fundamental object ρ or M in the endomorphism or bi-module descriptions respectively. This is illustrated in the Bratteli diagram examples of Figure 2, where the irreducibles ρ_i appear as one decomposes higher and higher powers of ρ and $\bar{\rho}$.

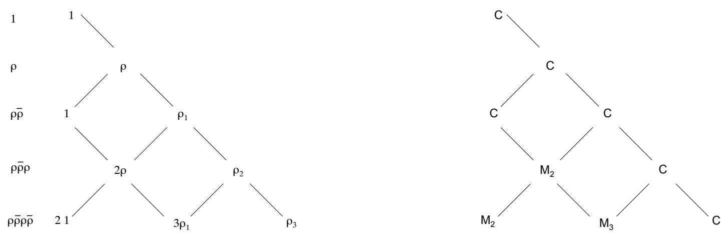


Figure 2. (i) Decomposing irreducibles (ii) Bratteli diagram

To give some concrete examples, suppose a finite group G acts outerly on a hyperfinite factor R. We can form the inclusion $R^G \subset R$ of fixed points which has the inclusion $R \subset R \rtimes G$ as the natural extension. We can iterate using dual actions, and the principal graphs in the case of the symmetric group S_3 is precisely as in Figure 1 (i). The upper vertices are labelled by group elements $g \in G$ and the lower ones by group representations $\pi \in \hat{G}$, with multiplicity $\dim(\pi)$.

There are of course two groups of cardinality 4, but at the integer index 4 we can construct examples, indeed all index 4 examples, via tensoring with 2×2 matrices and the natural adjoint $SU(2)$ group action. Taking the inclusion $R = \otimes_{\mathbb{N}} M_2 \subset R \otimes M_2$, the larger algebra is clearly 4 copies of the smaller as an R module and so the index is 4. The tower is the obvious one $R = \otimes_{\mathbb{N}} M_2 \subset R \otimes M_2 \subset R \otimes M_2 \otimes M_2 \subset R \otimes M_2 \otimes M_2 \otimes M_2 \subset \cdots$ with the relative commutants being finite dimensional tensors of two by two matrices. Taking fixed point actions under the product adjoint action of say $G = SU(2)$ the tower is $\mathbb{C} = M_2^G \subset (M_2 \otimes M_2)^G \subset (M_2 \otimes M_2 \otimes M_2)^G \subset \cdots$. The relative commutants are just the fixed point algebras $(\otimes^n M_2)^G$ generated through Weyl duality by transposition matrices in $End(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2)$ or a representation of the symmetric group. The eigen-projections of these transpositions are precisely the Temperley-Lieb projections at index 4. Comparing with the template of Figure 2, the graphs drawn there are precisely what appears for this $SU(2)$ example, and the irreducibles ρ_i of $SU(2)$ are the natural labelling. Deforming the action of $SU(2)$ to a quantum group reduces the index and yields certain representations of a Hecke algebra and related integrability or braid group Yang-Baxter type relations as in Figure 1 (iii).

At index four there is a classification of subfactors by affine ADE diagrams corresponding to subgroups of $SU(2)$ and twisted by cohomology. In the deformed case, with indices less than 4, there is an ADE classification but E_7 and D_{odd} do not appear. There is an analogous story for $SU(3)$, with subgroups of $SU(3)$ providing index 9 subfactors though the corresponding subfactors of index less than 9 are not so closely related. Figure 4, an embellishment of an atlas of [34] summarizes the possible values of indices but also maps other classifying graphs, namely the nimrep graphs of modular invariants which we will come to shortly.

The classification between 4 and 5 was recently completed by Izumi, Jones, Morrisson, Penneys, Peters [30,35] following the fundamental work of Haagerup [28]. At index 5 there are certain group-like subactors but between 4 and 5 there are only 10 finite depth subfactors. The first is that of Haagerup [28] at index $(5+\sqrt{13})/2$ and its dual, followed by that of Asaeda-Haagerup and its dual with index value a root of some cubic, the extended Haagerup $(5+\sqrt{17})/2$ and its dual whose existence was shown in [28]. The conformal embedding subfactor of $SU(2)_{10}$ in $SO(5)_1$ has principal graph the star shaped graph 3311 (where $n_1 n_2 \ldots n_m$ has m arms of length $n_1, n_2, \ldots n_m$) of index $3+\sqrt{3}$ and its dual. Finally there is the self dual Izumi subfactor 2221 of index $(5+\sqrt{21})/2$ and its opposite.

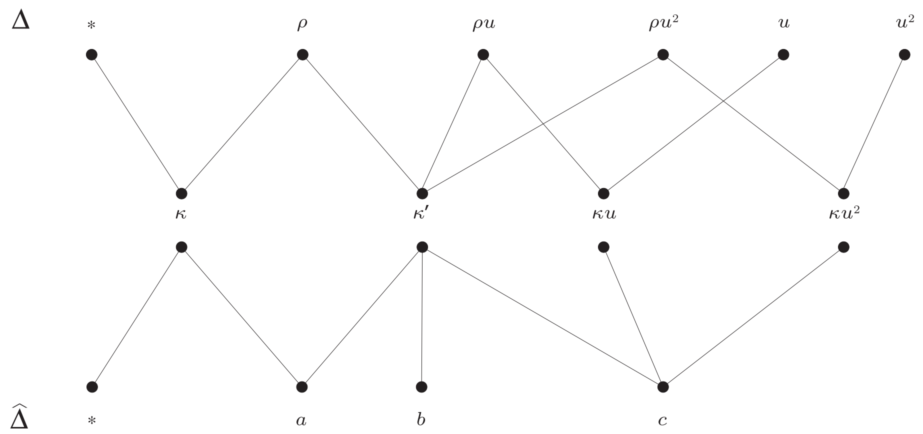


Figure 3. Principal and dual principal graphs for the Haagerup subfactor

The Haagerup subfactor is the first finite depth subfactor of index bigger than 4. It can be regarded as a deformation of the symmetric group S_3, with even vertices satisfying the non-commutative fusion rules: $\alpha^3 = 1\,, \rho\alpha = \alpha^2\rho\,, \rho^2 = 1 + \rho + \rho\alpha + \rho\alpha^2$. The statistical dimension $d_\rho = [M, \rho M]^{1/2}$ satisfies the relation $d_\rho^2 = 1 + 3d_\rho$ and so $d_\rho = (3 + \sqrt{13})/2$. The index $d_\kappa^2 = d_\rho + 1$ of the Haagerup subfactor $\kappa M \subset M$ is then $(5 + \sqrt{13})/2$. There are currently three ways to construct this subfactor. One is by bare hands – Haagerup constructed basically 6j-symbols or Boltzmann weights. Izumi showed the existence of this subfactor by constructing endomorphisms on Cuntz algebras satisfying these fusion rules [29]. More recently [5] found the Haagerup subfactor by constructing the planar algebra or relative commutants. Izumi [29] put the Haagerup in a potential series of subfactors for the graphs 33...3 ($2n + 1$ arms) and an abelian group of order $2n + 1$, and established existence and uniqueness for $\mathbb{Z}_3$ and $\mathbb{Z}_5$. We showed [15] that there are (respectively) 1, 2, 0 subfactors of Izumi type $\mathbb{Z}_7$, $\mathbb{Z}_9$ and $\mathbb{Z}_3^2$, and found strong numerical evidence for at least 2, 1, 1, 1, 2 subfactors of Izumi type $\mathbb{Z}_{11}, \mathbb{Z}_{13}, \mathbb{Z}_{15}, \mathbb{Z}_{17}, \mathbb{Z}_{19}$. We are confident there will be at least one subfactor of Izumi type, for the cyclic group $\mathbb{Z}_{2n+1}$ (any n), and more than one whenever $4n^2 + 4n + 5$ is composite. More recently [17], we generalised Izumi's framework, weakening his equations and allowing solutions for even order abelian groups. In particular, we have constructed new subfactors at indices $3 + \sqrt{5}$ and $4 + \sqrt{10}$ and with graphs 3333 and 333333, and expect these to again fall into an infinite series.

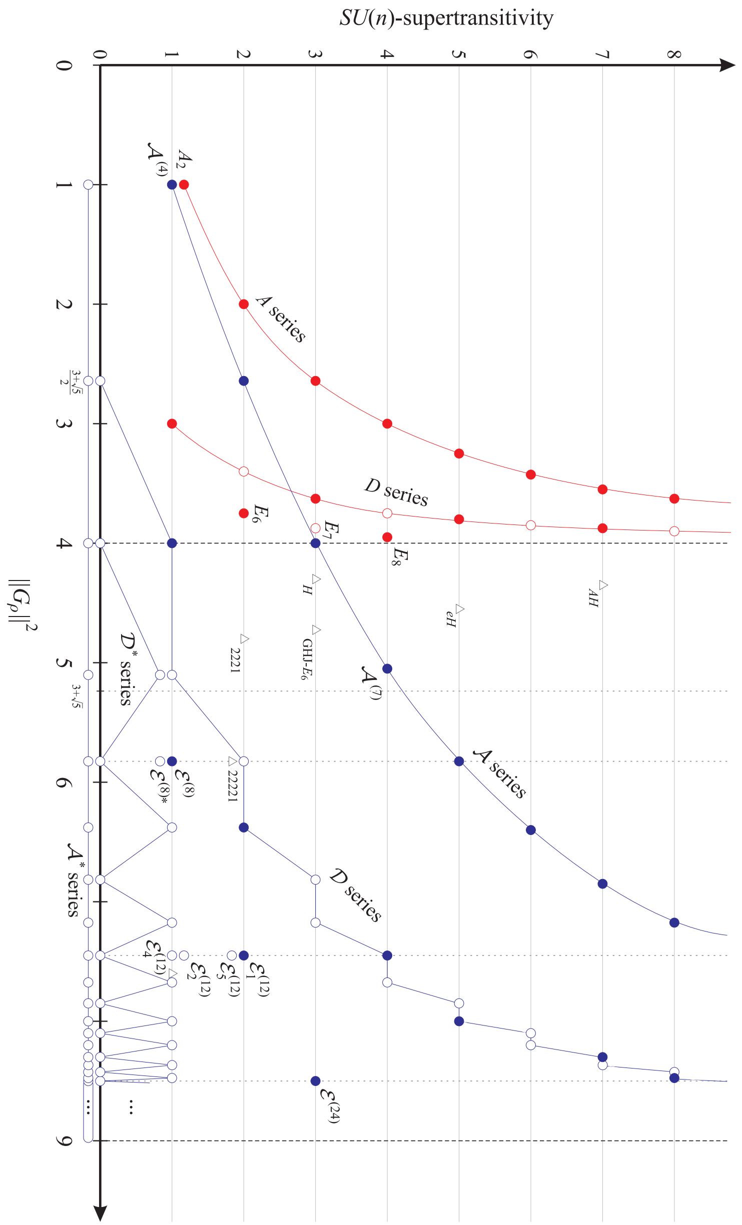


Figure 4. Plotting the $SU(n)$-supertransitivity and the norm squared of the $SU(n)$ N-M nimrep graphs G_ρ, $n = 2, 3$

2 Statistical Mechanical models at criticality

We are not only interested in subfactors but braided systems – in the type III setting with systems of endomorphism reproducing the Verlinde fusion ring. Before we indulge in the mathematical aspects of this, let us see how conformal field theories naturally throw up such structures, beginning with statistical mechanical models at criticality.

We can look in detail at the case of the Ising model which will exhibit many of the features we wish to highlight. Take the nearest neighbour Ising hamiltonian on the configuration space $\mathcal{P} = \{\pm\}^{\mathbb{Z}^2}$, $H(\sigma) = \Sigma_{\alpha,\beta} J\sigma_\alpha\sigma_\beta$ for $\sigma \in \mathcal{P}$. Then the partition function decomposes as $Z = \Sigma_\sigma exp(-H(\sigma)) = \Sigma\ \Pi$ Boltzmann weights for a Boltzmann weight involving local interactions on a plaquette. We can compute this by first taking the partition function $T_{\zeta\eta}$ of a column, with boundary distributions ζ, η. This can be computed using vertical and horizontal interactions in the nearest neighbour Hamiltonian:

$$T \;=\; V^{1/2}\, W\; V^{1/2} = e^{-\mathcal{H}}$$

Here $V = exp\, K\Sigma\sigma^x_j\sigma^x_{j+1}$ and $W = exp\, L^*\Sigma\sigma^z_j$ are the partition functions or transfer matrices for interactions along columns and rows respectively, in terms of Pauli matrices, where σ^x is the diagonal matrix with ± 1 eigenvalues with eigenvectors $|\pm\rangle$, and σ^z interchanges these vectors, and K and K^* are temperature dependant interaction constants. At zero temperature K^* is zero and $T = V$ has a degenerate 2-dimensional largest eigenspace, whilst at infinite temperature K vanishes and $T = W$ has a non-degenerate largest eigenspace spanned. To relate this to the operator algebraic approach to the phase transition and subsequent algebraic conformal field theory, it is slicker to work with a half lattice $\mathbb{Z} \times \mathbb{N}$, but see [18] for a discussion of the full lattice. Then by this transfer matrix formalism, the classical one-dimensional lattice model is understood via a two-dimensional noncommutative quantum model $C\{+,-\}^{\mathbb{Z}\times\mathbb{N}} = \otimes_{\mathbb{Z}\times\mathbb{N}}(\mathbb{C}^2) \to M_2 \otimes M_2 \otimes \cdots$ where classicial expectation values are computed via quantum ones $\mu(F) = \phi_\mu(F_\beta)$ with time development α_t given by the quantum Hamiltonian $\mathcal{H} = \log T$,
$\alpha_t = T^{it}(\ \ )T^{-it} = Ad\, e^{i\mathcal{H}t}$. Equilibrium states in the classical model correspond to ground states in the quantum model. At zero temperature, there are two extremal states given by $\phi_0^+ = \otimes_{\mathbb{Z}}\omega_+$ $\phi_0^- = \otimes_{\mathbb{Z}}\omega_-$; and at infinite temperature $\phi_\infty^+ = \otimes_{\mathbb{Z}}\omega$. Here $\omega_\pm A = \langle A\pm, \pm\rangle$ are the vector states on M_2 for the $\pm$ eigenspaces of σ^x and w

is the vector state for the equi-distribution $(|+\rangle + |-\rangle)/\sqrt{2}$, the largest eigenspace for σ^z. What interests us here is that the Kramers-Wannier high temperature - low temperature duality, which interchanges the roles of V and W, relates the ground states at infinite and zero temperature $\phi_0^+ = \phi_\infty \nu$ if ν is the automorphism which switches $\sigma_j^x \sigma_{j+1}^x \leftrightarrow \sigma_j^z$. More precisely, define $\nu \sigma_j^x \sigma_{j+1}^x = \sigma_{j+1}^z$ and $\nu \sigma_j^z = \sigma_j^x \sigma_{j+1}^x$. Here ν is only defined on the even part of the Pauli algebra, if we grade σ^x as odd and σ^z as even. Squaring ν^2 is not the identity but a shift, the restriction of $\sigma_j^\gamma \to \sigma_{j+1}^\gamma$ to the even Pauli algebra. We can extend ν to the whole Pauli algebra by defining a Jordan-Wigner formulation it to be $\nu \sigma_j^x = \sigma_1^z \sigma_2^z \cdots \sigma_j^z$, but ν^2 is no longer the shift. To understand the key role of ν it is convenient to extend to a larger ambient algebra which is infinite with no trace - a Cuntz algebra $\mathcal{O}_2$ which is the semi-direct product of the Pauli algebra by the shift $\otimes^{\mathbb{N}} M_2 \rtimes \mathbb{N}$. The algebra $\mathcal{O}_2$ is generated by two isometries s_+, s_- with orthogonal ranges summing to the identity, $s_+ s_+^* + s_- s_-^* = 1$. The Pauli algebra is naturally contained in the Cuntz algebra, e.g. $s_+ s_+^*, s_- s_-^*, s_- s_+^*, s_+ s_-^*$ form a copy of the matrix units of M_2.

This formalism enables amongst other things one to handle non-rectangular transfer matrices algebraically with for example s_+ on the right below:

We can extend ν to the Cuntz algebra with $\nu(s_+ \pm s_-) = \sqrt{2}(s_+ s_\pm s_\pm^* + s_- s_\mp s_\mp^*)$ with the property on generators $\nu^2(s_\sigma) = s_+ s_\sigma s_+^* + s_- s_{-\sigma} s_-^*$, and hence for any $x \in \mathcal{O}_2$ we get $\nu^2(x) = s_+ x s_+^* + s_- \alpha x s_-^*$ if α denotes the automorphism of $\mathcal{O}_2$ which interchanges $+$ and $-$, i.e. $s_+ \leftrightarrow s_-$. What this means is that we can decompose the underlying Hilbert space $\mathcal{K}$ on which $\mathcal{O}_2$ acts by $\mathcal{K} = s_+\mathcal{K} + s_-\mathcal{K}$ so that $\nu^2(x)$ is represented as in Figure 5.(i), i.e. $\nu^2 = id + \alpha$. We are naturally led to systems of endomorphisms on infinite algebras (type III if completed appropriately) with fusion rules represented by Figure 5.(ii) so that vertices represent endomorphisms and edges multiplication by the fundamental generator ν.

Figure 5: (i) $\nu^2 = 1 + \alpha$ (ii) Ising endomorphisms

Taking lattice models, periodic in one direction, leads to cylinders with boundary conditions, and then a torus with defect lines.

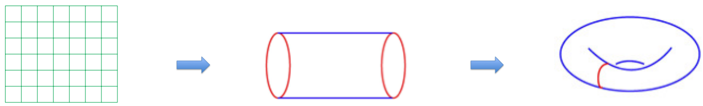


Figure 6: (i) lattice model (ii) cylinder with boundary conditions (iii) torus with defects

In the continuum limit we may expect to get a field theory with a partition function Z which decomposes as relative to some underlying symmetry (the underlying vertex operator algebra):

$$Z = \operatorname{tr} e^{2\pi i\tau(L_0 - c/24)} e^{-2\pi i\overline{\tau}(\overline{L_0} - c/24)} = \sum Z_{\lambda\mu}\chi_\lambda(\tau)\chi_\mu(\tau)^*,$$

where $\chi_\lambda = \operatorname{tr} q^{L_0 - c/24}$, $q = e^{2\pi i\tau}$, are the characters corresponding to irreducible λ. It was argued by Cardy that the parition function is invariant under re-parameterisations of the torus: $Z(\tau) = Z((a\tau + b)/(c\tau + d))$. Since typically the characters themselves transform linearly under the action of $SL(2,\mathbb{Z})$, a modular invariant gives rise to a matrix of multiplicities $Z_{\lambda\mu} \in \{0, 1, 2, \ldots\}$, satisfying $Z = [Z_{\lambda\mu}] \in SL(2,\mathbb{Z})'$ and $Z_{00} = 1$, where 0 denotes the vacuum.

In the case of the two-dimensional Ising model, there are three irreducibles corresponding to the vertices of the A_3 Dynkin diagram of Figure 5 (ii), with $\pm$ labelling the end points and $\bullet$ the internal vertex. The transfer matrix formalism allows a description in terms of fermion operators $g_a : a \in \mathbb{N} - 1/2$ *or* $\mathbb{N}$ with half integer or integer labels and corresponding Hamiltonians and characters: $L_0 = \Sigma_{r\in\mathbb{N}-1/2} r g_r^* g_r \to \chi_\pm$, $L_0 = \Sigma_{n\in\mathbb{N}} n g_n^* g_n \to \chi_\bullet$. The half integer Hamiltonian is reducible according to a parity with corresponding characters:

$$\chi_+ \pm \chi_- = q^{-1/48}\Pi_{n\in\mathbb{N}}(1 \pm q^{n-1/2}), \quad \chi_\bullet = q^{1/24}\Pi_{n\in\mathbb{N}}(1 + q^n)$$

The corresponding action of $SL(2,\mathbb{Z})$ is given by:

$$\begin{aligned} \tau \to -1/\tau \qquad S &= \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \\ \tau \to \tau + 1 \qquad T &= \operatorname{diag}(e^{-\pi i/24}, e^{-\pi i/12}, e^{-\pi i 23/24}) \end{aligned}$$

What we need are braided systems of endomorphisms – not necessarily commutative but which commute up to an adjustment which can be chosen to satisfy the Yang-Baxter or braid relations and braiding fusion relations. Crossings represent intertwiners, from which one can form S and T matrices as scalar intertwiners from the Hopf link and a twist.

There are two principal sources of examples. The first arises from loop groups, e.g. that of $SU(n)$, developed by Wassermann and his students [42]. Restricting to loops concentrated on an interval $I \subset S^1$ (proper, i.e. $I \neq S^1$ and non-empty), the corresponding subgroup denoted by $L_I SU(n) = \{f \in LSU(n) : f(z) = 1\,, z \notin I\}$, one finds that in each positive energy representation π_λ we obtain hyperfinite type III_1 subfactors $\pi_\lambda(L_I SU(n))'' \subset \pi_\lambda(L_{I^c} SU(n))'$, where I^c denotes the complementary interval [42]. In the vacuum representation, labelled by $\lambda = 0$, we have Haag duality in that the inclusion collapses to a single factor $N(I) = N(I)$. More generally, the inclusion can be read as providing an endomorphism λ of the local algebra $N(I)$ such that the inclusion $\pi_\lambda(L_I SU(n))'' \subset \pi_\lambda(L_{I^c} SU(n))'$ is isomorphic to $\lambda(N(I)) \subset N(I)$. In this way we obtain systems of endomorphisms – which are braided from locality considerations where to compare two endomorphisms on the same interval we move one away to another disjoint interval, where commutativity holds, and then back again.

The second class comes from taking the double of systems of endomorphisms which themselves may not be braided nor even commutative, such as the quantum double of a finite group, Haagerup subfactor etc. If ${}_N\mathcal{X}_N$ denotes a system of endomorphisms on a type III factor, then there is a subfactor $\iota : A \subset N \otimes N^{opp}$, whose canonical endomorphism $\bar{\iota}\iota$ is expressible as $\Sigma_{\lambda \in \mathcal{X}} \lambda \otimes \lambda^{opp}$, with a non-degenerately braided system of endomorphisms on A. Thus doubles naturally come with braided inclusions.

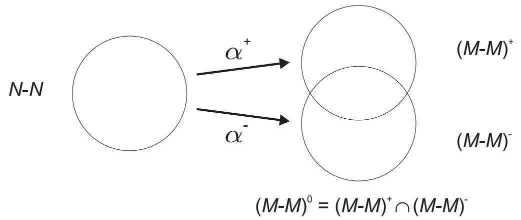
2.1 Subfactor framework for modular invariants and RCFT

To understand modular invariants of the form $\sum Z_{\lambda\mu} \chi_\lambda \chi_\mu^*$, let us first consider the obvious one: the diagonal invariant $\sum \chi_\lambda \chi_\lambda^*$ or more generally $\sum \chi_\tau \chi_{\sigma\tau}^*$ for suitable permutations σ of the irreducibles. In some sense, made precise in [7], every modular invariant is of this form in some extended system. In subfactor language, the factor N which carries the Verlinde algebra $\mathcal{A}$ as a system of endomorphims is embedded in a larger von Neumann algebra with a system $\mathcal{B}$ of endomorphisms. When we restrict to the smaller system, σ-restriction on characters $\chi_\tau = \sum b_{\tau\lambda} \chi_\lambda$ should be interpreted as $\sigma_\tau = \sum b_{\tau\lambda} \lambda$ as endomorphisms. In particular this will certainly mean that ${}_N M_N$ thought of as an endomorphism decomposes as a sum of λ's. Moreover the diagonal modular invariant for the ambient $\mathcal{B}$ system decomposes as

$$Z = \sum \chi_\tau \chi_{\sigma\tau}^* = \sum \left(\sum b_{\tau\lambda} \chi_\lambda\right) \left(\sum b_{\sigma\tau\,\lambda} \chi_\lambda\right)^* = \sum Z_{\lambda\mu}\, \chi_\lambda \chi_\mu^*$$

to yield a possibly non-trivial $Z_{\lambda\mu} = \sum b_{\tau\lambda} b_{\sigma\tau\,\mu}$.

However, in practice we will not be given the ambient extended system $\mathcal{B}$ but instead will start with an inclusion $N \subset M$ such that ${}_N M_N$ decomposes as a sum of λ's in $\mathcal{A}$. In such a situation we can induce the system on N to systems on M. Using the braiding and its opposite we get two ways of getting endomorphisms on M, namely $\alpha^{\pm} : \lambda \longrightarrow \alpha^{\pm}_{\lambda}$. What is important is their intersection.

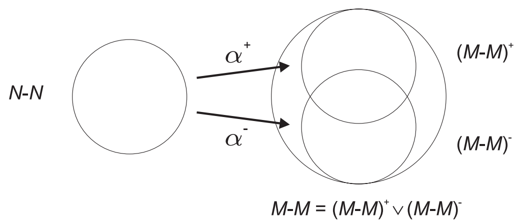


When we decompose $\alpha^{+}\lambda, \alpha^{-}\mu$ into irreducibles, we count the number of common sectors and get a multiplicity $Z_{\lambda\mu} = \langle \alpha^{+}_{\lambda}, \alpha^{-}_{\mu} \rangle$. The resulting $\sum Z_{\lambda\mu}\chi_{\lambda}\chi^{*}_{\mu}$ is a modular invariant. By associativity, we can regard the multiplication of the N-N system on itself as a representation of the Verlinde algebra $\lambda\mu = \sum_{\nu} N^{\nu}_{\lambda\mu}\nu$ by commuting matrices $N_{\lambda} = [N^{\nu}_{\lambda\mu}]_{\mu\nu}$.

Such a family of commuting matrices can be straightforwardly diagonalised: $N_{\lambda} = \sum_{\kappa} S_{\lambda\kappa}/S_{0\kappa}|S_{\kappa}\rangle\langle S_{\kappa}|$. What is not straightforward is that the diagonalising matrix is the same as the S matrix in the representation of $SL(2,\mathbb{Z})$.

We can form a system of N-M sectors ${}_{N}\mathcal{X}_{M}$ from $\iota\lambda$, where $\lambda \in {}_{N}\mathcal{X}_{N}$ and $\iota : N \subset M$. Now, multiplication of N-N on N-M gives a nimrep – a representation of the Verlinde algebra by positive integer matrices $G_{\lambda} = [G^{b}_{\lambda a}]_{ab}$. These can likewise be diagonalised: $G_{\lambda} = \sum_{\kappa} S_{\lambda\kappa}/S_{0\kappa}|\psi_{\kappa}\rangle\langle\psi_{\kappa}|$, with spectrum $\sigma(G_{\lambda}) = \{S_{\lambda\mu}/S_{0\lambda} : \text{multiplicity } Z_{\lambda\lambda}\}$ coinciding precisely with the diagonal part of the modular invariant. In the case of $SU(2)$ modular invariants, this is the conceptual explanation of the ADE classification of Capelli-Itzykson-Zuber [10]. All $SU(2)$ [10] and $SU(3)$ [25] modular invariants can be realised by subfactors following work of Ocneanu, Feng-Xu, Böckenhauer, Evans, Kawahigashi and Pugh. We refer to the review article [21] for precise references.

The map of Figure 4 describes a map of nimrep index values, i.e. the squares of the norms of nimrep generators λ =fundamental weight, for $SU(2)$ (roman) and $SU(3)$ (script). The $SU(n)$-supertransitivity measures how far the nimrep graph remains alike to the identity nimrep graph before diverging following Jones [32] in the bi-partite or $SU(2)$ case, with a precise definition in the review [21].



The larger family ${}_M\mathcal{X}_M$ of M-M sectors is obtained from the irreducibles of $\iota\lambda\bar{\iota}$ and co-incides with those generated by the images of the two inductions by decomposing $\alpha_\lambda^+\alpha_\nu^-$ when the braiding is non-degenerate. Remarkably, this can be identified with the nimrep graph for the (usually non-normalized) modular invariant ZZ^*.

In the cases we are interested in, the factor N is obtained as a local factor $N = N(I_\circ)$ of a conformally covariant quantum field theoretical net of factors $\{N(I)\}$ indexed by proper intervals $I \subset \mathbb{R}$ on the real line arising from current algebras defined in terms of local loop group representations, and the N-N system is obtained as restrictions of Doplicher-Haag-Roberts morphisms (cf. [26]) to N. Taking two copies of such a net and placing the real axes on the light cone, then this defines a local conformal net $\{A(\mathcal{O})\}$, indexed by double cones $\mathcal{O}$ on two-dimensional Minkowski space (cf. [40] for such constructions). A braided subfactor $N \subset M$, determining in turn two subfactors $N \subset M_\pm$ obeying chiral locality, will provide two local nets of subfactors $\{N(I) \subset M_\pm(I)\}$. Arranging $M_+(I)$ and $M_-(J)$ on the two light cone axes defines a local net of subfactors $\{A(\mathcal{O}) \subset A_{\text{ext}}(\mathcal{O})\}$ in Minkowski space. The embedding $M_+ \otimes M_-^{\text{op}} \subset B$ gives rise to another net of subfactors $\{A_{\text{ext}}(\mathcal{O}) \subset B(\mathcal{O})\}$, where the conformal net $\{B(\mathcal{O})\}$ satisfies locality. As shown in [40], there exist a local conformal two-dimensional quantum field theory such that the coupling matrix Z describes its restriction to the tensor products of its chiral building blocks $N(I)$. There are chiral extensions $N(I) \subset M_+(I)$ and $N(I) \subset M_-(I)$ for left and right chiral nets which are indeed maximal and should be regarded as the subfactor version of left- and right maximal extensions of the chiral algebra.

2.2 Exotic possibilities

The most natural place to look for exotic possibilities of subfactors and hence of conformal field theories is with the Haagerup subfactor and its siblings. However to get braided systems we need to take the doubles. The upper part of Fig. 7 shows the double of the even part of the principal graph Δ of the Haagerup subfactor, computed by Izumi, and the lower part comes from the double of the even part of the dual principal graph, computed in [15].

This was the first time the dual graph was computed – using the theory of modular invariants for the double which as we have noted come equipped with canonical braided inclusions and hence canonical modular invariants. These should be compared with the corresponding objects for the doubles of the symmetric group and its dual. Note how we can recover the graph and dual graph for S_3 from this diagram by tracing from the vacuum sector on the bottom to the top, and vice versa respectively.

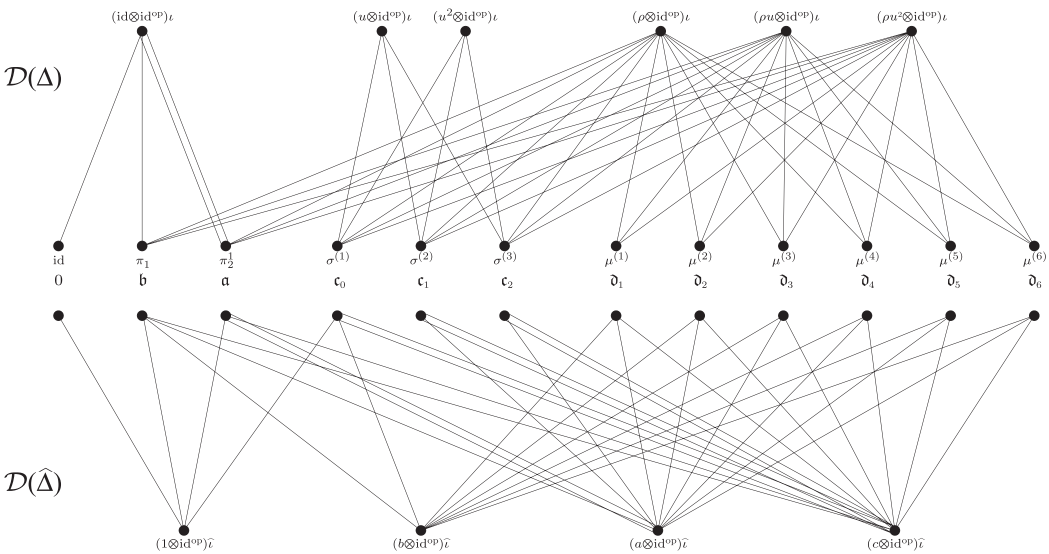


Figure 7: Dual principal graphs of the double of the Haagerup subfactor

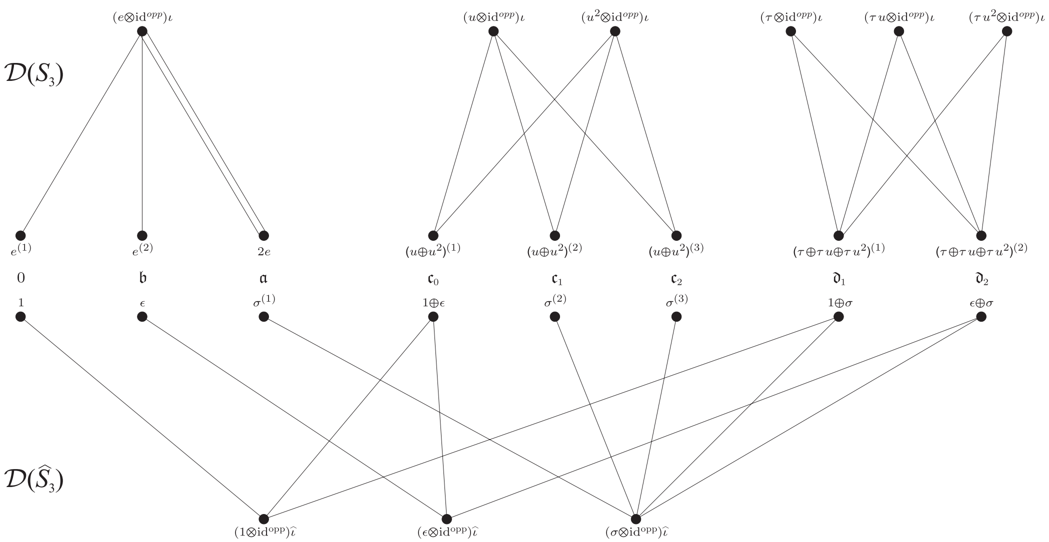


Figure 8: Dual principal graphs of the double of the S_3 subfactor

The Haagerup modular data was computed by Izumi [29], with T being the diagonal matrix $\mathrm{diag}(1,1,1,1,\xi_3,\overline{\xi_3},\xi_{13}^6,\xi_{13}^{-2},\xi_{13}^2,\xi_{13}^5,\xi_{13}^{-6},\xi_{13}^{-5})$. His S matrix though was obscure and involved a complicated rational function in $e^{2\pi ß/13}$ and $(1+ß\sqrt{5+2\sqrt{13}})/(1+\sqrt{13})$. We derived an explicit simple description for the S matrix:

$$S=\frac{1}{3}\begin{pmatrix} x & 1-x & 1 & 1 & 1 & 1 & y & y & y & y & y & y \\ 1-x & x & 1 & 1 & 1 & 1 & -y & -y & -y & -y & -y & -y \\ 1 & 1 & 2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & -y & 0 & 0 & 0 & 0 & c(1) & c(2) & c(3) & c(4) & c(5) & c(6) \\ y & -y & 0 & 0 & 0 & 0 & c(2) & c(4) & c(6) & c(5) & c(3) & c(1) \\ y & -y & 0 & 0 & 0 & 0 & c(3) & c(6) & c(4) & c(1) & c(2) & c(5) \\ y & -y & 0 & 0 & 0 & 0 & c(4) & c(5) & c(1) & c(3) & c(6) & c(2) \\ y & -y & 0 & 0 & 0 & 0 & c(5) & c(3) & c(2) & c(6) & c(1) & c(4) \\ y & -y & 0 & 0 & 0 & 0 & c(6) & c(1) & c(5) & c(2) & c(4) & c(3) \end{pmatrix},$$

for $x=(13-3\sqrt{13})/26$, $y=3/\sqrt{13}$ and $c(j)=-2y\cos(2\pi j/13)$. That this bears some relation with the double of S_3 may not be surprising given the relations between the Haagerup fusion rules and those of S_3 and $\widehat{S}_3$. There is however also a striking relationship with the affine algebra modular data $B_{6,2}$ which has central charge $c=12$, and 10 primaries. The T-matrix is $\mathrm{diag}(-1,-1;-ß,ß;-\xi_{13}^{6l^2})$, while the S-matrix is [33]

$$S=\frac{1}{3}\begin{pmatrix} y/2 & y/2 & 3/2 & 3/2 & y & y & y & y & y & y \\ y/2 & y/2 & -3/2 & -3/2 & y & y & y & y & y & y \\ 3/2 & -3/2 & 3/2 & -3/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3/2 & -3/2 & -3/2 & 3/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & y & 0 & 0 & -c(1) & -c(2) & -c(3) & -c(4) & -c(5) & -c(6) \\ y & y & 0 & 0 & -c(2) & -c(4) & -c(6) & -c(5) & -c(3) & -c(1) \\ y & y & 0 & 0 & -c(3) & -c(6) & -c(4) & -c(1) & -c(2) & -c(5) \\ y & y & 0 & 0 & -c(4) & -c(5) & -c(1) & -c(3) & -c(6) & -c(2) \\ y & y & 0 & 0 & -c(5) & -c(3) & -c(2) & -c(6) & -c(1) & -c(4) \\ y & y & 0 & 0 & -c(6) & -c(1) & -c(5) & -c(2) & -c(4) & -c(3) \end{pmatrix},$$

where y and $c(j)$ is as before. Ignoring the first 4 primaries, the only difference with the Haagerup modular data are some signs.

The question then arises as to whether there is a corresponding conformal field theory. Some evidence in support of this is that characters $\chi_\lambda(\tau)$ (with nonnegative integer coefficients) which transform among themselves according to this $SL(2,\mathbb{Z})$ representation were found following the procedures developed by [3].

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HIDDEN SYMMETRIES IN INTEGRABLE DYNAMICAL SYSTEMS

A. T. Fomenko

Key words: integrable systems with two degrees of freedom, Morse functions, atoms, molecules

AMS Mathematics Subject Classification: 11B39

Abstract. In case of integrable Hamiltonian dynamical systems with two degrees of freedom (restricted, for simplicity, on 3-dimesional isoenergy surface Q) the author in collaboration with H.Zieschang discovered an important invariant. The invariant is called "marked molecule" W^* and is some graph with specific vertices ("atoms") and numerical marks on its edges. Two such systems are said to be Liouville equivalent if their phase spaces are foliated in the same way into Liouville tori (tori are the closures of integral trajectories).

1 Introduction

It is well known that many systems of differential equations that appear in physics, geometry, and mechanics and describe quite different phenomena, turn out nevertheless to be closely connected and sometimes can be transformed one to another by some diffeomorphism (or homeomorphism). But the recognition of such "equivalent systems" is needed in some topological invariants. In case of integrable Hamiltonian dynamical systems with two degrees of freedom (restricted, for simplicity, on 3-dimesional isoenergy surface Q) such invariant was discovered by the author in collaboration with H.Zieschang. Integrability means that in some sense the dynamical system has "hidden symmetries". The invariant is called "marked molecule" W^* and is some graph with specific vertices ("atoms") and numerical marks on its edges. Two such systems are said to be Liouville equivalent if their phase spaces are foliated in the same way into Liouville tori (tori are the closures of integral trajectories). The molecule W^* can be naturally considered as a portrait of the integrable Hamiltonian system. The basic theorem of this theory is as follows (A.T.Fomenko, H.Zieschang).

Theorem 1. *Let (v, Q) and (v', Q') be two integrable systems, and let W^* and W^*' be the corresponding marked molecules. Then the systems v and v' are Liouville equivalent if and only if the molecules W^* and $W^{*\prime}$ coincide.*

Let us recall here the result by A.V.Bolsinov and A.T.Fomenko: well known Jacobi problem (geodesic flow on 2-ellipsoid) and the Euler case (in the dynamics of rigid body motion) 1) are Liouville equivalent, 2) are topologically (continuosly) orbitally conjugate, but 3) are not (in general case) smoothly orbitally conjugate.

Recently a new results were obtained in the problem of calculation of these topological invariants and classification of many concrete integrable systems up to their equivalence. It was discovered that many well known integrable systems (which usually are considered as "different") are Liouville equivalent on some isoenergy 3-surfaces. These results are based on the recent topological analysis of integrable systems, which was done by A.A.Oshemkov, P.V.Morozov, A.Yu.Moskvin, Zotiev D.B., D.G.Hagigatdust B.G., H.Horshidi.

2 Morse functions and 2-atoms as the description of bifurcations. Simple atoms and molecules

Consider a smooth function $f(x)$ on a smooth manifold X^n, and let $x_1, \ldots, x_n$ be a smooth regular coordinates in a neighborhood of a point $p \in X^n$. The point p is

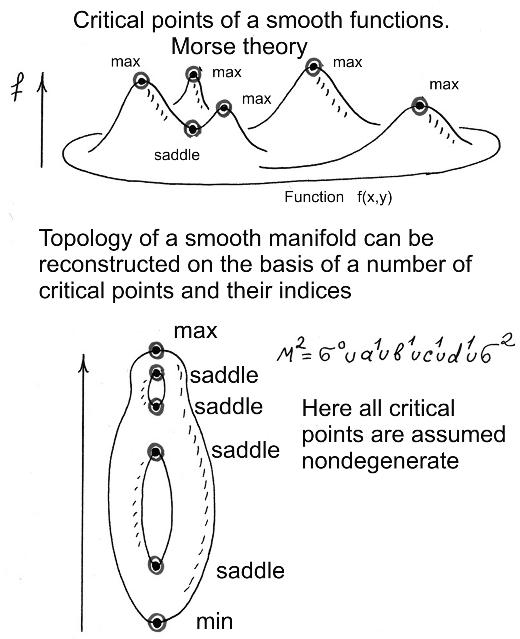


Figure 1. General approach of Morse theory

called *critical* for the fucntion f if the differential

$$df = \Sigma \frac{\partial f}{\partial x_i} dx_i$$

vanishes at the point p.

The critical point is called $non-degenerate$ if the second differential

$$d^2 f = \Sigma \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j$$

is non-degenerate at this point (See Fig. 1). There are three possible types of non-degenerate critical points for functions on two-dimensional surfaces: maximum, minimum, and saddle (See Fig. 2)

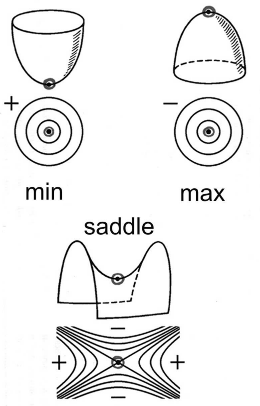


Figure 2. Types of critical points

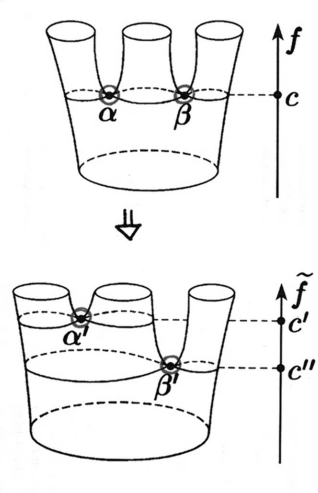


Figure 3. Small perturbation of a function

A smooth function is called a *Morse function* if all its critical points are non-degenerate. The following important statement holds: the Morse functions are everywhere dense in the space of all smooth functions on a smooth manifold.

By c we shall denote critical values of f, i.e. those in whose preimage there is at least one critical point. By arbitrary small perturbation, one can do so that, on every critical level c (i.e., on the set of x's for which $f(x) = c$), there is exactly one critical point. In other words, the critical points which occur in the same level can be moved to close but different levels (See Fig. 3). If each critical level $f^{-1}(c)$ contains exactly one critical point, then f is called *simple Morse function.*

Let f be a Morse function on a compact smooth manifold X^n. For any $a \in \mathbb{R}$ consider the level surface $f^{-1}(a)$ and its connected components, which will be called *fibers.* As a result, on the manifold there appears the structure of a foliation with singularities. By declaring each fiber to be a point and introducing the natural quotient topology in the space $\Gamma$of fibers, we obtain some quotient space. It can be considered as the base of the foliation. For Morse function, the space Γ is a finite graph. The graph is called the *Reeb graph* of the Morse function f on manifold X^n. Consider, for instance, the two-dimensional torus in $\mathbb{R}^3$ embedded as shown in Fig. 4, and take the natural height function to be a Morse function on the torus.

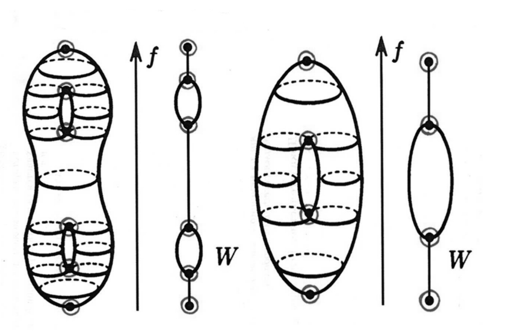


Figure 4. Reeb graphs for height functions

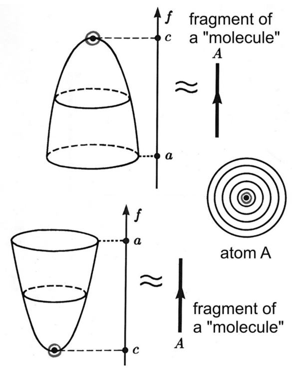


Figure 5. Atom A

Then its Reeb graph has the form, shown next to the torus in the same figure. In addition in Fig. 4 one can see another example of Morse function (again a height function) on the sphere with two handles and its Reeb graph.

It is a natural problem to give the classification of Morse functions on two-dimensional surfaces up to the fiber equivalence. To solve it, at first we need to study the local question, namely, to describe the local topological structure of singular fibers.

We begin with the informal definition. An *atom* is defined to be the topological type of a two-dimensional Morse singularity. In other words, this is the topological type of singular fiber of the foliation defined on a two-dimensional surface by a Morse function. More precisely, we can reformulate this as follows.

Definition 1. An *atom* is a neighborhood P^2 of a critical fiber (which is defined by inequality $c-\epsilon \leqslant f(x) \leqslant c+\epsilon$ for sufficiently small ϵ), foliated into level lines of

f and considered up to the fiber equivalence. In other words, an atom is the germ of the foliation on a singular fiber.

The atom P^2 is called *simple*, if the Morse function f in the pair (P^2, f) is simple. The other atoms are called *complicated.* The *complexity* of an atom is a number of critical points on its critical level $f(x) = c$. The atom is called *orientable (oriented)* or *non-orientable* depending on whether the surface P^2 is orientable (oriented) or non-orientable.

First consider a non-singular level line which is close to a local maximum point. This line ia s circle. As the regular value tends to the local maximum, the circle shrinks into a point (Fig. 5). Let us represent this evolution and the bifurcation in the following conventional, but quite visual manner. Every regular level line (a circle) we represent as one point which is located on the level a (Fig. 5). As a changes, this point moves running through a segment. At the moment, when the value of the function becomes critical (equal to c), a circle has shrunk into a point. Denote this event by the letter A with a segment going out of it. This segment is directed downwards. In the case of minimum we proceed the similar way (Fig. 5).

If c is a critical saddle value, then the singular level line looks like a figure eight curve. As a tends to c, two circles are getting closer and, finally, touch at a point. After this, the level line bifurcation happens and, instead of two, we obtain just one circle. This process is also shown in Fig. 6 and Fig. 7.

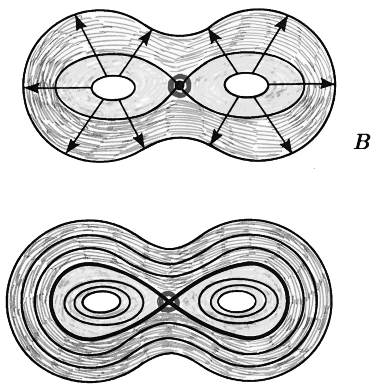


Figure 6. Atom B. Level curves of a function f are shown

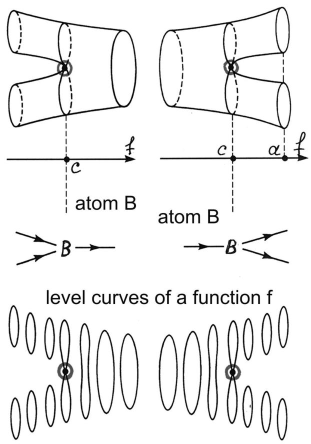


Figure 7. Level line transformation

Let f be a simple Morse function on a compact closed surface X^2 (orientable or non-orientable). Consider Reeb graph Γ. The vertices of Γ correspond to critical fibers of f. Let us replace these vertices by corresponding atoms (either A, or B, or $\tilde{B}$, which is non-orientable version of B). The graph obtained is called a *simple molecule* W. In fact, the notion of the simple molecule does not differ yet from that of the Reeb graph. However, for complicated Morse functions the molecule W will carry more information than the Reeb graph Γ.

A minimal simple Morse function on the pretzel, i.e., on the sphere with two handles, is realized as the height function on the embedding of the pretzel, presented in Fig. 8. The corresponding simple molecule is also shown here.

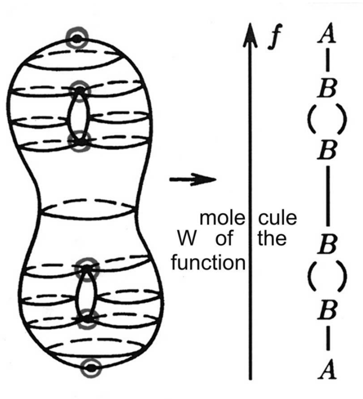


Figure 8. Minimal simple Morse function on the pretzel and its simple molecule

3 Complicated atoms and molecules

Recall that atom is called complicated if critical connected level surface of function f contains several critical points. Such objects naturally arise in many problems in geometry and physics (See Fig. 9).

We give now a simple example. Suppose that a finite group G acts smoothly on a surface X^2, and let f be a $G-$invariant Morse function. Then, as a rule, such function will be complicated.Indeed, if, for instance, the orbit of a critical point x belongs entirely to a connected component of the level line $\{f(x) = const\}$, then this level contains several critical points. An example is shown in Fig. 10.

Of course, a small perturbation can make the function into a simple one by moving critical points into different levels. However, this destroys the $\mathbb{Z}_5-$symmetry, as is seen from Fig. 11. Thus, in the problems that require studying symmetries of different kinds, one has to investigate complicated Morse functions as an independent object.

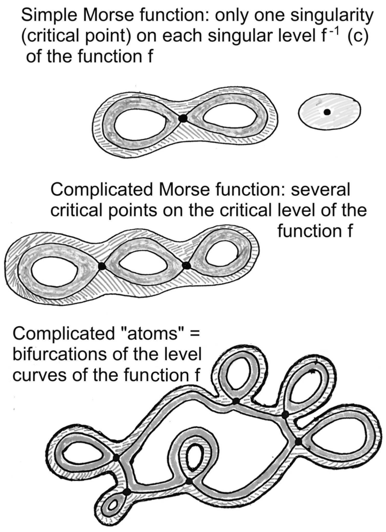


Figure 9. Complicated Morse functions

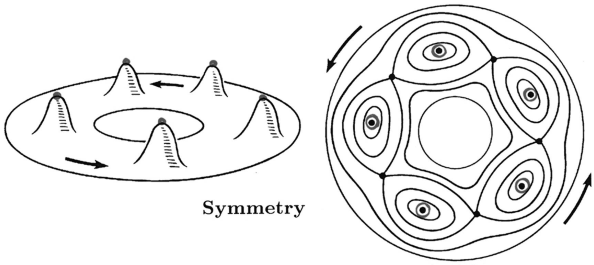


Figure 10. Complicated Morse functions as functions of symmetries

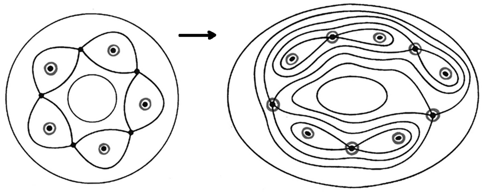


Figure 11. Perturbation leads to the loss of symmetry

It is convenient to denote every atom (P^2, K) by some letter with number of incoming and outgoing edges. The end of each edge corresponds to a certain boundary circle of the surface P. It is important to emphasize that, generally speaking, the ends of an atom (P, K) are not equivalent, because the boundary circles of the surface P are not equivalent in the sense that not every two of them can be matched by a homeomorphism of the pair (P, K) onto itself. Some atoms of low complexity (both orientable and non-orientable) are listed in Fig. 12. In the same table one can see the corresponding pairs of $f-$graphs, as well as the surface $\tilde{P}$ obtained from P by gluing discs to all of its boundary circles.

4 Integrable Hamiltonian systems with two degrees of freedom

Consider symplectic manifold M^{2n} with closed non-degenerate skewsymmetric two-form ω. It defines on the space of smooth functions on M^{2n} Poisson bracket

$$\{f, g\} = \omega^{-1}(df, dg).$$

This operation is bilinear, skewsymmetric and satisfies the Jacobi identity. The system of ordinary differential equations defined by vector field v is called *Hamiltonian system* if there exists such function H (called *Hamiltonian*) that for any function g on M^{2n} the equality holds

$$v(g) = \{g, H\}.$$

We denote $v = \mathrm{sgrad} H$. It's easy to see that the function f is an integral of the flow v iff $\{f, H\} = 0$. The Hamiltonian system is called *integrable* if its Hamiltonian H is "symmetrical", i.e., has the sufficient number of integrals.

The system is integrable in Liouville sense if there exists exactly n functionally independent commuting integrals $f_1, \ldots, f_n$. They define the so called Liouville foliation (See Fig. 13). The Liouville theorem states, that if the regular (i.e. f_i are functionally independent on it) common level surface of these functions is compact, then it is a torus T^n. The solution (integral trajectory) in general case determines almost periodic motion on this torus.

This class of Hamiltonian systems contains many important examples from physics an classical mechanics: the different cases of motion of rigid body (Euler case, Lagrange top), geodesic flow on ellipsoid, interaction of the material points, located on the line or on the circle S^1. The classification of such systems is a very difficult task. It turns out, however, that in case of $n = 2$ the full classification exists.

Consider a symplectic manifold M^4 with an integrable Hamiltonian system $v = \mathrm{sgrad} H$; let Q_h^3 be a non singular compact connected isoenergy 3-surface in M^4.

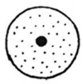
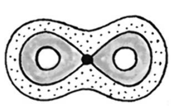
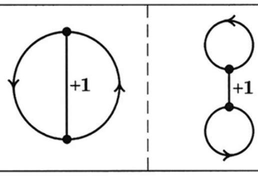
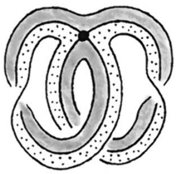
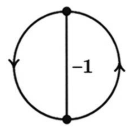
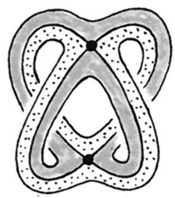
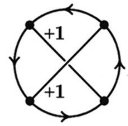
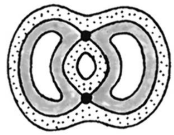
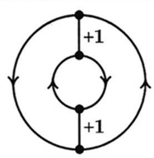
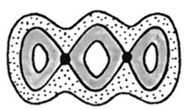
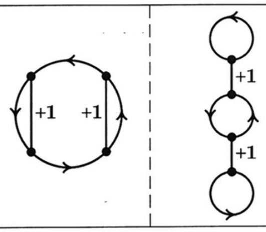
No	ATOM	f-GRAPHS	CODE	GENUS
	Complexity 1, orientable			
1			$-A$	S^2
2			$-B<$	S^2
	Complexity 1, non-orientable			
$\widetilde{1}$			$-\widetilde{B}-$	$\mathbb{R}P^2$
	Complexity 2, orientable			
1			$-C_1-$	T^2
2			$>C_2<$	S^2
3			$-D_1\leqq$	S^2

Figure 12. Several atoms of low-complexity

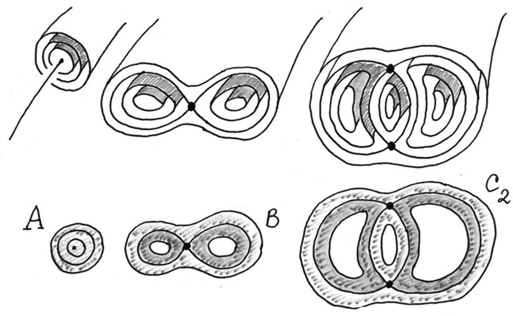


Figure 13. The two-dimensional cross-section of Liouville foliation in three-dimensional invariant submonifold around singular leaf

Let f be an additional integral of the system v that is independent of H. We denote its restriction to Q_h^3 by the same letter f. It is assumed to be a Bott function on Q_h^3. Our aim is to investigate the topology of the Liouville foliation on Q_h^3 defined by the given integrable system. Its non-singular leaves are Liouville tori, and the singular ones correspond to critical levels of the integral f on Q_h^3.

Now consider a topologically stable integrable system with Bott integral f on an isoenergy $3-$surface Q_h^3 and take some singular leaf L of the corresponding Liouville foliation on Q_h^3. Consider a neighborhood of this leaf, i.e., a three-dimensional manifold $U(L)$ with the Liouville foliation structure and fixed orientation. By analogy with the two-dimensional case, as neighborhood $U(L)$, we take the connected component of the set $c - \epsilon \leqslant f(x) \leqslant c + \epsilon$ that contains the singular leaf L (same as in previous section $f(L) = c$ is a critical value of f). Such an object is naturally called a $3-$*atom*. However, from the formal viewpoint, we have to be more careful. We shall assume two such $3-$manifolds $U(L)$ and $U'(L)$ with the structure of the Liouville foliation to be fiberwise equivalent if
1) there exists a diffeomorphism between them that maps the leaves of the first Liouville foliation into those of the second one,
2) this diffeomorphism preserves both the orientation on $3-$manifolds and the orientation on the critical circles defined by the Hamiltonian flows.

Definition 2. The equivalence class of the three-dimensional manifold $U(L)$ is called a $3-$*atom*. The number of critical circles in the $3-$atom is called its *atomic weight* or *complexity*.

Consider $3-$atom $U(L)$ with the structure of a Seifert fibration on it. Let

$$\pi : U(L) \to P^2$$

denote its projection onto a two-dimensional bas P^2 with the embedded graph $K = \pi(L)$. Let us mark those points on the base P^2, into which the singular fibers of the Seifert fibration (i.e., the fibers of type (2,1)) are projected. Recall that the base P^2 has a canonical orientation. The point is that an orientation is already fixed on $U(L)$, as well as on the fibers of the Seifert fibration. It is clear that, as a result, we obtain some oriented 2-atom (P^2, K).

Theorem 2 (Fomenko). *Under the projection $\pi : U(L) \to P^2$, the $3-$atom $U(L)$ turns into the 2-atom (P^2, K), and moreover, the singular fibers of the Seifert fibration on the $3-$atom are in one-to-one correspondence with the star-vertices of the $2-$atom. This correspondence between $2-$atoms and $3-$atoms is bijection.*

The example of 3-atom is shown in the Fig. 14.

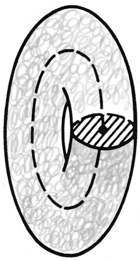


Figure 14. 3-atom A

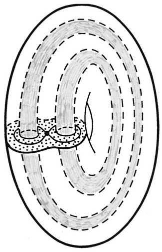


Figure 15. 3-atom B

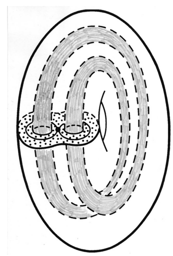


Figure 16. 3-atom B^*

Let us describe $3-$atom A (See fig. 14). This $3-$atom is presented as a solid torus foliated into concentric tori, shrinking into the axis of the solid torus. In other words, the $3-$atom A is the direct product of a circle and a disc foliated into concentric circles. From the viewpoint of the corresponding dynamical system, A is a neighborhood of a stable periodic orbit. The examples of saddle 3-atoms are presented in the Fig.15 and Fig.16.

Now we make more precise the definition of the Liouville equivalence for integrable Hamiltonian systems. From now on, we shall assume that two Liouville foliations are Liouville equivalent if and only if there exists a diffeomorphism that sends the leaves of the first foliation to those of the second one and satisfies two conditions related to the orientation. Namely it preserves the orientation of $3-$manifolds Q^3_h and ${Q'}^3_h$, and moreover, it also preserves the orientation on the critical circles given by the Hamiltonian flows. The molecule W contains a lot of essential information on the structure of the Liouville foliation on Q^3_h. However, this information is not quite complete. Indeed, the molecule of the form $A - A$, for example, informs us that the manifold Q^3_h is glued from two solid tori foliated into concentric tori in a natural way. However, it does not tell us how this gluing is made, and what three-dimensional manifold is obtained as a result. Therefore, we have to add some additional information to the molecule W, namely, the rules that clarify how to glue the isoenergy surface Q^3_h from individual 3−atoms. As it was discovered by A.T.Fomenko and H.Zieschang, the molecule W, which corresponds to integrable Hamiltonian system, can be endowed by some numerical marks in such a way, that these *marked molecules* W^* will classify such a systems up to Liouville equivalence. In short, the *marked molecule* W^* is molecule W, equipped with three sets of numbers r_i, ϵ_i and n_k called *marks*.

Theorem 3. *Two integrable systems* (v, Q^3_h) *and* (v', Q'^3_h) *are Liouville equivalent if and only if their marked molecule* W^* *and* $W^{*\prime}$ *coincide.*

The marks can not be chosed orbitrary, as there are several important conditions on them. Any marked molecule W^* with marks, which satisfy these conditions, is called *abstract marked molecule.*

Theorem 4. *Any abstract marked molecule* W^* *is realized as a marked molecul of some integrable Hamiltonian system.*

Corollary. *1) There exist a one-to-one correspondence between the Liouville equivalence classes of integrable systems and marked molecules. In particular, the set of Liouville equivalence classes of integrable systems is discrete (countable) and has no continuous parameters.*
2) There exists an enumeration algorithm for marked molecules (i.e. classes of integrable systems)
3) There exists an algorithm for comparison of marked molecules, i.e., the algorithm that gives answer to the question whether two integrable systems corresponding to given molecule are Liouville equivalent or not.

The table shown in the Fig. 17 represents, for example, the marked molecules calculated for well known integrable Zhukovskii case in the dynamics of rigid body in 3-dimensional Euclidean space.

5 Jacobi problem and Euler case

Consider an ellipsoid X in three-dimensional Euclidean space given by

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1,$$

where $a < b < c$.

The geodesic flow on the ellipsoid is a Hamiltonian system on the cotangent bundle T^*X with standard symplectic structure. The Hamiltonian of this system

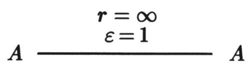
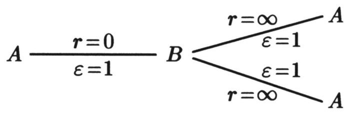
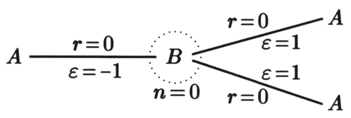
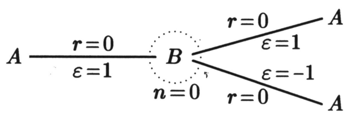
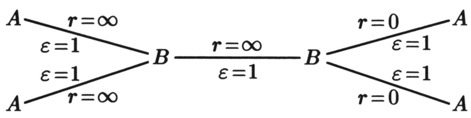
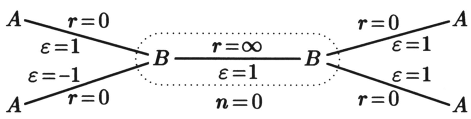
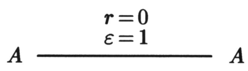
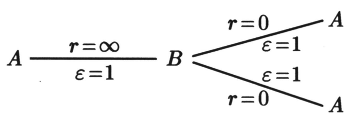
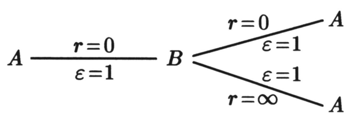
No	MOLECULE	Q^3
1		$S^1 \times S^2$
2		$S^1 \times S^2$
3		$S^1 \times S^2$
4		$S^1 \times S^2$
5		$S^1 \times S^2$
6		$S^1 \times S^2$
7		S^3
8		S^3
9		S^3

Figure 17. Zhukovskii case

is

$$H(q,p) = \frac{1}{2}\sum_{i,j} g^{ij}(q)p_ip_j = \frac{1}{2}|p|^2,$$

where $g_{ij}(q)$ is the induced Riemannian metric on the ellipsoid X, and $(q,p) \in T^*X$, $q \in X$, $\in T_q^*X$. The isoenergy surface $Q^3 = \{2H = |p|^2 = 1\}$ in this case is a S^1-fibration over X (unit covector bundle). The geodesic flow on the ellipsoid admits an additional integral

$$f_J = abc\Big(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\Big)\Big(\frac{\dot{x}^2}{a} + \frac{\dot{y}^2}{b} + \frac{\dot{z}^2}{c}\Big).$$

Here $(\dot{x}, \dot{y}, \dot{z})$ is a tangent vector to a geodesic (we identify tangent and cotangent vectors in the usual way).

The second system (Euler case) is given by the standard Euler-Poisson equations and describes the motion of a rigid body fixed at its center of mass:

$$\frac{dK}{dt} = [K, \Omega],\ \frac{d\gamma}{dt} = [\gamma, \Omega].$$

Here vector $K = (s_1, s_2, s_3)$ is a kinetic momentum vector of the body, $\Omega = (As_1, Bs_2, Cs_3)$ is its angular velocity vector, $\gamma = (r_1, r_2, r_3)$ is the unit vertical vector (the coordinates of these vectors are written in the orthonormal basis which is fixed in the body and whose axes coincide with the principal axes of inertia). The parameters A, B, C of the problem are the inverse of the principal moments of inertia of the rigid body. We suppose they are all different $A < B < C$. By $v_J(a, b, c)$ and $v_E(A, B, C)$ we denote the restrictions of the Jacobi and Euler systems to their isoenergy surface $Q_J = \{2H_J = 1\}$ and $Q_E = \{2H_E = 1\}$ respectively, where H_J and H_E are the Hamiltonians of the Jacobi problem and the Euler case indicated above. Thus we have two dynamical systems on diffeomorphic isoenergy three-dimensional manifolds. We want to find out whether these systems are similar in some sense. In particular, are they orbitally equivalent? If yes, then topologically or smoothly? The following theorems were proved by A.V.Bolsinov and A.T.Fomenko.

Theorem 5. *The Liouville foliation related to the Hamiltonian systems $v_J(a, b, c)$ and $v_E(A, B, C)$ on isoenergy surfaces are diffeomorhic. In other words, the Jacobi problem and the Euler case are Liouville equivalent.*

Recall that the $t-$molecule is obtained from the usual marked molecule W^* by adding the rotation vectors on all of its edges and the $\Lambda-$invariant on atoms. It turns out that there are no other orbital invariant for the Euler and Jacobi systems besides the $t-$molecule.

Theorem 6. *The Jacobi problem (geodesic flow on the ellipsoid) and the Euler case (in rigid body dynamics) are topologically orbitally equivalent in the following exact sense. For any rigid body there exists an ellipsoid (and vice versa, for any ellipsoid there exists a "rigid body") such that the corresponding systems* $v_J(a, b, c)$ *and* $v_E(A, B, C)$ *are topologically orbitally equivalent. The parameters* a, b, c *and* A, B, C *related to equivalent systems are uniquely defined up to proportionality.*

Discussing the equivalence of the Euler and Jacobi problems, we can ask another question: can they be topologically conjugate for some values of their parameters? In other words, does there exist a homeomorphism between isoenergy surfaces which sends one Hamiltonian flow to the other and preserves parametrization on integral curves?

Theorem 7. *The geodesic flow on any ellipsoid (different from the sphere), restricted to its constant energy three-dimensional manifold, is not topologically conjugate to any system of the Euler case. In other words, for any values of parameters* a, b, c *and* A, B, C *(except for* $a = b = c$ *and* $A = B = C$*) the systems* $v_J(a, b, c)$ *and* $v_E(A, B, C)$ *are not topologically conjugate.*

Theorem 8 (P. V. Morozov) *1) The dynamical systems known as Euler case, Clebsch case and Steklov case in the rigid-body dynamics in 3-space are Liouville equivalent for sufficiently large values of energy (i.e. energy integral).*
2) If the value of area integral g *is sufficiently large, than Steklov case and Clebsch case are Liouville equivalent (as systems on four-dimensional symplectic manifolds) to Euler case with non-zero value of area integral.*

Theorem 9 (N.V. Korovina) *Let us consider dynamical systems of Lagrange case and Euler case for zero value of area integral:*
1) Let us recall that Lagrange case is characterized by the parameter β *and the smooth function* $V(x)$ *called a potential. Then for sufficiently large value* h *of energy* H*, the Lagrange system of the isoenergy 3-surface* $Q = \{H = h = const\}$ *is orbitally equivalent to the Euler system, corresponding to some value of parameter* $G(\beta)$*, where* G *determines the Euler system.*
2) Let us consider Euler system with parameter G*. Then for any value* $\beta > 0$ *and for sufficiently large value* h *of energy* H *there exist potentials* $V(x)$ *such that Lagrange system with parameters* β *and* $V(x)$ *(on the isoenergy three-dimensional level surface* $H = h = const$*) is orbitally equivalent to Euler system with parameter* G*.*

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RESOLVENT ESTIMATES, SMOOTHING PROPERTIES AND SCATTERING FOR SCHRÖDINGER AND WAVE EQUATIONS

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Abstract. We survey some basic problems of Schrödinger, Klein-Gordon and wave equations in the framework of general scattering theory. The following topics are treated under suitable decay and/or snallness conditions on the perturbation term: Growth estimates of generalized eigenfunctions, Resolvent estimates, Scattering theory, Smoothing properties and Strichartz estimates. Due to our formulation of the weighted energy method, some topics are naturally extended to time-dependent and/or non-selfadjoint perturbations.

1 Introduction

This article will summarize with some addendum and modification the following 4 works which remain to the author as a personal history of particitation in ISAAC: [13] (August 2001, Berlin), [14] (July 2005, Catania), [15] (August 2007, Ankara), [16] (July 2009, London).

In §3 an inverse scattering problem of [13] is generalized to wave equations with both "dissipation" and potential terms. We give a reconstruction procedure of both coefficients from the scattering amplitude with a fixed energy. In §6 and §7 are respectively treated scattering and Strichartz estimates for Schrdinger, Klein-Gordon and wave equations under time dependent small perturbations. We did not enter into the Strichartz estimates in [15] and excluded Klein-Gordon equayions there. Moreover, in §5 decay-nondecay properties of solutions in L^2 are illustrated to dissipative Schrödinger evolution equations.

2 Selfadjoint magnetic Schrödinger operators

Let Ω be an exterior domain in $\mathbf{R}^n$ with smooth compact boundary $\partial\Omega$ (the case $\Omega = \mathbf{R}^n$ is not excluded). We consider in Ω the Schrödinger operator

$$Lu = -\sum_{j=1}^{n} \{\partial_j + ib_j(x)\}^2 u + c(x)u, \tag{1}$$

where $b_j(x)$ are real valued C^1-function of $x \in \mathbf{R}^n$ and $c(x)$ is a real valued continuous function of $x \in \mathbf{R}^n \backslash \{0\}$. $b(x) = (b_1(x), \cdots, b_n(x))$ represents a magnetic potential. Thus the magnetic field is defined by its rotation $\nabla \times b(x)$. The external potential $c(x)$ may have a singularity like $O(|x|^{-2})$ at $x = 0$ when $\Omega = \mathbf{R}^n$.

In the following we put $\nabla_b = \nabla + b(x)$, $\Delta_b = \nabla_b \cdot \nabla_b$, $r = |x|$, $\tilde{x} = x/r$ and $\partial_r = \tilde{x} \cdot \nabla$. The inner product and norm of the Hilbert space $L^2 = L^2(\Omega)$ are defined by

$$(f, g) = \int f(x)\overline{g(x)}dx \quad \text{and} \quad \|f\| = \sqrt{(f, f)}.$$

Here we specify by $\displaystyle\int dx$ the integration over Ω. For function $\mu = \xi(r) > 0$ let L^2_μ be the weighted L^2-space with norm $\|f\|^2_\mu = \displaystyle\int \mu(r)|f(x)|^2 dx < \infty$.

We assume

$$(A1) \qquad \exists c_\infty(x) \in L^\infty \text{ such that } c(x) - c_\infty(x) \geqslant \frac{\beta}{r^2} \quad \text{with} \quad \beta > -\frac{(n-2)^2}{4}.$$

Theorem 1. *Under (A1) let L be defined by*

$$\begin{cases} Lu = -\Delta_b u + c(x)u \text{for} u \in \mathcal{D}(L), \\ \mathcal{D}(L) = \big\{u \in L^2 \cap H^2_{\mathrm{loc}}(\overline{\Omega}\backslash\{0\}); (-\Delta_b + c)u,\ r^{-1}u \in L^2,\ u|_{\partial\Omega} = 0\big\}. \end{cases} \tag{2}$$

Then it gives a lower semibounded selfadjoint operator in L^2.

The essential spectrum $\sigma_e(L)$ of L is included in the half line $[0, \infty)$ if $c(x) \to 0$ as $|x| \to \infty$. To investigate further properties of the essential spectrum we first consider the homogeneous equation

$$-\Delta_b u + c(x)u - \lambda u = 0, \quad \lambda > 0, \tag{3}$$

with $b(x)$ and $c(x)$ satisfying the additional condition

$$(A2) \qquad \max\Big\{|\nabla \times b(x)|,\ |c(x)|\Big\} \leqslant \mu(r), \quad r = |x| > \exists R_0,$$

where $\mu = \mu(r)$ is a smooth, positive, non-increasing L^1-function of $r \in \mathbf{R}_+ = (0, \infty)$.

Theorem 2. *Under* $(A1)$, $(A2)$ *let* $u \in H^2_{\rm loc}(\overline{\Omega}\backslash\{0\})$ *solves* (2.1). *If the support of* u *is not compact, then*

$$\liminf_{t\to\infty} \int\limits_{S_t} |\tilde{x}\cdot\vartheta(x,\pm\sqrt{\lambda})|^2 dS \neq 0,$$

where $\vartheta(x,\kappa) = \nabla_b u + \tilde{x}\left(\frac{n-1}{2r} - i\kappa\right)u$ *for* $\kappa \in \mathbf{C}$.

If we additionary require the following $(A2.2)$, the unique continuation property holds and non-existence of positive eigenvalues results from this theorem.

$(A3)$ $\nabla b_j(x)$ $(j = 1, \cdots, n)$ and $c(x)$ are locally Hölder continuous.

By contradiction, Theorem 2 directly shows the following assertion.

Theorem 3. *Assume* $(A1)$, $(A2)$ *with* μ *satisfying also*

$$\int\limits_r^\infty \mu(s)ds \geqslant r\mu(r) \quad for \quad r \geqslant R_0, \tag{4}$$

and $(A3)$. *Then for any* $0 < a < b < \infty$, *the resolvent* $R(\kappa^2) \in \mathcal{B}(L^2_{\mu^{-1}}, L^2_\mu)$ *restricted in* $\kappa \in K_\pm = \{\kappa; a \leqslant \pm\mathrm{Re}\kappa \leqslant b, 0 < \mathrm{Im}\kappa \leqslant 1\}$ *is continuously extended to* $K_\pm \cup [a,b]$ *as an operator from* $L^2_{\mu^{-1}}$ *to* L^2_μ. *Thus, the positive spectrum of* L *is absolutely continuous with respect to the Lebesgue measure.*

$C(1+r)^{-1-\delta}$ and $C(1+r)^{-1}[\log(1+r)]^{-1-\delta}$ $(C > 0,\ 0 < \delta < 1)$ are typical examples of μ satisfying (4). These examples also satisfy (26) given later.

Cf., Kalf et ar. [8] for Theorem 1. Theorem 2 is a real generalization of Rellich [20] (cf. Kato [9]). A more general oscillating long-range potential is treated in [14] (also Jäger-Rejto [8]). Theorem 3 states the principle of limiting absorption, the proof of which is originated by Eidus [8].

3 Spectral representations and scattering

The Fourier transform $\hat{f}(\xi) = (2\pi)^{-n/2} \int e^{-ix\cdot\xi} f(x)dx$ determines the spectral representation of L_0. Namely, put

$$[F_0(\sigma)f](\omega) = \sigma^{(n-1)/2}\hat{f}(\sigma\omega), \quad \omega \in S^{n-1},$$

$$[F_0^*(\sigma)h](x) = \sigma^{(n-1)/2}(2\pi)^{-n/2} \int\limits_{S^{n-1}} e^{i\sigma x\cdot\omega} h(\omega)dS_\omega, \quad h \in L^2(S^{n-1}).$$

Then $[F_0 f](\sigma,\omega) = [F_0(\sigma)f](\omega)$ gives a unitary operator from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}_+ \times S^{n-1})$ and its adjoint F_0^* is given by

$$[F_0^* h](x) = \int_0^\infty [F_0^*(\sigma)h(\sigma,\cdot)](x)d\sigma \quad \text{for} \quad h(\sigma,\omega) \in L^2(\mathbf{R}_+ \times S^{n-1}).$$

In this section we require

$$(A4) \qquad \max\left\{|b(x)|,\ |\nabla b(x)|,\ |c(x)|\right\} \leqslant \mu(r), \quad r = |x| > \exists R_0 > 0.$$

The decay condition on $b(x)$ itself is used to compare L with free Lapalacian $-\Delta$ in $L^2(\mathbf{R}^n)$. Let $j(r)$ be a C^∞-function of $r > 0$ such that $j(r) = 0$ $(r < R_0)$ and $= 1$ $(r > R_0 + 1)$, and define the operator $J:\ L^2(\mathbf{R}^n) \to L^2 = L^2(\Omega)$ and its adjoint J^* by

$$[Jf](x) = j(r)f(x), \quad x \in \Omega,$$

$$[J^*g](x) = j(r)g(x)\ (x \in \Omega) \text{ and } = 0\ (x \in \mathbf{R}^n \backslash \Omega).$$

Let $R_0(\kappa^2) = (L_0 - \kappa^2)^{-1}$. Then we have the following resolvent equation

$$R(\kappa^2)J = \{J - R(\kappa^2)V\}R_0(\kappa^2), \quad V = LJ - JL_0,$$

$$J^*R(\kappa^2) = R_0(\kappa^2)\{J^* - V^*R(\kappa^2)\}, \quad V^* = J^*L - L_0J^*.$$

For each $\sigma \in \mathbf{R}_+$, we define

$$F_\pm(\sigma) = F_0(\sigma)\{J^* - V^*R(\sigma^2 \pm i0)\},$$

$$F_\pm^*(\sigma) = \{J - R(\sigma^2 \mp i0)V\}F_0^*(\sigma).$$

Theorem 4. *Assume* (*A*1), (*A*3) *and* (*A*4). *Then the operator*

$$[F_\pm f](\sigma,\omega) = [F(\pm\sigma)f](\omega), \quad (\sigma,\omega) \in \mathbf{R}_+ \times S^{n-1}.$$

is extended to a unitary operator from $\{I - P\}L^2$ *onto* $L^2(\mathbf{R}_+ \times S^{n-1})$:

$$F_\pm^* F_\pm = I - P \text{ in } L^2 \quad (\textit{completeness}),$$

$$F_\pm F_\pm^* = I \text{ in } L^2(\mathbf{R}_+ \times S^{n-1}) \quad (\textit{orthogonality}),$$

where P is the orthogonal projection onto the eigenspace of L.

We define the operators $U_\pm$ and $\mathcal{S}$ by $U_\pm = F_\pm^* F_0$, $\quad S = U_+^* U_- = F_0^* F_+ F_-^* F_0$.

Proposition 1. $U_\pm : \; L^2(\mathbf{R}^n) \to (I-P)L^2(\Omega)$ are unitary operators which intertwine L_0 and L:

$$LU_\pm f = U_\pm L_0 f, \quad f \in \mathcal{D}(L_0).$$

$\mathcal{S}$ is a unitary operator in $L^2(\mathbf{R}^n)$ which commutes with L_0.

Now, consider the Schrödinger evolution operators e^{-itL} and e^{-itL_0}. Theorem 4 implies that for $f \in (I-P)L^2(\Omega)$ and $f_0 \in L^2(\mathbf{R}^n)$, $e^{-itL}f = F_\pm e^{-i\sigma^2 t}F_\pm f$, $e^{-itL_0}f_0 = F_0 e^{-i\sigma^2 t}F_0 f_0$.

Theorem 5. *Assume (A1), (A3) and (A4). Then the Mϕller wave operator exists and coincides with $U_\pm$:*

$$s - \lim_{t\to\pm\infty} e^{itL} J e^{-itL_0} = U_\pm.$$

Thus, $S = U_+^* U_-$ defines the Mϕller scattering operator, the representation of which in the momentum space $L^2(\mathbf{R}_+ \times S^{n-1})$ is given by

$$F_0 S F_0^* = \hat{I} - \hat{T}, \quad [\hat{T}\hat{f}](\sigma,\omega) = \frac{1}{2\sigma}[F_+(\sigma) V F_0^* \hat{f}(\sigma,\cdot)](\omega).$$

The kernel of $\hat{T}$ is called the scattering amplitude.

A survey of classical stationary approach on short-range scattering is given above. We can find a detailed description e.g. in Mochizuki [12].

4 Inverse scattering for small nonselfadjoint perturbation of wave equations

We consider the wave equation of the form

$$w_{tt} + b(x)w_t - \Delta w + c(x)w = 0, \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}, \tag{5}$$

where $n \geqslant 3$ and $b(x)$ and $c(x)$ are real, continuous functions satisfying

$$(A5) \qquad |b(x)| \leqslant \epsilon_0 \mu(r), \quad \frac{\beta}{r^2} < c(x) \leqslant \mu(r)$$

with $\epsilon_0 > 0$ (small) and $\beta > -\dfrac{(n-2)^2}{4}$. $\mu(r)$ is a positive L^1-function satisfying (5).

We rewrite (5) in the form

$$i\partial_t u = \Lambda u \equiv \Lambda_0 u + Vu, \quad u = \{w, w_t\};$$

$$\Lambda_0 = i \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \quad \text{and} \quad V = -i \begin{pmatrix} 0 & 0 \\ c(x) & b(x) \end{pmatrix}.$$

Let $\mathcal{H}_E = \dot{H}^1 \times L^2$ be the Hilbert space with energy norm

$$\|f\|_E^2 = \frac{1}{2}\left\{\|\nabla f_1\|^2 + \|f_2\|^2\right\}, \quad f = \{f_1, f_2\}.$$

Λ_0 is selfadjoint in $\mathcal{H}_E$, and its spectral representation is determined by

$$\mathcal{F}_0(\lambda) = \frac{1}{2} F_0(|\lambda|) \begin{pmatrix} 1 & i\lambda^{-1} \\ -i\lambda & 1 \end{pmatrix} \quad (\lambda \neq 0).$$

The spectral representation of Λ is then given by

$$\mathcal{F}_\pm(\lambda) = \mathcal{F}_0(\lambda)\{I - V\mathcal{R}(\lambda \pm i0)\}, \quad \mathcal{F}_\pm^{(*)}(\lambda) = \{I - \mathcal{R}(\lambda \mp i0)V\}\mathcal{F}_0^*(\lambda),$$

where $\mathcal{R}(\zeta)$ is the resolvent of Λ. Since the coefficient $b(x)$ of nonselfadjoint part is small, $\mathcal{R}(\zeta) \in \mathcal{B}(\mathcal{H}_{E,\mu^{-1}}, \mathcal{H}_{E,\mu})$ is extended continuously to $\zeta = \lambda \pm i0$ $(\lambda \in \mathbf{R}\backslash\{0\})$.

Proposition 2. There exists the strong limit

$$\mathcal{W}_\pm = s - \lim_{t \to \pm\infty} e^{it\Lambda} e^{-it\Lambda_0}.$$

It is expressed as $\mathcal{W}_\pm = \mathcal{F}_\pm^{(*)}\mathcal{F}_0$, and defines a bijection in $\mathcal{H}_E$. The scattering operator exists and is given by

$$\mathcal{S} = \mathcal{W}_+^{-1}\mathcal{W}_- = \mathcal{F}_0^*\mathcal{F}_+^{(*)-1}\mathcal{F}_-^{(*)}\mathcal{F}_0.$$

The last assertion gives us $\mathcal{F}_0(I - \mathcal{S})\mathcal{F}_0^* = \mathcal{F}_+(\mathcal{F}_+^{(*)} - \mathcal{F}_-^{(*)})$. Thus the scattering amplitude $\mathcal{A}(\lambda)$ with energy $\lambda \neq 0$ is expressed as

$$2\pi i \mathcal{A}(\lambda) \equiv \mathcal{F}_0(\lambda)\{I - \mathcal{S}(\lambda)\}\mathcal{F}_0^* = \frac{\pi i}{2}\hat{T}(\lambda) \begin{pmatrix} 1 & i\lambda^{-1} \\ -i\lambda & 1 \end{pmatrix},$$

where $\hat{T}(\lambda)$ is the scaler amplitude given by

$$\hat{T}(\lambda) = \lambda^{-1} F_0(|\lambda|)\{1 + q(\cdot, \lambda)R(\lambda^2 - i0, \lambda)\}q(\cdot, \lambda)F_0^*(|\lambda|)$$

with $R(\zeta^2, \alpha) = (-\Delta + c - i\alpha b - \zeta^2)^{-1}$ and $q(x, \alpha) = c - i\alpha b$.

$\hat{T}(\lambda)$ is an integral operator on S^{n-1} with kernel

$$a(\lambda,\omega,\omega') = (2\pi)^{-n}\lambda^{n-2}\Bigg[\int e^{-i\lambda(\omega-\omega')\cdot x}q(x,\lambda)dx+$$

$$+\int e^{-i\lambda\omega\cdot x}q(x,\lambda)R(\lambda^2-i0,\lambda))\{q(\cdot,\lambda)e^{i\lambda\vartheta'\cdot}\}(x)dx\Bigg]. \tag{6}$$

Our aim is to derive the reconstruction procedure of $b(x)$ and $c(x)$ from this $a(\lambda,\omega,\omega')$.

The following result is well known as the high energy Born approximation.

Theorem 6. *In case $b(x)\equiv 0$, if we further require $c(x)\in L^1(\mathbf{R}^n)$, then for any $\xi\in\mathbf{R}^n$ we can choose $\omega(\lambda)$, $\omega'(\lambda)\in S^{n-1}$ to satisfy $\lambda\{\omega(\lambda)-\omega'(\lambda)\}=\xi$ and*

$$\lim_{\lambda\to\infty}(2\pi)^n\lambda^{-n+2}a(\lambda,\omega(\lambda),\omega'(\lambda)) = \int e^{-i\xi\cdot x}c(x)dx.$$

In case $b(x)\not\equiv 0$, however, $\|R(\lambda^2-i0,\lambda)\|_{L^2_{\mu-1},L^2_\mu}$ does not in general decay as $|\lambda|\to\infty$. To fill up, we restrict $b(x)$, $c(x)$ to exponentially decreasing functions, and introduce the so called nonphysical Faddeev resolvent ([5]).

Let $k\in\mathbf{R}^n$, $\gamma\in S^{n-1}$, $\epsilon\geqslant 0$. We simply write $\zeta^2=\zeta\cdot\zeta$ for $\zeta\in\mathbf{C}^n$, and both the resolvent and its kernel by $R_0(\kappa^2)$. Then since

$$R_0((k+i\epsilon\gamma)^2) = (2\pi)^{-n}\int\frac{e^{i(x-y)\cdot\xi}}{\xi^2-k^2+\epsilon^2-2i\epsilon\gamma\cdot k}d\xi,$$

choosing γ to satisfy $t=\gamma\cdot k\geqslant 0$ and putting $\xi=\eta+t\gamma$, we have

$$= (2\pi)^{-n}\int\frac{e^{i(x-y)\cdot(\eta+t\gamma)}}{\eta^2+2t\gamma\cdot\eta-(k^2-\epsilon^2-t^2)-2i\epsilon\gamma\cdot k}d\eta.$$

We let $\epsilon\to+0$ and define the Faddeev unperturbed resolvent depending on γ by

$$R_{\gamma,0}(k^2,t) = e^{it\gamma\cdot x}G_{\gamma,0}((k-t\gamma)^2,t)e^{-it\gamma\cdot x},$$

$$G_{\gamma,0}(\sigma^2,t) = (2\pi)^{-n}\int\frac{e^{i(x-y)\cdot\eta}}{\eta^2+2t\gamma\cdot\eta-\sigma^2-i0}d\eta. \tag{7}$$

Lemma 1. (see Isozaki [6]) *Let $\Phi_\gamma(t)=\chi(\gamma\cdot\vartheta\geqslant t/\lambda)$ (defining function of $\vartheta\in S^{n-1}$). Then*

$$R_{\gamma,0}(\lambda,t) = R_0((\lambda+i0)^2)-2\pi F_0(\lambda)^*\Phi_\gamma(t)F_0(\lambda).$$

Lemma 2. (see Weder [21]) *In the expression of $G_{\gamma,0}(\sigma^2,t)$ we replace t by $z \in \mathbf{C}_+$. Then*

(*i*) *$G_{\gamma,0}(\sigma^2,z)$ is continuou in $\{|\sigma|,\gamma\} \in \mathbf{R}_+ \times S^{n-1}$ and analytic in $z \in \overline{\mathbf{C}_+}$.*

(*ii*) *$\forall \epsilon_0 > 0$, $\exists C > 0$ such that*

$$\|G_{\gamma,0}(\sigma^2,z)\|_{\mathcal{B}(L^2_{\mu-1},L^2_\mu)} \leqslant C(|\sigma|+|z|)^{-1} \textit{ for } |\sigma|+|z| > \epsilon_0.$$

For $a \in \mathbf{R}$ let $\mathcal{H}_a = \{f; e^{a|x|}f(x) \in L^2\}$, and for $\epsilon > 0$ let $D_\epsilon = \{z \in \mathbf{C}_+; |\mathrm{Re} z| < \epsilon/2\}$.

Lemma 3. (see Eskin-Ralston [4]) *There exists an operator $U_{\gamma,0}(\lambda^2,z)$ satisfying the following properties.*

(*i*) *$\forall \delta > 0$, $\exists \epsilon > 0$ such that $U_{\gamma,0}(\lambda^2,z) \in \mathcal{B}(\mathcal{H}_\delta, \mathcal{H}_{\delta^{-1}})$ and is analytic in $z \in D_\epsilon$.*

(*ii*) *As $z \to t \in (-\epsilon/2, \epsilon/2)$ $U_{\gamma,0}(\lambda^2,z)$ has a boundary value $G_{\gamma,0}(\lambda^2-t^2,t)$, and $U_{\gamma,0}(\lambda^2,i\tau) = G_{\gamma,0}(\lambda^2+\tau^2,i\tau)$ for $\tau > 0$.*

The perturbed Faddeev resolvent is defined for a.e. $t \in (-\epsilon/2, \epsilon/2)$ as follows.

$$R_\gamma(\lambda,t) = \{I - R_{\gamma,0}(\lambda,t)(c - i\lambda b)\}^{-1} R_{\gamma,0}(\lambda,t).$$

Then $U_\gamma(\lambda,t) = e^{-it\gamma\cdot x} R_\gamma(\lambda,t) e^{it\gamma\cdot x}$ has a unique meromorphic continuation on D_ϵ and

$$\|U_\gamma(\lambda,i\tau)\|_{B(L^2_\mu,L^2_{\mu-1})} \leqslant C/\tau \text{ for large } \tau. \tag{8}$$

Theorem 7. *Assume* (*A*5) *and also*

$$(A6) \qquad b(x),\ c(x) = O(e^{-\delta_0|x|}) \quad (|x| \to \infty) \textit{ for some } \delta_0 > 0.$$

Then $a(\lambda,\omega,\omega')$ with a fixed energy $\lambda \neq 0$ uniquely determines $b(x)$ and $c(x)$.

Proof. In (6) we replace $R(\lambda^2 - i0, \lambda)$ by the Faddeev resolvent $R_\gamma(\lambda,t)$, and define the kernel of Faddeev scattering amplitude

$$a_\gamma(\lambda,\vartheta,\vartheta';t) = (2\pi)^{-n}\lambda^{n-1}\Bigg[\int e^{-i\lambda(\vartheta-\vartheta')\cdot x}\{\lambda^{-1}c(x) - ib(x)\}dx +$$

$$+\lambda\int e^{-i\lambda\vartheta\cdot x}\{\lambda^{-1}c(x) - ib(x)\}R_\gamma(\lambda,t)\{(\lambda^{-1}c - ib)e^{i\lambda\vartheta'\cdot}\}(x)dx\Bigg]. \tag{9}$$

Lemma 1 implies that this expression is rewritten by use of the physical scattering amplitude (6).

We choose $\omega, \omega' \in S^{n-1}$ to satisfy $\omega \cdot \gamma = \omega' \cdot \gamma = 0$ and put

$$\lambda\vartheta = \sqrt{\lambda^2 - t^2}\omega + t\gamma, \quad \lambda\vartheta' = \sqrt{\lambda^2 - t^2}\omega' + t\gamma.$$

Then (9) is reduced to

$$(2\pi)^n \lambda^{-n+1} a_\gamma(\lambda, \vartheta, \vartheta'; t) = \int e^{-i\sqrt{\lambda^2 - t^2}(\omega - \omega')\cdot x}\{\lambda^{-1}c(x) - ib(x)\}dx +$$

$$+\lambda \int e^{-i\sqrt{\lambda^2 - t^2}\omega\cdot x}\{\lambda^{-1}c(x) - ib(x)\}U_\gamma(\lambda, t)\{(\lambda^{-1}c - ib)e^{i\sqrt{\lambda^2 - t^2}\omega'\cdot}\}(x)dx.$$

The analytic continuation makes possible to replace t by $i\tau$ in this equation. It then follows from (8) that

$$(2\pi)^n \lambda^{-n+1} a_\gamma(\lambda, \vartheta, \vartheta'; i\tau) \simeq \int e^{-i\sqrt{\lambda^2 + \tau^2}(\omega - \omega')\cdot x}\{\lambda^{-1}c(x) - ib(x)\}dx \tag{10}$$

as $\tau \to \infty$. For any $\xi \in \mathbf{R}^n$ we choose $\gamma, \eta \in S^{n-1}$ to satisfy $\xi \cdot \gamma = \xi \cdot \eta = \gamma \cdot \eta = 0$, and put

$$\omega(\tau) = \left(1 - |\xi|^2/4\tau^2\right)^{1/2}\eta + \xi/2\tau, \quad \omega'(\tau) = \left(1 - |\xi|^2/4\tau^2\right)^{1/2}\eta - \xi/2\tau.$$

Then $\omega(\tau), \omega'(\tau) \in S^{n-1}$ and

$$\sqrt{\lambda^2 + \tau^2}(\omega(\tau) - \omega'(\tau)) = \sqrt{(\lambda/\tau)^2 + 1}\xi \simeq \xi(\tau \to \infty).$$

Thus, from (10) it is concluded that

$$\lim_{\tau\to\infty}(2\pi)^n \lambda^{-n+1} a_\gamma(\lambda, \vartheta(\tau), \vartheta'(\tau); i\tau) = \int e^{-i\xi\cdot x}\{\lambda^{-1}c(x) - ib(x)\}dx. \qquad \square$$

5 Uniform resolvent estimates and smoothing properties

We return to the magnetic Schrödinger operator (1). In the following we restrict ourselves to the case $n \geqslant 3$ and $\mathbf{R}^n \backslash \Omega$ being empty or starshaped with respect to the origin $x = 0$.

Theorem 8. *(i) Assume that*

$$\text{(A7)} \qquad \max\{|\nabla \times b(x)|, |c(x)|\} \leqslant \epsilon_0 r^{-2}, \quad \textit{in} \quad \Omega.$$

Then there exists $C_1 > 0$ *such that* $u = R(\kappa^2)f$ *satisfies*

$$\int \frac{1}{r^2}|u|^2 dx \leqslant C_1^2 \int r^2 |f|^2 dx \text{ for each} \kappa \in \Pi_{\pm}.$$

(*ii*) *Assume that*

$$(A8) \qquad \max\{|\nabla \times b(x)|, |c(x)|\} \leqslant \epsilon_0 \min\{\mu(r), r^{-2}\}, \quad \text{in} \quad \Omega,$$

where $\mu(r)$ *is a smooth, positive, non-incleasing* L^1*-function of* $r \in \mathbf{R}_+$. *Then there exists* $C_2 > 0$ *such that for each* $\kappa \in \Pi_{\pm}$,

$$\int \left\{ \mu(|\nabla_b u|^2 + |\kappa u|^2) - \mu' \frac{n-1}{2r}|u|^2 \right\} dx \leqslant C_2^2 \int \max\{\mu^{-1}, r^2\}|f|^2 dx.$$

As a corollary of Theorem 8 we are able to obtain space-time weighted estimates (smoothing properties) for the Schrödinger, and relativistic Schrödinger evolution equations

$$i\partial_t u + Lu = 0, \qquad u(0) = f \in L^2, \tag{11}$$

$$i\partial_t u + \sqrt{L + m^2}u = 0 \quad (m \geqslant 0), \quad u(0) = f \in L^2, \tag{12}$$

For an interval $I \subset \mathbf{R}$ and a Banach space X, we denote by $L_t^p(I, X)$ the space of X-valued L^p-functions of t, and simply write $L_t^p X$ for $L^p(\mathbf{R}, X)$.

Theorem 9. (*i*) *Under* (*A*7) *we have for* $h(t) \in L_t^2 L_{r^2}^2$, *and* $f \in L^2$,

$$\left\| \int_0^t e^{-i(t-\tau)L} h(\tau) d\tau \right\|_{L_t^2 L_{r^{-2}}^2} \leqslant C_1 \|h\|_{L_t^2 L_{r^2}^2}, \tag{13}$$

$$\|e^{itL} f\|_{L_t^2 L_{r^{-2}}^2} \leqslant \sqrt{2C_1}\|f\|. \tag{14}$$

(*ii*) *Under* (*A*8) *put* $\tilde{\mu}(r) = \min\{r^{-2}, \mu(r)\}$. *Then we have for* $g \in L^2$,

$$\|e^{it\sqrt{L+m^2}} g\|_{L_t^2 L_{\tilde{\mu}}^2} \leqslant \sqrt{mC_1 + C_2}\|g\|. \tag{15}$$

The above two theorems are the main part of [4].

6 Deacy-nondecay problems for time dependent complex potential

Consider the Schrödinger evolution equation in $L^2(\mathbf{R}^n)$,

$$i\partial_t u - \Delta u + c_1(x,t)u = 0, \quad u(x,0) = f(x), \tag{16}$$

where $c_1(x,t) = c(1+t)^{-\alpha}(1+r)^{-\beta}$ with some $c \in \mathbf{C}$ and α, $\beta \geqslant 0$. We denote by $U(t,s)$ the evolution operator which mapps solutons at time s to those at time t.

Theorem 10. *(i) (L^2 decay) If* $\mathrm{Im}c > 0$ *and* $\alpha + \beta \leqslant 1$*, then*

$$\|u(t)\|^2 \geqslant C\varphi(t)^{-1}\left\{\|\sqrt{\varphi(r)}f\|^2 + \|f\|^2_{H^1}\right\}; \quad \varphi(\sigma) = \int_0^\sigma (1+s)^{-\alpha-\beta}ds.$$

(ii) (L^2 nondecay) If $\mathrm{Im}c \geqslant 0$ *and* $\alpha + \beta > 1$*, then for each* $f \in L^2 \cap L^q$, $\exists s_0 > 0$ *such that for* $\forall s \geqslant s_0$,

$$U(t,s)e^{-i\Delta s}f \not\to 0 \quad \text{as } t \to \infty.$$

(iii) (existence of the scattering states) If $\mathrm{c} \geqslant 0$ *and* $\alpha + \dfrac{\beta}{2} > 1$*, then for any* $s \geqslant 0$ *and* $f \in L^2$, $\exists f_0 \in L^2$ *such that*

$$\lim_{t\to\infty} \|U(t,s)f - e^{-i(t-s)\Delta}f_0\| = 0.$$

See Mochizuki-Motai [17] for details. Similar properties are also proved for wave equations (e.g., Mochizuki-Nakazawa [18]).

Assertions (i) and (iii) hold for a more general equation with free Laplacian $-\Delta$ replaced by the magnetic Schrödinger operator L satisfying $(A7)$. In fact, under conditions of (iii), $c_1(x,t)$ satisfies

$$|c_1(x,t)| \leqslant |c|\left\{\frac{2-\beta}{2}(1+t)^{-2\alpha/(2-\beta)} + \frac{\beta}{2}(1+r)^{-2}\right\}.$$

Here without loss of generality we have assumed $\alpha+\beta \leqslant 2$. Since $(1+t)^{-2\alpha/(2-\beta)} \in L^1(\mathbf{R}_+)$, Theorem 8 (i) is applied to generalize the result.

7 Scattering for time dependent perturbations

Let $\mathcal{H}$ be a Hilbert space with innerproduct $(\cdot,\cdot)$ and norm $\|\cdot\|$, and consider in $\mathcal{H}$ the evolution equation

$$i\partial_t u + \Lambda_0 u + V(t)u = 0, \quad u(s) = f \in \mathcal{H}, \tag{17}$$

with initial time $s \in \mathbf{R}$, where Λ_0 is a selfadjoint operator in $\mathcal{H}$ with dense domain $\mathcal{D}(\Lambda_0)$ and $V(t)$ is a Λ_0-bounded operator which depends continuously on $t \in \mathbf{R}$. Let $e^{it\Lambda_0}$ be the unitary group in $\mathcal{H}$ which represents the solution of the free equation $i\partial_t u_0 + \Lambda_0 u_0 = 0$. Then the perturbed problem (17) reduces to the integral equation

$$u(t,s) = e^{i(t-s)\Lambda_0} f + \int_s^t e^{i(t-\tau)\Lambda_0} V(\tau) u(\tau,s) d\tau. \tag{18}$$

$(H1)$ There exist a Banach space X and $C_3 > 0$ such that

$$\|e^{i(t-s)\Lambda_0} f_0\|_{L^2_t X} \leqslant C_4 \|f_0\| \quad \text{for any } (s, f_0) \in \mathbf{R} \times \mathcal{H}.$$

Further, there exist a positive $\eta(t) \in L^1(\mathbf{R})$ and small $\epsilon_0 > 0$ such that

$$|(V(t)u, v)| \leqslant \eta(t)\|u\|\|v\| + \epsilon_0 \|u\|_X \|v\|_X.$$

$(H2)$ (18) has a unique solution $u(t,s) = U(t,s)f \in C(\mathbf{R}, \mathcal{H})$, which also satisfies

$$\|U(t,s)f\|_{L^2_t X} \leqslant C_4 \|f\|,$$

where $C_4 > 0$ is independent of $(s,f) \in \mathbf{R} \times \mathcal{H}$.

Theorem 11. *Assume $(H1)$ and $(H2)$ with ϵ_0 satisfying $\epsilon_0 C_3 C_4 < 1$. Then we have (i) $\{U(t,s)\}_{t,s\in\mathbf{R}}$ is a family of uniformly bounded operators. (ii) For every $s \in \mathbf{R}_\pm = \{t : \pm t > 0\}$, there exits the strong limit*

$$Z^\pm(s) = s - \lim_{t\to\pm\infty} e^{i(-t+s)\Lambda_0} U(t,s).$$

(iii) The operator $Z^\pm = Z^\pm(0)$ satisfies

$$w - \lim_{s\to\pm\infty} Z^\pm U(0,s) e^{is\Lambda_0} = I \quad (\textit{weak limit}).$$

(iv) If ϵ can be chosen smaller to satisfy $C_3 C_6 \epsilon_0 < 1$, then $Z^\pm : \mathcal{H} \longrightarrow \mathcal{H}$ is a bijection on $\mathcal{H}$. Moreover, the scattering operator $S = Z^+(Z^-)^{-1}$ is also a bijection.

A typical example is the Schrödinger equation

$$i\partial_t u + Lu + c_1(x,t)u = 0, \quad u|_{t=s} = f \in L^2, \tag{19}$$

where L is the selfadjoint operator in §7 and $c_1(x,t)$ is a complex function satisfyng

$$(A9) \qquad |c_1(x,t)| \leqslant \eta(t) + \epsilon_0 r^{-2}, \quad \text{with small } \epsilon_0 > 0.$$

We choose $\mathcal{H} = L^2(\Omega)$, $\Lambda_0 = L$, $V(t) = c_1(s,t)$ and $X = L^2_{r^{-2}}$. Then $(H1)$ is obvious from (13) and $(A9)$. To verify $(H2)$, put $Y(I) = L^\infty_t(I;L^2) \cap L^2_t(I;L^2_{r^{-2}})$. Then by use of (12) we have

Proposition 3. For $I_{+,s} = (s,T)$ $(s < T \leqslant \infty)$ or $I_{-,s} = (T,s)$ $(-\infty \leqslant T < s)$ let

$$\Phi_{\pm,s}v(t) = \int_s^t e^{i(t-\tau)L}c_1(\tau)v(\tau)d\tau, \quad v(t) \in Y(I_{\pm,s}).$$

Then we have $\|\Phi_{\pm,s}v\|_{Y(I_{\pm,s})} \leqslant C_5\|v\|_{Y(I_{\pm,s})}$ for some $C_5 = C_5(C_3,\eta,\epsilon_0) > 0$.

We choose $|T-s|$ so small or $|s|$ so large, and ϵ_0 so small that $C_5 < 1$. Then this lemma guarantees the solvability of (19) in $Y(I_{\pm,s})$ and we have

$$\|U(t,s)f\|_{Y(I_{\pm,s})} \leqslant C_6\|f\|, \quad C_6 = \frac{1+\sqrt{2C_3}}{1-C_5}.$$

Note that $\mathbf{R}$ is covered by a finite number $2N$ of such $I_{\pm,s}$. Then we see that (19) with $s=0$ has a unique global solution satisfying $(H2)$:

$$\|U(t,0)f\|_{L^\infty_t L^2} + \|U(t,0)f\|_{L^2_t L^2_{r^{-2}}} \leqslant C_7\|f\|, \quad C_7 = 2\sum_{k=1}^N C_6^k. \tag{20}$$

As we see, the inhomogeneous smoothing property (13) plays an important role to establish the scattering theory for time dependent perturbation. As for Klein-Gordon equations, we have the homogeneous smoothing property (15). However, it is insufficient to develop the scattering theory. So, we restrict ourselves to the simpler problem in the whole $\mathbf{R}^n$:

$$\partial_t^2 w - \Delta w + m^2 w + \sum_{j=1}^n b_j(x,t)\partial_j w + b_0(x,t)\partial_t w + c(x,t)w = 0, \tag{21}$$

$$w|_{t=s} = f_1(x), \quad \partial_t|_{t=s} = f_2(x).$$

Here $m > 0$, $b_j(x,t)$ $(j = 0, 1, \cdots, n)$ and $c(x,t)$ are complex functions satisfying

$$(A10) \qquad \max\{|b_j(x,t)|, m^{-1}|c(x,t)|\} \leqslant \eta(t) + \epsilon_0\tilde{\mu}(r), \quad \tilde{\mu} = \min\{\mu(r), r^{-2}\}.$$

Let $\mathcal{H}_E$ and X_E be the spaces with norms

$$\|\{f_1, f_2\}\|_E^2 = \frac{1}{2}\int \{|\nabla f_1|^2 + m^2|f_1|^2 + |f_2|^2\}dx < \infty,$$
$$\|f\|_{X_E}^2 = \frac{1}{2}\left\{\|\nabla f_1\|_X^2 + m^2\|f_1\|_X^2 + \|f_2\|_X^2\right\} < \infty,$$

where $X = L^2_{\tilde{\mu}}$. Then as an evolution equation in $\mathcal{H}E$, (21) is rewritten to the integral equation

$$u(t,s) = e^{i(t-s)\Lambda_0}f + \int_s^t e^{i(t-\tau)\Lambda_0}V(\tau)u(\tau,s)d\tau, \quad f = \{f_1, f_2\} \in \mathcal{H}_E; \tag{22}$$

$$\Lambda_0 = i\begin{pmatrix} 0 & 1 \\ \Delta - m^2 & 0 \end{pmatrix} \quad \text{and} \quad V(t) = -i\begin{pmatrix} 0 & 0 \\ b(x,t)\cdot\nabla + c(x,t) & b_0(x,t) \end{pmatrix}.$$

For $\kappa \in \mathbf{C}\backslash\mathbf{R}$, let $R_{0m}(\kappa^2) = (-\Delta + m^2 - \kappa^2)^{-1}$. Then the resolvent of Λ_0 is given by

$$\mathcal{R}_0(\kappa) = \begin{pmatrix} -\kappa & i \\ i(\Delta - m^2) & -\kappa \end{pmatrix} R_{0m}(\kappa^2)$$

and hence, we have for f, $g \in X_E'$,

$$|(\mathcal{R}_0(\kappa)f, g)_E| \leqslant \sum_{j=1}^n \left\{\|\kappa R_{0m}(\kappa^2)\partial_j f_1\|_X + \|\partial_j R_{0m}(\kappa^2)f_2\|_X\right\}\|\partial_j g_1\|_{X'} +$$
$$+ m^2\left\{\|\kappa R_{0m}(\kappa^2)f_1\|_X + \|R_{0m}(\kappa^2)f_2\|_X\right\}\|g_1\|_{X'} +$$
$$+ \left\{\sum_{j=1}^n \|\partial_j R_{0m}(\kappa^2)\partial_j f_1\|_X + m^2\|R_{0m}(\kappa^2)f_1\|_X + \|\kappa R_{0m}(\kappa^2)f_2\|_X\right\}\|g_2\|_{X'}.$$

Both inequalities of Theorem 8 imply that

$$\|\nabla R_{0m}(\kappa^2)h\|_X^2 + (1 + |\sqrt{\kappa^2 - m^2}|^2)\|R_{0m}(\kappa^2)h\|_X^2 \leqslant C\|h\|_{X'}^2$$

for any $h \in X'$ and κ^2 in the resolvent set of $-\Delta + m^2$. Thus, we conclude the existence of suitable $C_8 > 0$ verifying

$$|(\mathcal{R}_0(\kappa)f, g)_E| \leqslant C_8 \|f\|_{X'_E} \|g\|_{X'_E}, \tag{23}$$

or equivalently, we obtain the inhomogeneous smoothing property

$$\left\| \int_0^t e^{i(t-\tau)\Lambda_0} h(\tau) d\tau \right\|_{L_t^2 X_E} \leqslant C_8 \|h\|_{L^2 X'_E}.$$

Then as in the case of the Schrödinger equation, this and the smallness assumption $(A10)$ show the unique existence of solutions to (21) with $s = 0$ satisfying

$$\|U(t,0)f\|_{L_t^\infty \mathcal{H}_E} + \|U(t,0)f\|_{L_t^2 X_E} \leqslant C_9 \|f\|_E. \tag{24}$$

The above treatment is possible also in the mass less case $m = 0$ if $n \geqslant 4$. However, more general results in exterior domain, including the 3-dimensional problem, are guaranteed if we apply weighted energy methods. We consider in Ω the wave equation (21) with $m = 0$ and the initial-boundary conditions

$$w|_{t=s} = f_1(x), \quad w_t|_{t=s} = f_2(x), \quad w|_{\partial\Omega} = 0, \tag{25}$$

where $b_j(x,t)$ $(j = 0, 1, \cdots, n)$ and $c(x,t)$ are real functions satisfying

$$(A11) \qquad \max\left\{ |b_j(x,t)|, \frac{2r}{n-2}|c(x,t)| \right\} \leqslant \eta(t) + \epsilon_0 \mu(r).$$

Here $\mu(r) \in L^1(\mathbf{R})$ is chosen to satisfy also

$$\mu(r) > 0, \quad \mu'(r) \leqslant 0, \quad \mu'(r)^2 \leqslant 2\mu(r)\mu''(r). \tag{26}$$

We choose $m = 0$, $\tilde{\mu}(r) = \mu(r)$ and f_1 (the first component) verifying the zero boundary condition $f_1|_{\partial\Omega} = 0$ in the definition of $\mathcal{H}_E$ and X_E. Then $(A11)$ and the following proposition verify both $(H1)$ and $(H2)$ since the unique existence of solution in $C(\mathbf{R}; \mathcal{H}_E)$ is evident,

Proposition 4. Under $(A11)$ with sufficiently small $\epsilon_0 > 0$, let $u(t) = \{w(t), w_t(t)\}$ be the solution of (21) with $m = 0$ and (25). Then

$$\|u(t)\|_E \leqslant C_{10} \|f\|_E, \quad f = \{f_1, f_2\},$$

$$\frac{1}{2}\int_s^t \int_\Omega \Big\{\mu(|\nabla w|^2 + w_t^2) - \mu' \frac{n-1}{2r} w^2\Big\} dx d\tau \leqslant C_{11}^2 \|u\|_E^2,$$

where $C_{10} > 0$ and $C_{11} > 0$ are independent of $(s, f) \in \mathbf{R} \times \mathcal{H}_E$.

For the proof of Theorem 11 and Proposition 4 see [15]. Shrödinger equations (16) with $c_1(x,t) \in L_t^\nu L^r$ $(0 < 1/r \leqslant 2/n,\ 1/\nu = 1-n/2r)$ and the above wave equations are studied there as examples. But Klein-Gordon equations are not treated there.

8 Strichartz estimates

In the rest of this article we discuss the so called Strichartz estimates. As will be seen, Strichartz estimates of free equations and smoothig properites of perturbed solutions (i.e., $(H2)$) lead us to the Strichartz estimates for pertubed equations.

First consider Schrödinger equations in $\mathbf{R}^n$. Let $p \geqslant 2$, q be the admissible exponents $\dfrac{2}{p} + \dfrac{n}{q} = \dfrac{n}{2}$. Then as is well known, there exists a constant $C > 0$ such that

$$\|e^{-it\Delta} f(x)\|_{L_t^p L^q} \leqslant C\|f\|_{L^2}. \tag{27}$$

More precislely, the end poit estimate is given by

$$\Big\|\int_0^t e^{-i(t-s)\Delta} h(x,s) ds\Big\|_{L_t^2 L^{2n/(n-2),2}} \leqslant C\|h\|_{L_t^2 L^{2n/(n+2),2}}, \tag{28}$$

where $L^{\alpha,\beta}$ are Lorentz spaces.

Theorem 12. *Under $(A9)$ with $\eta(t) \equiv 0$, let $u(t) \in C(\mathbf{R}; L^2)$ be the solution of (16). Then for any admissible exponents p, q there exists $C > 0$ such that*

$$\|u\|_{L_t^p L^q} \leqslant C\|f\|_{L^2}, \quad \forall f \in L^2.$$

Proof. All we need to check is $F = cu \in L_t^2 L^{2n/(n+2),2}$. However, from (20) we have $r^{-1}u \in L_t^2 L_x^2$, while by assumption $rc_1(\cdot, t) \in L^{n,\infty}$. Thus, (28) and O'Neil's inequality ([19]) imply

$$\Big\|\int_0^t e^{-i(t-s)\Delta} c_1(s) u(s) ds\Big\|_{L_t^2 L^{2n/(n-2),2}} \leqslant C\|c_1 u\|_{L_t^2 L^{2n/(n+2),2}} \leqslant$$

$$\leqslant C\|rc_1\|_{L^{n,\infty}}\|r^{-1}u\|_{L_t^2L^2} \leqslant C\|f\|_{L^2}.$$

This and (27) prove the Strichartz estimate at the end point:

$$\|u\|_{L_t^2L^{2n/(n-2),2}} \leqslant C\|f\|_{L^2}.$$

Interpolatng between this and the uniform boundedness (cf. also (20)) $\|u\|_{L_t^\infty L^2} \leqslant C\|f\|$, one obtains the full range of the estimates in Theorem 12. $\square$

Next, the solution $w(t)$ of the Klein-Gordon equation (21) satisfies

$$w(t) = \dot{W}(t)f_1 + W(t)f_2 + \int_0^t W(t-s)[V(s)u(s)]_2 ds, \tag{29}$$

where $W(t) = \sqrt{-\Delta+m^2}^{-1}\sin(t\sqrt{-\Delta+m^2})$ with $m > 0$ and

$$[V(t)u(t)]_2 = b_0(x,t)w_t + b(x,t)\cdot\nabla w + c(x,t)w.$$

Let p, q be any admissible exponents of Schrödinger equations, and $\gamma = \frac{1}{p} + \frac{1}{2} - \frac{1}{q}$. Then the following estimate holds for the free solution (see e.g., D'Ancona-Fanelli [2]).

$$\|e^{it\sqrt{-\Delta+m^2}}g\|_{L_t^pH_q^{-\gamma}} \leqslant C\|g\|_{L^2}. \tag{30}$$

The following is the well known Christ-Kisherev lemma ([1]).

Lemma 4. *Let X, Y be Banach spaces and let $Tf(t) = \int_0^\infty K(t,s)f(s)ds$ be a bounded operator from $L^\alpha(\mathbf{R};X)$ to $L^\beta(\mathbf{R};Y)$. If $\alpha < \beta$, then $\tilde{T}f(t) = \int_0^t K(t,s)f(s)ds$ is also a bounded operator, and we have $\|\tilde{T}\| \leqslant C(\alpha,\beta)\|T\|$.*

Theorem 13. *Under (A10) with $\eta(t) \equiv 0$, let $w(t) \in C^1(\mathbf{R};H^1)$ be the solution of (29). Then for any Schrödinger admissible exponents p, q satisfying also $p > 2$, there exists $C > 0$ such that*

$$\|\sqrt{-\Delta+m^2}w\|_{L_t^pL^q} + \|w_t\|_{L_t^pL^q} \leqslant C\{\|f_1\|_{H^\gamma} + \|f_2\|_{H^{\gamma-1}}\}.$$

Proof. Let $h(t) \in L_t^2 L_{\tilde{\mu}}^2$. Then it follows from (28) and the above lemma that

$$\left\| \int_0^t e^{i(t-s)\sqrt{-\Delta+m^2}} h(s) ds \right\|_{L_t^p H_p^{-\gamma}} \leqslant C \left\| \int_0^\infty e^{-is\sqrt{-\Delta+m^2}} h(s) ds \right\|_{L^2} \leqslant C\|h\|_{L_t^2 L_{\tilde{\mu}-1}^2}.$$

In the last inequality we have applied the dual fomula of (15) of Theorem 9. Put $h(t) = [V(t)u]_2$. Then as is seen in (24) $\|[V(t)u]_2\|_{L_t^2 L_{\tilde{\mu}-1}^2} \leqslant C\|f\|_E$. Combining these inequalities and (30), we conclude the assertion. □

Finally, we consider the solution $w(t)$ of the wave equation (23) with 0-boundary condition requiring $\mathbf{R}^n \backslash \Omega$ is convex. Let $p \geqslant 2$, q be any admissible exponents of wave equations satisfying $\dfrac{2}{p} + \dfrac{n-1}{q} = \dfrac{n-1}{2}$ $(q \neq \infty)$, and $\gamma = \dfrac{1}{p} + \dfrac{1}{2} - \dfrac{1}{q}$. Then the following estimate is known to hold (see e.g., Metacalfe [10]).

$$\|e^{it\sqrt{-\Delta_D}} g\|_{L_t^p H_q^{-\gamma}} \leqslant C\|g\|_{L^2}. \tag{31}$$

Theorem 14. *Under* (*A*11) *with* $\eta(t) \equiv 0$, *let* $w(t) \in C^1(\mathbf{R}; \dot{H}^1)$ *be the solution of* (29) *with* $W(t) = \sqrt{-\Delta_D}^{-1} \sin(t\sqrt{-\Delta_D})$. *Then for any wave admissible exponents* p, q *satisfying also* $p > 2$, *there exists* $C > 0$ *such that*

$$\|\sqrt{-\Delta_D} w\|_{L_t^p L^q} + \|w_t\|_{L_t^p L^q} \leqslant C\{\|f_1\|_{\dot{H}^\gamma} + \|f_2\|_{\dot{H}^{\gamma-1}}\}.$$

Proof. With (31) we can follow the above argument to obtain

$$\left\| \int_0^t e^{i(t-s)\sqrt{-\Delta_D}} [V(s)u(s)]_2 ds \right\|_{L_t^p(\dot{H}_q^{-\gamma})} \leqslant$$

$$\leqslant C\|\{|\partial_t w| + |\nabla w| + r^{-1}|w|\}\|_{L_t^2(L_{x,\tilde{\mu}}^2)} \leqslant C\|f\|_E. \tag{32}$$

In the last inequality we have used Proposition 4 and the Hardy inequality. Combining (31) and (32) we conclude the assertion. □

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SMOOTHING ESTIMATES OF AN INVARIANT FORM

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Abstract. For operators $a(D_x)$ of order m satisfying the dispersiveness $\nabla a(\xi) \neq 0$, the smoothing estimate

$$\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \leqslant C \|\varphi\|_{L^2(\mathbf{R}^n_x)} \quad (s > 1/2)$$

is well known, while it is known to fail for general non-dispersive operators. We suggest a form

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \leqslant C \|\varphi\|_{L^2(\mathbf{R}^n_x)} \quad (s > 1/2)$$

which is equivalent to the usual estimate in the dispersive case and also invariant under canonical transformations for the operator $a(D_x)$. It does continue to hold for a variety of non-dispersive operators $a(D_x)$, where $\nabla a(\xi)$ may become zero on some set.

1 Introduction

This survey article is a collection of results and arguments from author's papers [9] and [10]. Let us consider the following Cauchy problem to Schrödinger equation:

$$\begin{cases} (i\partial_t - \triangle_x)\, u(t,x) = 0 \\ u(0,x) = \varphi(x) \in L^2(\mathbf{R}^n). \end{cases}$$

By Plancherel's theorem, the solution $u(t,x) = e^{it\triangle_x}\varphi(x)$ preserves the L^2-norm of the initial data φ, that is, we have $\|u(t,\cdot)\|_{L^2(\mathbf{R}^n_x)} = \|\varphi\|_{L^2(\mathbf{R}^n)}$ for any fixed time $t \in \mathbf{R}$. But if we integrate the solution in t, we get an extra gain of reguralily of order $1/2$ in x:

$$\left\| \langle x \rangle^{-s} |D_x|^{1/2} u \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \leqslant C \|\varphi\|_{L^2(\mathbf{R}^n)} \quad (s > 1/2),$$

where $\langle x\rangle = \sqrt{1+|x|^2}$. This type of estimate is called smoothing estimate, and its local version was first proved by Sjörin [11], Constantin & Saut [4], and Vega [12]. The global version was proved by Kenig, Ponce, &, Vega 1991 ($n = 1$), Ben-Artzi & Klainerman 1992 ($n \geqslant 3$), and Chihara 2002 ($n \geqslant 2$).

Historically, such smoothing estimates was first shown to Korteweg-de Vries equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, \\ u(0,x) = \varphi(x) \in L^2(\mathbf{R}) \end{cases}$$

and Kato [6] proved that the solution $u = u(t,x)$ $(t, x \in \mathbf{R})$ satisfies

$$\int_{-T}^{T}\int_{-R}^{R} |\partial_x u(x,t)|^2 \, dxdt \leqslant c(T, R, \|\varphi\|_{L^2}).$$

The purpose of this article is to present a similar kind of smoothing estimate for more general equation $(i\partial_t + a(D_x))\, u(t,x) = 0$ which corresponds to principal parts of many important equations from physics:

— $a(\xi) = |\xi|^2$ $\cdots$ Schrödinger

$$i\partial_t u - \Delta_x u = 0$$

— $a(\xi) = \sqrt{|\xi|^2+1}$ $\cdots$ Relativistic Schrödinger

$$i\partial_t u + \sqrt{-\Delta_x + 1}\, u = 0$$

— $a(\xi) = \xi^3$ $(n = 1)$ $\cdots$ Korteweg-de Vries (shallow water wave)

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0$$

— $a(\xi) = |\xi|\xi$ $(n = 1)$ $\cdots$ Benjamin-Ono (deep water wave)

$$\partial_t u - \partial_x |D_x| u + u\partial_x u = 0$$

— $a(\xi) = \xi_1^2 - \xi_2^2$ $(n = 2)$ $\cdots$ Davey-Stewartson (shallow water wave of 2D)

$$\begin{cases} i\partial_t u - \partial_x^2 u + \partial_y^2 u = c_1|u|^2 u + c_2 u\partial_x v \\ \partial_x^2 v - \partial_y^2 v = \partial_x |u|^2 \end{cases}$$

— $a(\xi) = \xi_1^3 + \xi_2^3$, $\xi_1^3 + 3\xi_2^2$, $\xi_1^2 + \xi_1\xi_2^2$ $\cdots$ Shrira (deep water wave of 2D)

— $a(\xi)$ = quadratic form ($n \geqslant 3$) $\cdots$ Zakharov-Schulman (interaction of sound wave and low amplitudes high frequency wave)

We investigate smoothing estimates for them by using new methods of comparison and canonical transfrom which are quite strong to this problem. It works not only for all the dispersive equations (that is, the case $\nabla a \neq 0$) but also for some non-dispersive equations, and induces smoothing estimates of an invariant form.

2 Smoothing estimate for dispersive equations

We consider smoothing estimates for solutions $u(t,x) = e^{ita(D_x)}\varphi(x)$ to general equations

$$\begin{cases} (i\partial_t + a(D_x))\, u(t,x) = 0 \\ u(0,x) = \varphi(x) \in L^2(\mathbf{R}^n). \end{cases}$$

Let $a_m(\xi)$ be the principal term of $a(\xi)$ satisfiing

$$a_m(\xi) \in C^\infty(\mathbf{R}^n \setminus 0), \quad \text{real-valued}, \quad a_m(\lambda\xi) = \lambda^m a_m(\xi) \ \ (\lambda > 0, \xi \neq 0),$$

and we assume that $a(\xi)$ is *dispersive* in the following sense:

$$(\mathbf{H}) \quad a(\xi) = a_m(\xi), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \neq 0),$$

otherwise

$$\begin{aligned} (\mathbf{L}) \quad & a(\xi) \in C^\infty(\mathbf{R}^n), \quad \nabla a_m(\xi) \neq 0 \ (\xi \neq 0), \quad \nabla a(\xi) \neq 0 \ (\xi \in \mathbf{R}^n), \\ & |\partial^\alpha (a(\xi) - a_m(\xi))| \leqslant C\langle\xi\rangle^{m-1-|\alpha|} \quad (|\xi| \geqslant 1). \end{aligned}$$

Example 1. $a(\xi) = \xi_1^3 + \cdots + \xi_n^3 + \xi_1$ satisfies (L).

The dispersiveness means that classical orbit, that is, the solution of

$$\begin{cases} \dot{x}(t) = (\nabla a)(\xi(t)), \quad \dot{\xi}(t) = 0 \\ x(0) = 0, \quad \xi(0) = k, \end{cases}$$

does not stop, and the singularity of $u(t,x) = e^{ita(D_x)}\varphi(x)$ travels to the infinity along this orbit. Hence we can expect the smoothing, and indeed we have

Theorem 1. *Assume* (H) *or* (L). *Suppose $m > 0$ and $s > 1/2$. Then we have*

$$\left\| \langle x\rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)}\varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \leqslant C\|\varphi\|_{L^2(\mathbf{R}^n)}.$$

Remark 1. Any polynomial $a(\xi)$ which satisfies the estimate in Theorem 1 has to be dispersive, that is $\nabla a_m(\xi) \neq 0 \quad (\xi \neq 0)$ (Hoshiro [5]).

Remark 2. Chihara [3] proved Theorem 1 with the case (H) and $m > 1$.

3 Usual and new methods of the approach

The following are equivalent to each other:

- Smoothing estimate

$$\left\| A e^{-it\Delta_x} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \leqslant C \|\varphi\|_{L^2(\mathbf{R}^n_x)} \quad \text{where } A = A(X, D_x),$$

- Restriction estimate

$$\left\| \widehat{A^* f}_{|S^{n-1}_\rho} \right\|_{L^2\left(S^{n-1}_\rho\right)} \leqslant C\sqrt{\rho} \|f\|_{L^2(\mathbf{R}^n)}, \quad \text{where } S^{n-1}_\rho = \{\xi;\, |\xi| = \rho\},\ (\rho > 0),$$

- Resolvent estimate

$$\sup_{\operatorname{Im}\zeta > 0} |(R(\zeta) A^* f, A^* f)| \leqslant C \|f\|^2_{L^2(\mathbf{R}^n)}, \quad \text{where } R(\zeta) = (-\triangle - \zeta)^{-1}.$$

Most of the literature so far use the above equivalence to show smoothing estimates for dispersive equations by showing restriction or resolvent estimates instead. But here we develop completely different methods

1. **Comparison principle** $\cdots$ comparison of the symbol implies the comparison of estimate,
2. **Canonical Transformation** $\cdots$ shift an equation to another simple one (Egorov's theorem),

and use them to show smoothing estimates for both dispersive and non-dispersive equations. We will explain them in due order.

4 Comparison Principle

Here we list theorems showing that the comparison principle is surely true.

Theorem 2 (1D case). *Let $f, g \in C^1(\mathbf{R})$ be real-valued and strictly monotone. If $\sigma, \tau \in C^0(\mathbf{R})$ satisfy*

$$\frac{|\sigma(\xi)|}{|f'(\xi)|^{1/2}} \leqslant A \frac{|\tau(\xi)|}{|g'(\xi)|^{1/2}}$$

then we have

$$\|\sigma(D_x)e^{itf(D_x)}\varphi(x)\|_{L^2(\mathbf{R}_t)} \leqslant A\|\tau(D_x)e^{itg(D_x)}\varphi(x)\|_{L^2(\mathbf{R}_t)}$$

for all $x \in \mathbf{R}$.

Theorem 3 (2D case). *Let* $f(\xi,\eta), g(\xi,\eta) \in C^1(\mathbf{R}^2)$ *be real-valued and strictly monotone in* $\xi \in \mathbf{R}$ *for each fixed* $\eta \in \mathbf{R}$. *If* $\sigma, \tau \in C^0(\mathbf{R}^2)$ *satisfy*

$$\frac{|\sigma(\xi,\eta)|}{|f_\xi(\xi,\eta)|^{1/2}} \leqslant A\frac{|\tau(\xi,\eta)|}{|g_\xi(\xi,\eta)|^{1/2}}$$

then we have

$$\left\|\sigma(D_x,D_y)e^{itf(D_x,D_y)}\varphi(x,y)\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}_y)} \leqslant A\|\tau(D_x,D_y)e^{itg(D_x,D_y)}\varphi(x,y)\|_{L^2(\mathbf{R}_t\times\mathbf{R}_y)}$$

for all $x \in \mathbf{R}$.

Theorem 4 (Radially Symmetric case). *Let* $f, g \in C^1(\mathbf{R}_+)$ *be real-valued and strictly monotone. If* $\sigma, \tau \in C^0(\mathbf{R}_+)$ *satisfy*

$$\frac{|\sigma(\rho)|}{|f'(\rho)|^{1/2}} \leqslant A\frac{|\tau(\rho)|}{|g'(\rho)|^{1/2}}$$

then we have

$$\|\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^2(\mathbf{R}_t)} \leqslant A\|\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^2(\mathbf{R}_t)}$$

for all $x \in \mathbf{R}^n$.

5 Low dimensional model estimates

By the comparison principal, we can show the equivalence of low dimensional estimates of various type:

In the 1D case, we have $(l, m > 0)$.

$$\sqrt{m}\left\||D_x|^{(m-1)/2}e^{it|D_x|^m}\varphi(x)\right\|_{L^2(\mathbf{R}_t)} = \sqrt{l}\left\||D_x|^{(l-1)/2}e^{it|D_x|^l}\varphi(x)\right\|_{L^2(\mathbf{R}_t)} \tag{1}$$

for all $x \in \mathbf{R}$. Here $\operatorname{supp}\widehat{\varphi} \subset [0,+\infty)$ or $(-\infty,0]$. In the 2D case, we have $(l, m > 0)$

$$\left\||D_y|^{(m-1)/2}e^{itD_x|D_y|^{m-1}}\varphi(x,y)\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}_y)} =$$

$$= \left\| |D_y|^{(l-1)/2} e^{itD_x|D_y|^{l-1}\varphi(x,y)} \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_y)} \tag{2}$$

for all $x \in \mathbf{R}$. On the other hand, in 1D case, we have trivially

$$\left\| e^{itD_x}\varphi(x) \right\|_{L^2(\mathbf{R}_t)} = \|\varphi(x+t)\|_{L^2(\mathbf{R}_x)} = \|\varphi\|_{L^2(\mathbf{R}_x)} \tag{3}$$

for all $x \in \mathbf{R}$. Using the equality (3), the right hand sides of (1) and (2) with $l = 1$ can be estimated, and we have for all $x \in \mathbf{R}$:

- 1D Case

$$\left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbf{R}_t)} \leqslant C\|\varphi\|_{L^2(\mathbf{R}_x)},$$

- 2D Case

$$\left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x,y) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_y)} \leqslant C\|\varphi\|_{L^2(\mathbf{R}^2_{x,y})}.$$

Remark 3. In the case $m = 2$, these estimates were proved by Kenig, Ponce & Vega [7] (1D case) and Linares & Ponce [8] (2D case).

The following is a straightforward consequence from these estimates:

Proposition 1. Suppose $m > 0$ and $s > 1/2$. Then for $n \geqslant 1$ we have

$$\left\| \langle x \rangle^{-s} |D_n|^{(m-1)/2} e^{it|D_n|^m} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \leqslant C\|\varphi\|_{L^2(\mathbf{R}^n_x)}$$

and for $n \geqslant 2$ we have

$$\left\| \langle x \rangle^{-s} |D_n|^{(m-1)/2} e^{itD_1|D_n|^{m-1}} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \leqslant C\|\varphi\|_{L^2(\mathbf{R}^n_x)},$$

where $D_x = (D_1, \ldots, D_n)$.

6 Canonical Transformation

For the change of variable $\psi : \mathbf{R}^n \setminus 0 \to \mathbf{R}^n \setminus 0$ satisfying $\psi(\lambda\xi) = \lambda\psi(\xi)$ for all $\lambda > 0$ and $\xi \in \mathbf{R}^n \setminus 0$, we set

$$Iu(x) = F^{-1}[(Fu)(\psi(\xi))](x)$$

Then we have the relation

$$a(D_x)\cdot I = I\cdot\sigma(D_x), \quad a(\xi)=(\sigma\circ\psi)(\xi).$$

By using this transform, the equation

$$\begin{cases}(i\partial_t + a(D_x))\,u(t,x)=0\\ u(0,x)=\varphi(x)\in L^2(\mathbf{R}^n)\end{cases}$$

can be transformed to

$$\begin{cases}(i\partial_t + \sigma(D_x))\,v(t,x)=0\\ v(0,x)=g(x)\in L^2(\mathbf{R}^n).\end{cases}$$

The canonical transformation is bounded on the weighted space $L^2_k(\mathbf{R}^n)$ defined by

$$\|f\|_{L^2_k(\mathbf{R}^n)} = \left(\int \left|\langle x\rangle^k f(x)\right|^2 dx\right)^{1/2}.$$

Indeed we have

Theorem 5. *I is $L^2_k(\mathbf{R}^n)$-bounded for $|k|<n/2$.*

7 Reduction of smoothing estimates to model estimates

On account of the boundedness result (Theorem 5), smoothing estimates for dispersive equations (Theorem 1) can be reduced to low dimensional model estimates (Proposition 1) by the canonical transformation if we find a homogeneous change of variable ψ such that

$$a(\xi)=(\sigma\circ\psi)(\xi), \quad \sigma(D)=|D_n|^m \quad \text{or} \quad \sigma(D)=D_1|D_n|^{m-1}.$$

We show how to select such ψ under the assumption (H). The argument for the case (L) is similar. By microlocalization and rotation, we may assume that the initial data φ satisfies $\operatorname{supp}\hat{\varphi}\subset\Gamma$, where $\Gamma\subset\mathbf{R}^n\setminus 0$ is a sufficiently small conic neighborhood of $e_n=(0,\dots 0,1)$. Furthermore, we have Euler's identity

$$a(\xi)=a_m(\xi)=\frac{1}{m}\xi\cdot\nabla a(\xi),$$

and the dispersiveness $\nabla a(e_n)\neq 0$ implies the following two cases:

(I) $\partial_n a(e_n) \neq 0 \cdots$ (elliptic). By Euler's identity, we have $a(e_n) \neq 0$. Hence, in this case, we may assume $a(\xi) > 0$ $(\xi \in \Gamma)$, $\partial_n a(e_n) \neq 0$

(II) $\partial_n a(e_n) = 0 \cdots$ (non-elliptic). By assumption $\nabla a(e_n) \neq 0$, there exits $j \neq n$ such that $\partial_j a(e_n) \neq 0$. Hence, in this case, we may assume $\partial_1 a(e_n) \neq 0$.

In the elliptic case (I), we take

$$\sigma(\eta) = |\eta_n|^m, \quad \psi(\xi) = (\xi_1, \ldots, \xi_{n-1}, a(\xi)^{1/m}).$$

Then we have $a(\xi) = (\sigma \circ \psi)(\xi)$, and ψ is surely a change of variables on Γ since

$$\det \partial\psi(e_n) = \begin{vmatrix} E_{n-1} & 0 \\ * & \frac{1}{m} a(e_n)^{1/m-1} \partial_n a(e_n) \end{vmatrix} \neq 0$$

where E_{n-1} is the identity matrix. In the non- elliptic case (II), we take

$$\sigma(\eta) = \eta_1 |\eta_n|^{m-1}, \quad \psi(\xi) = \left(\frac{a(\xi)}{|\xi_n|^{m-1}}, \xi_2, \ldots, \xi_n \right).$$

Then we have again $a(\xi) = (\sigma \circ \psi)(\xi)$ and

$$\det \partial\psi(e_n) = \begin{vmatrix} \partial_1 a(e_n) & * \\ 0 & E_{n-1} \end{vmatrix} \neq 0.$$

8 Non-dispersive case

What happens if the equation does not satisfy the dispersiveness assumption $\nabla a(\xi) \neq 0$ $(\xi \in \mathbf{R}^n)$? Although we cannot have smoothing estimates (see Remark 1), such case appears naturally in the physics. For example, let us consider a coupled system of Schrödinger equations

$$i\partial_t v = \Delta_x v + b(D_x) w, \quad i\partial_t w = \Delta_x w + c(D_x) v$$

which represents a linearized model of wave packets with two modes. Assume that this system is diagonalized and regard it as a single equations for the eigenvalues:

$$a(\xi) = -|\xi|^2 \pm \sqrt{b(\xi) c(\xi)}.$$

Then there could exist points ξ such that $\nabla a(\xi) = 0$ because of the lower order terms $b(\xi)$, $c(\xi)$. Another interesting examples are Shrira equations:

$$a(\xi) = \xi_1^3 + \xi_2^3, \quad \xi_1^3 + 3\xi_2^2, \quad \xi_1^2 + \xi_1\xi_2^2.$$

Although $a(\xi) = \xi_1^3 + \xi_2^3$ satisfies assumption (H), $a(\xi) = \xi_1^3 + 3\xi_2^2$ and $a(\xi) = \xi_1^2 + \xi_1\xi_2^2$ do not satisfy assumption (L) because $\nabla a(0) = 0$.

We suggest an estimate which we expect to have for non-dispersive equations:

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_x^n)} \leqslant C \|\varphi\|_{L^2(\mathbf{R}_x^n)} \quad (s > 1/2) \tag{4}$$

and let us call it *invariant estimate*. This estimate has a number of advantages:

— in the dispersive case $\nabla a(\xi) \neq 0$, it is equivalent to Theorem 1;
— it is invariant under canonical transformations for the operator $a(D_x)$;
— it does continue to hold for a variety of non-dispersive operators $a(D_x)$, where $\nabla a(\xi)$ may become zero on some set and when the usual estimate fails;
— it does take into account zeros of the gradient $\nabla a(\xi)$, which is also responsible for the interface between dispersive and non-dispersive zone (e.g. how quickly the gradient vanishes).

9 Secondary comparison

By using comparison principle again to the smoothing estimates obtained from the comparison principle, we can have new estimates. This is a powerful tool to induce the invariant estimates (4) for non-dispersive equations.

For example, we have just obtained the estimate

$$\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_x^n)} \leqslant C \|\varphi\|_{L^2(\mathbf{R}_x^n)}$$

(Theorem 1 with $a(\xi) = |\xi|^m$) from comparison principle and canonical transformation. If we set $g(\rho) = \rho^m$, $\tau(\rho) = \rho^{(m-1)/2}$, then we have $|\tau(\rho)|/|g'(\rho)|^{1/2} = 1/\sqrt{m}$. Hence by the comparison result again for radially symmetric case (Theorem 4), we have

Theorem 6. *Suppose $s > 1/2$. Let $f \in C^1(\mathbf{R}_+)$ be real-valued and strictly monotone. If $\sigma \in C^0(\mathbf{R}_+)$ satisfy*

$$|\sigma(\rho)| \leqslant A|f'(\rho)|^{1/2},$$

then we have

$$\|\langle x\rangle^{-s}\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^2(\mathbf{R}_t\times\mathbf{R}^n_x)} \leqslant C\|\varphi\|_{L^2(\mathbf{R}^n_x)}.$$

From this secondary comparison, we obtain immediately the following invariant estimate since a radial function $a(\xi) = f(|\xi|)$ always satisfies $|\nabla a(\xi)| = |f'(|\xi|)|$.

Theorem 7. *Suppose $s > 1/2$. Let $a(\xi) = f(|\xi|)$ and $f \in C^\omega(\mathbf{R}_+)$ be real-valued. Then we have*

$$\left\|\langle x\rangle^{-s}|\nabla a(D_x)|^{1/2}e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}^n_x)} \leqslant C\|\varphi\|_{L^2(\mathbf{R}^n_x)}.$$

Example 2. $a(\xi) = (|\xi|^2-1)^2$ is non-dispersive because

$$\nabla a(\xi) = 4(|\xi|^2-1)\xi = 0$$

if $|\xi| = 0, 1$. But we have the invariant estimate by Theorem 7.

For non-radially symmetric case, we compare again to the low dimensional model estimates (Proposition 1) and obtain

Theorem 8 (1D secondary comparison). *Suppose $s > 1/2$. Let $f \in C^1(\mathbf{R})$ be real-valued and strictly monotone. If $\sigma \in C^0(\mathbf{R})$ satisfies*

$$|\sigma(\xi)| \leqslant A|f'(\xi)|^{1/2},$$

then we have

$$\|\langle x\rangle^{-s}\sigma(D_x)e^{itf(D_x)}\varphi(x)\|_{L^2(\mathbf{R}_t\times\mathbf{R}_x)} \leqslant AC\|\varphi(x)\|_{L^2(\mathbf{R}_x)}.$$

Theorem 9 (2D secondary comparison). *Suppose $s > 1/2$. Let $f \in C^1(\mathbf{R}^2)$ be real-valued and $f(\xi,\eta)$ be strictly monotone in $\xi \in \mathbf{R}$ for every fixed $\eta \in \mathbf{R}$. If $\sigma \in C^0(\mathbf{R}^2)$ satisfies*

$$|\sigma(\xi,\eta)| \leqslant A|\partial f/\partial\xi(\xi,\eta)|^{1/2},$$

then we have

$$\left\|\langle x\rangle^{-s}\sigma(D_x,D_y)e^{itf(D_x,D_y)}\varphi(x,y)\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}^2_{x,y})} \leqslant AC\|\varphi(x,y)\|_{L^2(\mathbf{R}^2_{x,y})}.$$

Example 3. By using secondary comparison for non-radially symmetric case, we have invariant estimates for Shrira equations. In fact, for $a(\xi) = \xi_1^3 + 3\xi_2^2$, we have by 1D secondary comparison (Theorem 8)

$$\left\| \langle x_1\rangle^{-s}|D_1|e^{itD_1^3}\varphi(x)\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}_x^2)} \leqslant C\|\varphi\|_{L^2(\mathbf{R}_x^2)},$$
$$\left\| \langle x_2\rangle^{-s}|D_2|^{1/2}e^{it3D_2^2}\varphi(x)\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}_x^2)} \leqslant C\|\varphi\|_{L^2(\mathbf{R}_x^2)}$$

for $s > 1/2$. Hence by $\langle x\rangle^{-s} \leqslant \langle x_k\rangle^{-s}$ $(k = 1, 2)$ we have

$$\left\| \langle x\rangle^{-s}\Big(|D_1| + |D_2|^{1/2}\Big)e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}_x^2)} \leqslant C\|\varphi\|_{L^2(\mathbf{R}_x^2)}$$

and hence have

$$\left\| \langle x\rangle^{-s}|\nabla a(D_x)|^{1/2}e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}_x^2)} \leqslant C\|\varphi\|_{L^2(\mathbf{R}_x^2)}.$$

For $a(\xi) = \xi_1^2 + \xi_1\xi_2^2$, we have by 2D secondary comparison (Theorem 9)

$$\left\| \langle x_1\rangle^{-s}|2D_1 + D_2^2|^{1/2}e^{ita(D_1,\, D_2)}\varphi(x)\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}_x^2)} \leqslant C\|\varphi\|_{L^2(\mathbf{R}_x^2)},$$
$$\left\| \langle x_2\rangle^{-s}|D_1D_2|^{1/2}e^{ita(D_1,\, D_2)}\varphi(x)\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}_x^2)} \leqslant C\|\varphi\|_{L^2(\mathbf{R}_x^2)},$$

for $s > 1/2$, hence we have similarly

$$\left\| \langle x\rangle^{-s}|\nabla a(D_x)|^{1/2}e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}_x^2)} \leqslant C\|\varphi\|_{L^2(\mathbf{R}_x^2)}.$$

10 Non-dispersive case controlled by Hessian

We will show that in the non-dispersive situation the rank of $\nabla^2 a(\xi)$ still has a responsibility for smoothing properties.

First let us consider the case when dispersiveness (L) is true only for large ξ:

$$\begin{aligned} (\mathbf{L}') \quad & |\nabla a(\xi)| \geqslant C\langle\xi\rangle^{m-1} \quad (|\xi| \gg 1), \\ & |\partial^\alpha(a(\xi) - a_m(\xi))| \leqslant C\langle\xi\rangle^{m-1-|\alpha|} \quad (|\xi| \gg 1). \end{aligned}$$

Theorem 10. *Suppose $n \geqslant 1$, $m \geqslant 1$, and $s > 1/2$. Let $a \in C^\infty(\mathbb{R}^n)$ be real-valued and assume that it has finitely many critical points. Assume* (L′) *and*

$$\nabla a(\xi) = 0 \;\Rightarrow\; \det \nabla^2 a(\xi) \neq 0.$$

Then we have

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \leqslant C \|\varphi\|_{L^2(\mathbf{R}^n_x)}.$$

Example 4. $a(\xi) = \xi_1^4 + \cdots + \xi_n^4 + |\xi|^2$ satisfies assumptions in Theorem 10.

We outline the proof of Theorem 10. For the region where $\nabla a(\xi) \neq 0$, we can use a smoothing estimate for dispersive equations. Near the points ξ where $\nabla a(\xi) = 0$, there exists a change of variable ψ by Morse's lemma such that $a(\xi) = (\sigma \circ \psi)(\xi)$ where $\sigma(\eta)$ is a non-degenerate quadratic form, and satisfies dispersiveness (H). Hence the estimate can be reduced to the dispersive case by the method of canonical transformation.

Next we consider the case when $a(\xi)$ is homogeneous (of oder m). Then, by Euler's identity, we have

$$\nabla a(\xi) = \frac{1}{m-1} \xi \nabla^2 a(\xi) \quad (\xi \neq 0),$$

hence

$$\nabla a(\xi) = 0 \Rightarrow \det \nabla^2 a(\xi) = 0 \quad (\xi \neq 0).$$

Therefore assumption in Theorem 10 does not make any sense in this case, but we can have the following result if we use the idea of canonical transform again:

Theorem 11. *Suppose $n \geqslant 2$ and $s > 1/2$. Let $a \in C^\infty(\mathbb{R}^n \setminus 0)$ be real-valued and satisfy $a(\lambda\xi) = \lambda^2 a(\xi)$ $(\lambda > 0,\ \xi \neq 0)$. Assume that*

$$\nabla a(\xi) = 0 \Rightarrow \operatorname{rank} \nabla^2 a(\xi) = n - 1 \quad (\xi \neq 0).$$

Then we have

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \leqslant C \|\varphi\|_{L^2(\mathbf{R}^n_x)}.$$

Example 5. $a(\xi) = \dfrac{\xi_1^2 \xi_2^2}{\xi_1^2 + \xi_2^2} + \xi_3^2 + \cdots + \xi_n^2$ satisfies the assumptions in Theorem 2. In the case $n = 2$, this is an illustration of a smoothing estimate for the Cauchy problem for an equation like

$$i\partial_t u + D_1^2 D_2^2 \Delta^{-1} u = 0.$$

(A mixture of Davey-Stewartson and Benjamin-Ono type equations).

11 Summary

Finally we summarize what is explained in this article in a diagram below. Note that all the results of smoothing estimates here is derived from just the translation invariance of Lebesgue measure:

- Trivial estimate $\|\varphi(x+t)\|_{L^2(\mathbf{R}_t)} = \|\varphi\|_{L^2(\mathbf{R}_x)}$

$\Downarrow$ (comparison principle)

- Low dimensional model estimates (Proposition 1)

$\Downarrow$ (canonical transform)

- Smoothing estimates for dispersive equations (Theorem 1)

$\Downarrow$ (secondary comparison & canonical transform)

- Invariant estimates for non-dispersive equations at least for
  - $*$ radially symmetric $a(\xi) = f(|\xi|)$, $f \in C^\omega(\mathbf{R}_+)$,
  - $*$ Shrira equation $a(\xi) = \xi_1^3 + 3\xi_2^2,\ \xi_1^2 + \xi_1\xi_2^2$,
  - $*$ non-dispersive $a(\xi)$ controlled by its Hessian.

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I. Complex and Hypercomplex Analysis

I.1. Complex Variables and Potential Theory

(Sessions organizers: T. Aliyev, M. Lanza de Cristoforis, S. Plaksa, $\boxed{\text{P. Tamrazov}}$)

ANALYTIC FUNCTIONS OF VECTOR ARGUMENT AND PARTIALLY-CONFORMAL MAPPINGS IN MULTIDIMENSIONAL COMPLEX SPACES

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Key words: module and argument of multidimensional complex space, analogue of Riemann mapping theorem, holomorphic functions

AMS Mathematics Subject Classification: 517.55

Abstract. In the paper we propose an approach to the construction of analog ordinary complex analysis in the case of arbitrary multidimensional complex spaces.

1. Space $\mathbb{C}^n$. Let $\mathbb{N}, \mathbb{R}, \mathbb{C}$ be sets of natural, real and complex numbers, respectively. Let $\overline{\mathbb{C}}$ be a Riemann sphere (extended complex plane).

It is well known [1–3] that a complex space $\mathbb{C}^n$ is a linear vector space over the complex numbers with the Hermitian product

$$(\mathbb{Z} \cdot \mathbb{W}) = \sum_{k=1}^{n} z_k \overline{w}_k, \tag{1}$$

where $\mathbb{Z} = \{z_k\}_{k=1}^n \in \mathbb{C}^n$, $\mathbb{W} = \{w_k\}_{k=1}^n \in \mathbb{C}^n$.

2. Algebra $\mathbb{C}^n$. Binary operation acting from $\mathbb{C}^n \times \mathbb{C}^n$ in $\mathbb{C}^n$ by the rule

$$\mathbb{Z} \cdot \mathbb{W} = \{z_k w_k\}_{k=1}^n, \tag{2}$$

where $\mathbb{Z} = \{z_k\}_{k=1}^n \in \mathbb{C}^n$, $\mathbb{W} = \{w_k\}_{k=1}^n \in \mathbb{C}^n$, is called *vector multiplication of elements* $\mathbb{C}^n$.

This operation is converted $\mathbb{C}^n$ into a commutative, associative algebra [7, 8] with unit $\mathbf{1} = \underbrace{(1, 1, ..., 1)}_{n-\text{раз}} \in \mathbb{C}^n$.

Reversible with respect to defined operation of multiplication are precisely elements $\mathbb{Z} = \{z_k\}_{k=1}^n \in \mathbb{C}^n$ in which $z_k \neq 0$ for all $k = \overline{1, n}$.

Inverse to such elements $\mathbb{Z} \in \mathbb{C}^n$ are the elements $\mathbb{Z}^{-1} = \{z_k^{-1}\}_{k=1}^n \in \mathbb{C}^n$, since $\mathbb{Z} \cdot \mathbb{Z}^{-1} = \mathbb{Z}^{-1} \cdot \mathbb{Z} = 1$.

The set Θ of all elements $a = \{a_k\}_{k=1}^n \in \mathbb{C}^n$, in which at least one coordinate $a_k = 0$, is called the set of non-invertible elements $a \in \mathbb{C}^n$. The set Θ is an ideal in $\mathbb{C}^n$. When $n = 1$ equation (2) gives usual multiplication of complex numbers.

It is well known (see [7, p. 138; 8, p. 345]) that multiplication (2) can imagine $\mathbb{C}^n$ as a direct sum of n copies of the algebra complex numbers $\mathbb{C}$. The structure of the vector space $\mathbb{C}^n$ is fully consistent with the structure of the algebra $\mathbb{C}^n$.

We'll give several definitions of transforming algebra $\mathbb{C}^n$ into an algebra with properties similar to the ordinary algebra of complex numbers.

3. Conjugation. In algebra of complex numbers $\mathbb{C}$ a concept of complex conjugate number is an important concept. We will present the same object in the algebra $\mathbb{C}^n$.

Each element $\mathbb{W} = \{w_k\}_{k=1}^n \in \mathbb{C}^n$ we associate with vector-conjugate element $\overline{\mathbb{W}} = \{\overline{w}_k\}_{k=1}^n \in \mathbb{C}^n$, where $\overline{w}_k$ denote complex conjugate number w_k in usual sense.

Obtained correspondence gives an automorphism $\mathbb{C}^n$ which leaves fixed space $\mathbb{R}^n \subset \mathbb{C}^n$. When $n = 1$ vector-conjugate number coincides with the complex conjugate one.

4. Module (vector). In algebra $\mathbb{C}$ one of the most important concept is the concept of a module of complex number. The following definition gives an analogue of this concept in $\mathbb{C}^n$. Let $\mathbb{R}^n_+ = R_+ \times R_+ \times \ldots \times R_+$, $R_+ = [0, +\infty)$ (see [2, p. 16]).

Vector $|\mathbb{Z}| := \{|z_k|\}_{k=1}^n \in \mathbb{R}^n_+$ is called a vector modulus of any element $\mathbb{Z} = \{z_k\}_{k=1}^n \in \mathbb{C}^n$.

The operation of passing to the vector module defines the mapping $\mathbb{C}^n$ in $\mathbb{R}^n_+$. This mapping is used in complex analysis, in particular, to obtain Reinhart domains in $\mathbb{C}^n$ (see [2]).

It is important that for any $\mathbb{Z} = \{z_k\}_{k=1}^n \in \mathbb{C}^n$, we have equality

$$\mathbb{Z} \cdot \overline{\mathbb{Z}} = |\overline{\mathbb{Z}}|^2 = |\mathbb{Z}|^2.$$

5. Vector norm. Vector $\mathbb{X} = \{x_k\}_{k=1}^n \in \mathbb{R}^n$ is called nonnegative (strictly positive) and we will write $\mathbb{X} \geqslant \mathbb{O}$ $(\mathbb{X} > \mathbb{O})$, if $x_k \geqslant 0$ for all $k = \overline{1, n}$ ($x_k > 0$ at least one $k = \overline{1, n}$), $\mathbb{O} = \underbrace{(0, 0, \ldots 0)}_{n-\text{reps}}$.

We say that the vector $\mathbb{X} = \{x_k\}_{k=1}^n \in \mathbb{R}^n$ is more or equal (strictly more) a vector $\mathbb{Y} = \{y_k\}_{k=1}^n \in \mathbb{R}^n$, if $\mathbb{X} - \mathbb{Y} \geqslant \mathbb{O}$ $(\mathbb{X} - \mathbb{Y} > \mathbb{O})$.

These definitions for $n = 1$ coincide with those defined on the real line.

If $n > 1$ then situation is quite different from the case $n = 1$, for example vector $\mathbb{O} = \underbrace{(0, 0, \ldots 0)}_{n-\text{reps}}$ is more than or equal all vectors whose coordinates are all non-positive and is less than or equal all vectors from $\mathbb{R}^n_+$.

Others vectors $\mathbb{R}^n$ in which coordinates have different signs with the vector $\mathbb{O}$ is not comparable in the sense of these definitions.

Vector space $\mathbb{Y}$ is called vector normed space if each $y \in \mathbb{Y}$ is associated with a nonnegative vector $\|y\| \in \mathbb{R}_{+}^{n}$, $n \in \mathbb{N}$, satisfying the conditions:

1) $\|y\| \geqslant \mathbb{O}$, moreover $\|y\| = \mathbb{O} \Longleftrightarrow y = 0_{\mathbb{Y}}$, ($0_{\mathbb{Y}}$ is a zero of space $\mathbb{Y}$);

2) $\|\gamma y\| = |\gamma|\|y\|$, $\forall y \in \mathbb{Y}$, $\forall \gamma \in \mathbb{C}$;

3) $\|y_1 + y_2\| \leqslant \|y_1\| + \|y_2\|$, $\forall y_1, y_2 \in \mathbb{Y}$.

Similarly, we can introduce the concept of vector metric. Introduced concept of vector module of element $\mathbb{Z} \in \mathbb{C}^n$ satisfies the last definition.

Thus, vector module is vector norm in the algebra $\mathbb{C}^n : \|\cdot\| = |\cdot|$. Then the open unit ball in the algebra $\mathbb{C}^n$ is the open unit polydisk $\|z\| < 1$, $(\mathbf{1} = \underbrace{(1, 1, \ldots 1)}_{n-\text{reps}})$, and the unit sphere is n – dimensional torus $\mathbb{T}^n = \{\mathbb{Z} \in \mathbb{C}^n : \|\mathbb{Z}\| = 1\}$.

It is very important that

a) $|\mathbb{Z}_1 \cdot \mathbb{Z}_2| = \|\mathbb{Z}_1 \cdot \mathbb{Z}_2\| = \|\mathbb{Z}_1\|\|\mathbb{Z}_2\| = |\mathbb{Z}_1||\mathbb{Z}_2|$, $\forall \mathbb{Z}_1, \mathbb{Z}_2 \in \mathbb{C}^n$;

b) $|\mathbf{1}| = \|\mathbf{1}\| = \mathbf{1}$, $(\mathbf{1} = (1, 1, \ldots, 1))$.

6. Vector argument $a \in \mathbb{C}^n$. Vector argument of n-dimensional complex numbers $\mathbb{A} = \{a_k\}_{k=1}^{n} \in \mathbb{C}^n \backslash \Theta$ is n – dimensional real vector, defined by

$$\text{Arg } \mathbb{A} = \{\text{Arg } a_k\}_{k=1}^{n}.$$

7. Representation n – dimensional complex number in vector – cartesian form. Let $\mathbb{Z} = \{z_k\}_{k=1}^{n} \in \mathbb{C}^n$. Then

$$\mathbb{Z} = \{z_k\}_{k=1}^{n} = \{\operatorname{Re} z_k + i\operatorname{Im} z_k\}_{k=1}^{n} = \{\operatorname{Re} z_k\}_{k=1}^{n} + \{i\operatorname{Im} z_k\}_{k=1}^{n} =$$

$$= \{\operatorname{Re} z_k\}_{k=1}^{n} + i\{\operatorname{Im} z_k\}_{k=1}^{n} = \operatorname{Re}\mathbb{Z} + i\operatorname{Im}\mathbb{Z} = X + iY =$$

$$= \{x_k\}_{k=1}^{n} + i\{y_k\}_{k=1}^{n} \in \mathbb{R}^n + i\mathbb{R}^n,$$

where $X = \operatorname{Re}\mathbb{Z} = \{\operatorname{Re} z_k\}_{k=1}^{n} = \{x_k\}_{k=1}^{n}$, $Y = \operatorname{Im}\mathbb{Z} = \{\operatorname{Im} z_k\}_{k=1}^{n} = \{y_k\}_{k=1}^{n}$. That is $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$.

8. Presentation n–dimensional complex numbers in vector–polar form. Using the above definitions, we obtain the following chain of equalities:

$$\mathbb{Z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} |z_1|e^{i\alpha_1} \\ |z_2|e^{i\alpha_2} \\ \vdots \\ |z_n|e^{i\alpha_n} \end{pmatrix} = \begin{pmatrix} |z_1| \\ |z_2| \\ \vdots \\ |z_n| \end{pmatrix} \begin{pmatrix} e^{i\alpha_1} \\ e^{i\alpha_2} \\ \vdots \\ e^{i\alpha_n} \end{pmatrix} =$$

$$= |\mathbb{Z}| \left[\begin{pmatrix} \cos\alpha_1 \\ \cos\alpha_2 \\ \vdots \\ \cos\alpha_n \end{pmatrix} + i \begin{pmatrix} \sin\alpha_1 \\ \sin\alpha_2 \\ \vdots \\ \sin\alpha_n \end{pmatrix} \right] = |\mathbb{Z}| \left[\cos \mathrm{Arg}\ \mathbb{Z} + i \sin \mathrm{Arg}\ \mathbb{Z}\right] =$$

$$= |\mathbb{Z}| e^{i\mathrm{Arg}\ \mathbb{Z}} = |\mathbb{Z}| \exp i\mathrm{Arg}\ \mathbb{Z},$$

where

$$\cos\beta = \begin{pmatrix} \cos\beta_1 \\ \cos\beta_2 \\ \vdots \\ \cos\beta_n \end{pmatrix}, \quad \sin\beta = \begin{pmatrix} \sin\beta_1 \\ \sin\beta_2 \\ \vdots \\ \sin\beta_n \end{pmatrix},$$

$$\exp i\beta = \begin{pmatrix} \exp i\beta_1 \\ \exp i\beta_2 \\ \vdots \\ \exp i\beta_n \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} \in \mathbb{R}^n, \quad \mathbb{Z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{C}^n.$$

Similarly, $\ln \mathbb{Z}, \quad \mathbb{Z} = \{z_k\}_{k=1}^n \in \mathbb{C}^n \setminus \Theta$

$$\ln \mathbb{Z} = \ln |\mathbb{Z}| + i\mathrm{Arg}\ \mathbb{Z} = \begin{pmatrix} \ln|z_1| + i\mathrm{Arg}\ z_1 \\ \ln|z_2| + i\mathrm{Arg}\ z_2 \\ \dots\dots\dots\dots\dots\dots \\ \ln|z_n| + i\mathrm{Arg}\ z_n \end{pmatrix}.$$

Moreover, for any complex function $F(z)$ that is regular in domains $(B_1, B_2, \dots, B_n)$, $B_k \in \mathbb{C}$, $k = \overline{1, n}$ we define continuation of this function to the holomorphic mapping of domain $\mathbb{B} = B_1 \times B_2 \times \ldots \times B_n$ by the following rule

$$\mathbb{F}(\mathbb{W}) = \begin{pmatrix} F(W_1) \\ F(W_2) \\ \dots \\ F(W_n) \end{pmatrix}, \quad \mathbb{W} \in \mathbb{B}.$$

9. Compactification $\mathbb{C}^n$. By definition $\mathbb{C}^n = \underbrace{(\mathbb{C} \times \mathbb{C} \times \ldots \times \mathbb{C})}_{n-\text{reps}}$. Consider the compactification of the space $\mathbb{C}^n$, further so-called space of function theory (see, for example [1–3]) $\overline{\mathbb{C}}^n = \underbrace{(\overline{\mathbb{C}} \times \overline{\mathbb{C}} \times \ldots \times \overline{\mathbb{C}})}_{n-\text{reps}}$. It is clear that $\mathbb{C}^1 = \mathbb{C}$, $\overline{\mathbb{C}}^1 = \overline{\mathbb{C}}$.

Infinite points $\overline{\mathbb{C}}^n$ are points which have at least one infinite coordinate. The set of all infinite points has complex dimension $n-1$.

Topology in $\overline{\mathbb{C}}^n$ is introduced as in Cartesian product of topological spaces. In this topology $\overline{\mathbb{C}}^n$ is compact (see [1–3]).

10. Polycylindrical Riemann mapping theorem in $\overline{\mathbb{C}}^n$. Domain $B \subset \overline{\mathbb{C}}$ is called the domain of hyperbolic type, if ∂B is a connected set containing more than one point.

Domain $\mathbb{B} = B_1 \times B_2 \times \ldots \times B_n \subset \overline{\mathbb{C}}^n$, where each domain $B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$ is a domain of hyperbolic type, is called polycylindrical domain of hyperbolic type.

Directly from the classical Riemann theorem about mapping simply connected domain of hyperbolic type onto the unit circle (see [6]) implies the following result.

Riemann theorem (polycylindrical). Any polycylindrical domain $\mathbb{B} \subset \overline{\mathbb{C}}^n$ of hyperbolic type is biholomorphic equivalent to the unit polydisc $\mathbb{U}^n = \{\mathbb{W} \in \mathbb{C}^n : \|\mathbb{W}\| < 1\}$. This equivalence is realized by family of biholomorphic maps, which depends on $3 \cdot n$ real parameters.

Let $\mathbb{B} = B_1 \times B_2 \times \ldots \times B_n$ be a domain indicated in the Riemann theorem $\mathbb{A} = \{a_k\}_{k=1}^n \in \mathbb{B}$, $a_k \in B_k$, $k = \overline{1,n}$ and $w_k = f_k(z_k)$ be holomorphic function in B_k, which univalent and conformally maps the domain B_k, $k = \overline{1,n}$ onto the unit circle $|w_k| < 1$ such that $f(a_k) = 0$, $f'(a_k) > 0$. Then a biholomorphic mapping

$$\mathbb{F}_{\mathbb{B}}(\mathbb{Z}) = \begin{pmatrix} f_1(z_1) \\ f_2(z_2) \\ \ldots \\ f_n(z_n) \end{pmatrix}, \quad \mathbb{F}'_{\mathbb{B}}(\mathbb{Z}) = \begin{pmatrix} f'_1 \\ f'_2 \\ \ldots \\ f'_n \end{pmatrix},$$

satisfies conditions of normalization

$$\mathbb{F}_{\mathbb{B}}(\mathbb{A}) = \mathbb{O}, \quad \mathbb{F}'_{\mathbb{B}}(\mathbb{A}) = \begin{pmatrix} f'_1(a_1) \\ f'_2(a_2) \\ \ldots \\ f'_n(a_n) \end{pmatrix} > \mathbb{O}$$

and it will be unique mapping onto the unit circle. So, in algebra $\mathbb{C}^n$ norm is defined by equality $\|\mathbb{Z}\| := |\mathbb{Z}|$. Metric (vector) in $\mathbb{C}^n$ is given by usual way: $\rho(\mathbb{Z}_1, \mathbb{Z}_2) = \|\mathbb{Z}_1 - \mathbb{Z}_2\|$.

This (vector) norm and metric is polycylindrical.

Convergence with respect to polycylindrical norm is given by the relation $\mathbb{Z}_p \underset{p\to\infty}{\longrightarrow} 0 \iff \|\mathbb{Z}_p\| \underset{p\to\infty}{\longrightarrow} \mathbb{O} = \underbrace{(0, 0, \ldots 0)}_{n-\text{reps}} \iff |z_p^{(k)}| \underset{p\to\infty}{\longrightarrow} 0 \quad \forall k = \overline{1, n}$.

11. Differentiability. Consider domain $\mathbb{D} \subset \mathbb{C}^n$ and mapping $\mathbb{F} : \mathbb{D} \longrightarrow \mathbb{C}^m$, $\mathbb{F} = \{f_k(z_1, \ldots, z_n)\}_{k=1}^m$. Let $f_k = U^{(k)}(x_1, \ldots, x_n, y_1, \ldots, y_n) + iV^{(k)}(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a real functions which are differentiable everywhere in domain $\mathbb{D}$ if $k = \overline{1, m}$, $n, m \in \mathbb{N}$.

Consider Jacobi matrix of mapping $\mathbb{F}$, as a differentiable mapping of domain $\mathbb{D} \subset \mathbb{R}^{2n}$ in $\mathbb{R}^{2m}$ (matrix $2m \times 2n$)

$$\left(\begin{array}{ccc|ccc}
U_{x_1}^{(1)} & \ldots & U_{x_n}^{(1)} & U_{y_1}^{(1)} & \ldots & U_{y_n}^{(1)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \{\mathbb{U}_{\mathbb{X}}\} & \vdots & \vdots & \{\mathbb{U}_{\mathbb{Y}}\} & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
U_{x_1}^{(m)} & \ldots & U_{x_n}^{(m)} & U_{y_1}^{(m)} & \ldots & U_{y_n}^{(m)} \\
--- & --- & --- & --- & --- & --- \\
V_{x_1}^{(1)} & \ldots & V_{x_n}^{(1)} & V_{y_1}^{(1)} & \ldots & V_{y_n}^{(1)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \{\mathbb{V}_{\mathbb{X}}\} & \vdots & \vdots & \{\mathbb{V}_{\mathbb{Y}}\} & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
V_{x_1}^{(m)} & \ldots & V_{x_n}^{(m)} & V_{y_1}^{(m)} & \ldots & V_{y_n}^{(m)}
\end{array}\right), \tag{3}$$

where $U_{x_j}^{(k)} = \frac{\partial}{\partial x_j} U_k$, $V_{x_j}^{(k)} = \frac{\partial}{\partial x_j} V_k$, $k = \overline{1, m}$, $j = \overline{1, n}$.

Hatched lines divide Jacobian matrix (3) into four rectangular matrix of order $m \times n$, denoted by $\mathbb{U}_{\mathbb{X}}$, $\mathbb{U}_{\mathbb{Y}}$, $\mathbb{V}_{\mathbb{X}}$, $\mathbb{V}_{\mathbb{Y}}$, where $\mathbb{F} = \operatorname{Re}\mathbb{F} + i\operatorname{Im}\mathbb{F} = \mathbb{U} + i\mathbb{V}$, $\mathbb{Z} = \operatorname{Re}\mathbb{Z} + i\operatorname{Im}\mathbb{Z} = \mathbb{X} + i\mathbb{Y}$.

Then the matrix (3) can be represented as follows

$$\begin{pmatrix} \mathbb{U}_{\mathbb{X}} & \mathbb{U}_{\mathbb{Y}} \\ \mathbb{V}_{\mathbb{X}} & \mathbb{V}_{\mathbb{Y}} \end{pmatrix}.$$

Then the Cauchy-Riemann conditions for the mapping $\mathbb{F}$ can be written as

$$\begin{cases} \mathbb{U}_{\mathbb{X}} = \mathbb{V}_{\mathbb{Y}}, \\ \mathbb{U}_{\mathbb{Y}} = -\mathbb{V}_{\mathbb{X}}. \end{cases} \tag{4}$$

Taking into account (4), well-known definition of holomorphic mapping (see [1–5]) can be represented as follows.

Real and differentiable map $\mathbb{F} : \mathbb{D} \longrightarrow \mathbb{C}^m$ in $\mathbb{D}$ (as a map from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2m}$) which satisfies the matrix equation (4) everywhere in $\mathbb{D}$ is called holomorphic in $\mathbb{D}$.

For $n \in \mathbb{N}$ and $m = 1$ we obtain definition of holomorphic curve in domain $\mathbb{D} \subset \mathbb{C}^n$. In the case $n = 1$, $m \in \mathbb{N}$, we obtain definition a holomorphic curve.

It is known [1–3] that a holomorphic map $\mathbb{F} : \mathbb{D} \longrightarrow \mathbb{C}^m$, $\mathbb{D} \subset \mathbb{C}^n$ is called biholomorphic if it has an inverse mapping which is holomorphic in domain $\mathbb{F}(\mathbb{D})$.

12. Application. In connection with polycylindrical Riemann mapping theorem, we consider polycylindrical analog of class S from the theory of univalent functions (see, for example [6]).

Class $\mathbb{S}^{(n)}$ is the set of all biholomorphic maps of the unit polydisc $\mathbb{U}^n = \{\mathbb{Z} \in \mathbb{C}^n : \|\mathbb{Z}\| < 1\}$ of the form $\mathbb{F}(\mathbb{Z}) = \begin{pmatrix} f_1(z_1) \\ f_2(z_2) \\ \dots \\ f_n(z_n) \end{pmatrix}$, where $\mathbb{Z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{U}^n$, $f_k \in S$, $k = \overline{1, n}$.

It is clear that for $\mathbb{Z} \in \overline{\mathbb{U}}^n(r) := \{\|\mathbb{Z}\| \leqslant r < 1\}$, $r = \{r_k\}_{k=1}^n$, $0 < r_k < 1$, $k = \overline{1, n}$ series convergent absolutely and uniformly

$$\mathbb{F}(\mathbb{Z}) = \sum_{k=1}^{\infty} \mathbb{A}_k \mathbb{Z}^k = \sum \begin{pmatrix} a_k^{(1)} \\ a_k^{(2)} \\ \vdots \\ a_k^{(n)} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}^k = \begin{pmatrix} \sum a_k^{(1)} z_1^k \\ \sum a_k^{(2)} z_2^k \\ \vdots \\ \sum a_k^{(n)} z_n^k \end{pmatrix} = \begin{pmatrix} f_1(z_1) \\ f_2(z_2) \\ \dots \\ f_n(z_n) \end{pmatrix}.$$

Theorem 1. For any mapping $\mathbb{F} \in \mathbb{S}^{(n)}$ we have inequality

$$\frac{\|\mathbb{Z}\|}{(1 + \|\mathbb{Z}\|)^2} \leqslant \|\mathbb{F}(\mathbb{Z})\| \leqslant \frac{\|\mathbb{Z}\|}{(1 - \|\mathbb{Z}\|)^2},$$

where $\|\mathbb{Z}\| = r = \{|z_k|\}_{k=1}^n = \{|r_k|\} \in \mathbb{R}_+^n$, $\quad 0 \leqslant r_k < 1$, $\quad k = \overline{1, n}$.

Theorem 2. For any mapping $\mathbb{F} \in \mathbb{S}^{(n)}$ we have inequality

$$\frac{\|1-\mathbb{Z}\|}{(1+\|\mathbb{Z}\|)^3} \leqslant \|\mathbb{F}'(\mathbb{Z})\| \leqslant \frac{\|1+\mathbb{Z}\|}{(1-\|\mathbb{Z}\|)^3}\,,$$

where $\|\mathbb{Z}\| = r = \{|z_k|\}_{k=1}^{n} = \{|r_k|\} \in \mathbb{R}_+^n, \quad 0 \leqslant r_k < 1, \quad k = \overline{1,n}.$

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MEROMORPHIC PRYM DIFFERENTIALS ON A COMPACT RIEMANN SURFACE

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Key words: Prym differential, divisors, abelians differentials, variable Riemann surfaces, variable characters, meromorphic differentials

AMS Mathematics Subject Classification: 30F30

Abstract. The theory of multiplicative functions and Prym differentials for the case of special characters on a compact Riemann surface has found numerous applications in geometrical theory of function of complex variable, analytic number theory and equations of mathematical physics. In [5] we begun the construction of a general theory of multiplicative functions and Prym differentials on a compact Riemann surface for arbitrary characters. In this article some new properties of spaces of meromorphic Prym differentials on variable compact Riemann surfaces and for variable characters are obtained.

1 Introduction

Let F be a fixed compact oriented Riemann surface of genus $g \geqslant 2$ with framing $\{a_k, b_k\}_{k=1}^{g}$ in first fundamental group $\pi_1(F)$. Let F_0 be Riemann surface with a fixed complex analytic structure on F.

By the uniformization theorem there is a finitely generated Fuchsian group Γ of the first kind acting on the unit disk $U = \{z \in \mathbf{C} : |z| < 1\}$ such that U/Γ is conformally equivalent to F_0. The group Γ has the representation

$$\Gamma = \left\langle A_1, A_2, \ldots, A_g, B_1, \ldots, B_g : \prod_{j=1}^{g} A_j B_j A_j^{-1} B_j^{-1} = 1 \right\rangle.$$

Each complex analytic structure on F can be defined by Beltrami differential μ on F_0, that is by an expression $\mu(z)\frac{d\bar{z}}{dz}$ which does not depend on the choice of local parameters on F_0, where $\mu(z)$ is a complex function on F_0 and $\|\mu\|_{L_\infty}(F_0) = \operatorname{esssup}|\mu(z)| < 1$. Such structure on F will be denoted by F_μ.

Let $M(F)$ be the set of all complex analytic structures on F with C^∞-topology on F_0, and let $\mathrm{Diff}_0(F)$ be the group of orientation preserving diffeomorphisms for surface F, which consist of all diffeomorphisms homotopic to identity diffeomorphism on F_0. The action of the group $\mathrm{Diff}_0(F)$ on $M(F)$ is defined by the rule $\mu \to f^*\mu$. Then Teichmueller space $\mathbf{T}_g(F)$ is a quotient space $M(F)/\mathrm{Diff}_0(F)$.

Since the mapping $U \to F_0 = U/\Gamma$ is a local diffeomorphism, every Beltrami differential μ on F_0 lifts to a $\Gamma-$differential Beltrami μ on U, that is $\mu \in L_\infty(U), |\mu\|_\infty = \mathrm{esssup}|\mu(z)| < 1$ and $\mu(Tz)\overline{T'}(z)/T'(z) = \mu(z)$, $z \in U$, $T \in \Gamma$.

For the extension of the Γ-differential μ from U to $\overline{\mathbf{C}}$, equal to 0, on $\overline{\mathbf{C}}\backslash U$, there is an unique quasiconformal homeomorphism $w^\mu : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ with fixed points $+1$, -1, i satisfying the Beltrami equation $w_{\overline{z}} = \mu(z)w_z$. The mapping $T \to T^\mu = w^\mu T(w^\mu)^{-1}$ defines an isomorphism of the group Γ to the quasifuchsian group $\Gamma_\mu = w^\mu\Gamma(w^\mu)^{-1}$.

The classical results of L. Ahlfors, L. Bers [6] and other authors state, that: 1) $\mathbf{T}_g(F)$ is a complex analytic manifold of dimension $3g-3$ for $g \geqslant 2$; 2) $\mathbf{T}_g(F)$ has unique complex analytic structure such that the natural mapping $\Phi : M(F) \to M(F)/Diff_0(F) = \mathbf{T}_g(F)$ is holomorphic and local sections of Φ are holomorphic; 3) elements of Γ_μ are holomorphic with respect to $[\mu]$. It is natural, that the choice of generators $\{a_k, b_k\}_{k=1}^g$ for $\pi_1(F)$ is equivalent to the choice of the systems of generators $\{a_k(\mu), b_k(\mu)\}_{k=1}^g$ for $\pi_1(F_\mu)$ and $\{A_j^\mu, B_j^\mu\}_{j=1}^g$ for Γ_μ for every $[\mu]$ in $\mathbf{T}_g$.

Universal Jacobi variety of genus g is a bundle space over $\mathbf{T}_g$, whose fibre over $[\mu] \in \mathbf{T}_g$ is a Jacobian $J(F_\mu)$ for F_μ [8].

In the article [7] L. Bers has constructed the holomorphic forms $\zeta_1[\mu] = \zeta_1([\mu], \xi)d\xi, \ldots, \zeta_g[\mu] = \zeta_g([\mu], \xi)d\xi$. These forms are liftings to $w^\mu(U)$ of holomorphic abelian differentials $\zeta_1[\mu], \ldots, \zeta_g[\mu]$ on F_μ, which give canonical basis on F_μ, dual to canonical homotopic basis $\{a_k(\mu), b_k(\mu)\}_{k=1}^g$ on F_μ. This basis holomorphically depends on module $[\mu]$ of framed compact Riemann surface F_μ. Moreover, the matrix of $b-$periods $\Omega(\mu) = (\pi_{jk}[\mu])_{j,k=1}^g$ on F_μ consists of complex numbers $\pi_{jk}[\mu] = \int\limits_{\xi}^{B_k^\mu(\xi)} \zeta_j([\mu], w)dw$, $\xi \in w^\mu(U)$ and holomorphically depends on $[\mu]$.

For any fixed $[\mu] \in \mathbf{T}_g$ and $\xi_0 \in w^\mu(U)$ we define classical Jacobi mapping $\varphi : w^\mu(U) \to \mathbf{C}^g$ by the rule: $\varphi_j(\xi) = \int\limits_{\xi_0}^{\xi} \zeta_j([\mu], w)dw, j = 1, \ldots, g$. Then φ induces a fiberwise holomorphic embedding from F_μ to $J(F_\mu)$.

Next, for every natural number $n > 1$ there exists a bundle space over $\mathbf{T}_g$, whose fibre over $[\mu] \in \mathbf{T}_g$ is a space of all effective divisors of degree n on compact Riemann surface F_μ. Holomorphic sections of this bundle define on every F_μ an effective divisor D^μ of degree n, which holomorphically depends on $[\mu]$. Also, there exists a holomorphic mapping φ_n from this bundle to the universal Jacobi bundle, $n \geqslant 1$, whose restriction to fibres is the extension of classical Jacobi mapping $\varphi : F_\mu \to J(F_\mu)$. It is well known that for $n = g$ the mapping $\varphi : F_g[\mu]\backslash F_g^1[\mu] \to W_g[\mu]\backslash W_g^1[\mu]$

is an analytic isomorphism, where $F_g[\mu]$ is a $g-$multiple symmetric product of the surface F_μ, and that the complex dimension of $W_g^1[\mu] = \varphi(F_g^1[\mu])$ is at most $g-2$ [3]. The local holomorphic sections of these bundles over the neighbourhood $U([\mu_0]) \subset \mathbf{T}_g$ may be obtained (for every $n \geqslant 1$) from local holomorphic C. Earl's sections for $\Phi : M(F) \to \mathbf{T}_g$ over $U([\mu_0])$ [8].

A *character* ρ for F_μ is an arbitrary homomorphism $\rho : (\pi_1(F_\mu), \cdot) \to (\mathbf{C}^*, \cdot), \mathbf{C}^* = \mathbf{C} \setminus \{0\}$. The character is uniquely defined by an ordered collection $(\rho(a_1(\mu)), \rho(b_1(\mu)), \ldots, \rho(a_g(\mu)), \rho(b_g(\mu))) \in (\mathbf{C}^*)^{2g}$.

Definition 1. *A meromorphic function f on $w^\mu(U)$ such that $f(Tz) = \rho(T)f(z)$, $z \in w^\mu(U)$, $T \in \Gamma_\mu$ is called a multiplicative function f on F_μ for character ρ*

Definition 2. *A differential $\phi = \phi(z)dz^m$, such that $\phi(Tz)(T'z)^m = \rho(T)\phi(z), z \in U, T \in \Gamma$, $\rho : \Gamma \to \mathbf{C}^*$, is called a Prym $m-$differential with respect to the Fuchsian group Γ for ρ, or (ρ, m)-differential.*

A character ρ for multiplicative function f_0 without zeros and poles on F_μ has the form $\rho(a_k(\mu)) = \exp 2\pi i c_k([\mu], \rho)$, $\rho(b_k(\mu)) = \exp(2\pi i \sum\limits_{j=1}^{g} c_j([\mu], \rho)\pi_{jk}([\mu])), k = 1, \ldots, g$. Such characters ρ we call *unessential*, and a function f_0 with such character is called an *unit.* A character, which is not unessential, will be called *essential* on $\pi_1(F_\mu)$. Denote by $Hom(\Gamma, \mathbf{C}^*)$ the of group all characters on Γ with natural product operation. Unessential characters form a subgroup L_g of the group $Hom(\Gamma, \mathbf{C}^*)$.

Lemma 1 (see [5]). *Each holomorphic principal $Hom(\Gamma, \mathbf{C}^*)-$bundle E is biholomorphically isomorphic to the trivial bundle $\mathbf{T}_g(F) \times Hom(\Gamma, \mathbf{C}^*)$ over $\mathbf{T}_g(F)$.*

Definition 3. *A Prym differential ϕ of class C^1 on $F = U/\Gamma$ for ρ is called multiplicatively exact, if $\phi = df(z)$ and $f(Tz) = \rho(T)f(z), T \in \Gamma, z \in U$, that is f is a multiplicative function on F of class C^2 for ρ.*

A divisor on F_μ is a formal product $D = P_1^{n_1} \ldots P_k^{n_k}, P_j \in F_\mu, n_j \in \mathbf{Z}$, $j = 1, \ldots, k$. We have the following theorem.

Theorem (the Riemann-Roch theorem for characters [3]). *Let F be a compact Riemann surface of genus $g \geqslant 1$. Then for any divisor D on F and a for any character ρ the equality $r_\rho(D^{-1}) = \deg D - g + 1 + i_{\rho^{-1}}(D)$ is true.*

Theorem (H. Abel theorem for characters [3]). *Let D be a divisor on a marked variable compact Riemann surface $[F_\mu, \{a_1(\mu), \ldots, a_g(\mu), b_1(\mu), \ldots, b_g(\mu)\}]$ of genus $g \geqslant 1$ and let ρ be a character on $\pi_1(F_\mu)$. Then D is a divisor of a multiplicative function f on F_μ for the character $\rho \Leftrightarrow$ $\deg D = 0$ and $\varphi(D) = \frac{1}{2\pi i}\sum\limits_{j=1}^{g} \log \rho(b(\mu)_j) e^{(j)}[\mu] - \frac{1}{2\pi i}\sum\limits_{j=1}^{g} \log \rho(a_j(\mu))\pi^{(j)}[\mu] (= \psi(\rho, [\mu]))$ in $\mathbf{C}^g$ modulo integer lattice $L(F_\mu)$, which is generated by columns*

$e^{(1)}[\mu],\ldots,e^{(g)}[\mu],\pi^{(1)}[\mu],\ldots,\pi^{(g)}[\mu]$ of the matrix of $a(\mu)$, $b(\mu)-$periods on F_μ, where $\varphi[\mu]$ is Jacobi mapping from F_μ in Jacobi variety $J(F_\mu)$.

2 Divisors of Prym differentials on a compact Riemann surface

Theorem 1. *1) For any essential character ρ, a point $Q_1 \in F_\mu$, natural number $q \geqslant 1$, and unessential character ρ, point $Q_1 \in F_\mu$, natural number $q > 1$ there exists an elementary $(\rho, q)-$differential $\tau_{\rho,q;Q_1}$ of third kind with unique simple pole $Q_1 = Q_1[\mu]$ on F_μ, locally holomorphic with respect to ρ and $[\mu]$;*

2) For every unessential character ρ, and every point $Q_1 \in F_\mu$ for $q = 1$ there does not exist an elementary $(\rho, 1)-$differential $\tau_{\rho;Q_1}$ of third kind with unique simple pole Q_1 on F_μ.

Proof. 1) If ρ is an essential character and $q = 1$, then by Riemann-Roch theorem for characters we have the equality $i_\rho(\frac{1}{Q_1}) = -1 + g + 1 + r_{\rho^{-1}}(Q_1)$, $i_\rho(\frac{1}{Q_1}) = g$ and $i_\rho(1) = g - 1$. Hence $i_\rho(\frac{1}{Q_1}) = i_\rho(1) + 1$. Thus there exists a Prym differential $\tau_{\rho;Q_1}$ for ρ on F_μ with unique pole Q_1 of exactly first order.

If ρ is an arbitrary character and $q > 1$, then by Riemann-Roch theorem for (ρ, q)-differentials [5] we have $i_{\rho,q}(D) = (2q-1)(g-1) - \deg D + r((f)\frac{Z^{q-1}}{D})$ and $i_{\rho,q}(1) = (2q-1)(g-1)$, where f — multiplicative function for ρ and Z — canonical class of abelian differentials on F_μ. Hence $i_{\rho,q}(\frac{1}{Q_1}) = i_{\rho,q}(1) + 1 + r((f)Z^{q-1}Q_1)$. Thus we have the equality $i_{\rho,q}(\frac{1}{Q_1}) = i_{\rho,q}(1) + 1$, since $\deg((f)Z^{q-1}Q_1) = 0 + (q-1)(2g-2) + 1 > 0$. Therefore there exists a $(\rho, q)-$differential $\tau_{\rho,q;Q_1}$ for ρ on F_μ with unique pole Q_1 of exactly first order.

Let us construct such differentials which are locally holomorphic with respect to ρ and $[\mu]$:

a) Such (ρ, q)-differential $\tau_{\rho,q;Q_1}$ may be defined by $\tau_{\rho,q;Q_1} = f\omega_0^q$, where f is a multiplicative function for essential character ρ on F_μ, $q \geqslant 1$ and ω_0 — a holomorphic abelian differential on F_μ. Divisor $(\tau_{\rho,q;Q_1}) = \frac{R_1\ldots R_N}{(\omega_0)^q Q_1}(\omega_0)^q$, where $N = q(2g-2)+1$ and point Q_1 does not belong to divisor (ω_0). Hence we have an equality

$$\varphi(R_1\ldots R_g) = -2Kq + \varphi(Q_1) - \varphi(R_{g+1}\ldots R_N) + \psi(\rho) = a \qquad (*)$$

in the Jacobi variety $J(F_\mu)$ for F_μ;

b) In the case $\rho = 1$ or when ρ is an unessential character for $q > 1$ we find such differential in the form $\tau_{\rho,q;Q_1} = f_0 f_1 \omega_0^q$, where f_1 is a single-valued meromorphic function with divisor $(f_1) = \frac{R_1\ldots R_N}{(\omega_0)^q Q_1}$ and f_0 is a multiplicative unit for ρ on F_μ. Here $\psi(\rho) = 0$ and by Abel's theorem we have equality

$$\varphi(R_1\ldots R_g) = -2Kq + \varphi(Q_1) - \varphi(R_{g+1}\ldots R_N) = a. \qquad (**)$$

Under these conditions in both cases a) and b) we have unequalities $N-g \geqslant g-1$ and $\dim W_g^1 \leqslant g-2$. By slightly moving the divisors $R_{g+1} \dots R_N$ we can make sure that a does not belong to W_g^1 and the equations $(*)$ and $(**)$ have unique solutions $R_1 \dots R_g$ in Jacobi variety.

Chosing a section, which is locally holomorphic with respect to $[\mu]$ and ρ divisors $R_{g+1} \dots R_N$ over $\mathbf{T}_g$, we obtain a divisor $R_1 \dots R_g$ (as an unique solution of these equations in $J(F_\mu)$) which also holomorphically depend, on ρ and $[\mu]$. Moreover slightly moving divisors $R_{g+1} \dots R_N$ we may make sure, that the point Q_1 is different from the points $R_1, \dots, R_N$.

2) If there exists a differential $\tau_{\rho;Q_1}$ for unessential character ρ with residue $res_{Q_1}\tau_{\rho;Q_1} = c_{Q_1} \neq 0$ for some its branch, then $f_0^{-1}\tau_{\rho;Q_1}$ is an abelian differential with unique simple pole in Q_1 and f_0 is a multiplicative unit for ρ on F_μ. By residue theorem $f_0^{-1}(\widetilde{Q}_1)c_{Q_1} = 0$, where $\widetilde{Q}_1 \neq Q_1$. Contradiction.

This statement also follows from the Riemann-Roch theorem since $i_\rho(Q_1^{-1}) = g = i_\rho(1)$ for unessential character ρ. In fact, $0 = r_{\rho^{-1}}(Q_1) = \deg(\frac{1}{Q_1}) - g + 1 + i_\rho(\frac{1}{Q_1}) = -1 - g + 1 + i_\rho(\frac{1}{Q_1})$. Theorem 1 is proven.

It is clear, that a $(\rho, m)-$differential ω has an unique divisor $D = (\omega)$ obtained from his zero and poles, with regard to multiplicity, on F of genus $g \geqslant 2$, and $\deg D = (2g-2)m$, $m \geqslant 1$. Let us find, if the given divisor D on F of genus $g \geqslant 2$, $\deg D = (2g-2)m$, would define a $(\rho, m)-$differential ω up to the multiplication by a non-zero constant on F.

It is well known from [5], that any divisor D on F of genus $g \geqslant 2$ and of degree $(2g-2)m$, $g \geqslant 1$, $m \geqslant 1$ is a divisor for unique (up to the multiplication by nonzero constant) $(\rho, m)-$differential, which belong to uniquely normalized character ρ.

Theorem 2. *Let D be a divisor, $\deg D = (2g-2)m$, $m \geqslant 0$, on a compact Riemann surface F genus of $g \geqslant 2$, then:*

1) if there exist two differentials ω_1 and ω_2, $(\omega_1) = (\omega_2) = D$, for the same character ρ, then $\omega_1 = c\omega_2$, where $c = const \neq 0$ on F;

2) if there exist two differentials ω_1 and ω_2, $(\omega_1) = (\omega_2) = D$, for two different characters ρ_1 and ρ_2, $\rho_1 \neq \rho_2$, then $\omega_1 = \omega_2 g$, where g is a multiplicative units for an unessential character ρ_0 on F, where $\rho_1 = \rho_2\rho_0$;

3) if there exist two differentials ω_1 and ω_2, $(\omega_1) = (\omega_2) = D$, for normalized characters ρ_1 and ρ_2, then $\rho_1 = \rho_2$ and $\omega_1 = c\omega_2$, where $c = const \neq 0$ on F.

Proof. 1) Let us consider the ratio $g = \frac{\omega_1}{\omega_2}$. The divisor $(g) = 1$ and the character $\frac{\rho}{\rho} = 1$, therefore $g = c \neq 0$ on F, since g is a single-valued analytic function on F;

2) Also examine $g = \frac{\omega_1}{\omega_2}$. For this function $(g) = 1$, and its character $\rho_0 = \frac{\rho_1}{\rho_2}$ must be unessential. Hence, $\omega_1 = \omega_2 g$, where g is a multiplicative unit for ρ_0;

3) Again consider $g = \frac{\omega_1}{\omega_2}$. For this function $(g) = 1$, therefore its character $\rho_0 = \frac{\rho_1}{\rho_2}$ must be unessential. Then ρ_0 is simultaneously unessential and normalized. By the theorem [2], $\rho_0 \equiv 1$. Therefore $\rho_1 = \rho_2$ and by the statement 1) we have $\omega_1 = c\omega_2,\ c \neq 0$ on F. Theorem 2 is proven.

Denote by $\Omega_\rho(\frac{1}{Q_1 \dots Q_s}; F_\mu)$ the space of meromorphic Prym differential for ρ with poles of the order at most one in the points $Q_1, \dots, Q_s$, $s \geqslant 2$, and $\Omega_{e,\rho}(1; F_\mu)$ be the subspace of multiplicatively exact holomorphic differentials for ρ, where the divisor $Q_1 \dots Q_s$ on F_μ is viewed as a global real-analytic Earle's section [8] for the bundle of effective divisors of degree s over Teichmueller spaces $\mathbf{T}_g$ [5].

Introduce the collections of differentials, which represent cosets for our quotient space for essential characters

$$\widetilde{\zeta}_1, \dots, \widetilde{\zeta}_{g-1}, \widetilde{\tau}_{\rho;Q_2Q_1}, \dots, \widetilde{\tau}_{\rho;Q_sQ_1}, \widetilde{\tau}_{\rho;Q_1}. \tag{1}$$

Theorem 3. *The vector bundle* $E = \bigcup\limits_{[\mu],\rho} \Omega_\rho(\frac{1}{Q_1 \dots Q_s}; F_\mu)/\Omega_{e,\rho}(1; F_\mu)$ *over* $\mathbf{T}_g \times (Hom(\Gamma, \mathbb{C}^*) \setminus L_g)$ *is a holomorphic vector bundle of rank* $g+s-1$. *In this case the collection* (1) *of cosets of differentials is a basis of local holomorphic sections for this vector bundle, where* $s \geqslant 1, g > 1$.

Proof. Let ρ be an essential character. Consider the space $\Omega_{e,\rho}(1; F_\mu)$. Suppose, that there exist $\omega \neq 0, \omega \in \Omega_{e,\rho}(1; F_\mu)$, then $\omega = df$ - holomorphic multiplicatively exact differential for ρ. Consequently f is a global holomorphic multiplicative function for essential character ρ. It is known, that $deg(f) = 0$ therefore f is a multiplicative unit for ρ, and ρ is a unessential. We obtain contradiction. Therefore $\Omega_{e,\rho}(1; F_\mu) = \{0\}$.

Thus,

$$\dim_{\mathbf{C}} \Omega_\rho(\frac{1}{Q_1 \cdots Q_s}; F_\mu)/\Omega_{e,\rho}(1; F_\mu) = \dim_{\mathbf{C}} \Omega_\rho(\frac{1}{Q_1 \cdots Q_s}; F_\mu) = g+s-1.$$

By the Theorem 2.1, for essential character ρ we have $g+s-1$ linear independent differentials $\widetilde{\zeta}_1, \dots, \widetilde{\zeta}_{g-1}, \widetilde{\tau}_{\rho;Q_2Q_1}, \dots, \widetilde{\tau}_{\rho;Q_sQ_1}, \widetilde{\tau}_{\rho;Q_1}$.

In fact, if there a exist linear combination form with non-zero coefficients

$$c_1\widetilde{\zeta}_1 + \ldots + c_{g-1}\widetilde{\zeta}_{g-1} + c_1'\widetilde{\tau}_{\rho;Q_1} + c_2'\widetilde{\tau}_{\rho;Q_2Q_1} + \ldots + c_s'\widetilde{\tau}_{\rho;Q_sQ_1} = df,$$

where df is holomorphic multiplicatively exact Prym differential for ρ, then the coefficients $c_2' = \ldots = c_s' = 0$, since points $Q_2, \dots, Q_s$ are not singularity points for the right side. Next $c_1' = 0$, since in the opposite case the function f is not a local single-valued in the punctured neighbourhood of the point Q_1, that contradicts to the condition $\rho(\gamma_1) = 1$, where γ_1 is loop around the point Q_1. We obtain equality $c_1\widetilde{\zeta}_1 + \ldots + c_{g-1}\widetilde{\zeta}_{g-1} = df$. In this case $df = 0$, and coefficients $c_1 = \ldots = c_{g-1} = 0$. Theorem 3 is proven.

Corollary 1. *On compact Riemann surface F_μ of genus $g \geqslant 2$ for the first holomorphic de Rham cohomology group $H^1_{hol,\rho}(F_\mu) = \Omega_\rho(1; F_\mu)/\Omega_{e,\rho}(1; F_\mu)$ the characters satisfy $dim_{\mathbf{C}} H^1_{hol,\rho}(F_\mu) = g - 1$, $ifrho \neq 1$.*

For these quotient spaces on F_μ :

for any unessential character $\rho_0 \neq 1, \rho_0(a_k) \neq 1$, the cosets of differentials $[f_0\zeta_1], \ldots, \widehat{[f_0\zeta_k]}, \ldots, [f_0\zeta_g]$ form a basis, which locally holomorphically depend of $[\mu]$ and ρ;

for an essential character ρ the cosets of differentials $[\widetilde{\zeta}_1], \ldots, [\widetilde{\zeta}_{g-1}]$ form a basis, which locally holomorphically depend on $[\mu]$ and ρ, where $\widetilde{\zeta}_1, \ldots, \widetilde{\zeta}_{g-1}$ is a basis of holomorphic differentials in the space $\Omega_\rho(1; F_\mu)$, which locally holomorphically depend of $[\mu]$ and ρ.

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BIHARMONIC SCHWARTZ INTEGRAL FOR A HALF-PLANE

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Key words: biharmonic equation; biharmonic commutative Banach algebra; monogenic function; Schwartz integral.

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Abstract. We consider an analogue of the Schwartz integral in a commutative Banach algebra associated with the biharmonic equation.

1 Introduction

We say that an associative commutative two-dimensional algebra $\mathbb{B}$ with the unit 1 over the field of complex numbers $\mathbb{C}$ is *biharmonic* if in $\mathbb{B}$ there exists a *biharmonic* basis, i.e a bases $\{e_1, e_2\}$ satisfying the conditions

$$(e_1^2 + e_2^2)^2 = 0, \quad e_1^2 + e_2^2 \neq 0. \tag{1}$$

V. F. Kovalev and I. P. Mel'nichenko [1] found a multiplication table for a biharmonic basis $\{e_1, e_2\}$:

$$e_1 = 1, \quad e_2^2 = e_1 + 2ie_2, \tag{2}$$

where i is the imaginary complex unit. In the paper [2] I. P. Mel'nichenko proved that there exists the unique biharmonic algebra $\mathbb{B}$ with a non-biharmonic basis $\{1, \rho\}$ for which $\rho = 2e_1 + 2ie_2$ and $\rho^2 = 0$, and he constructed all biharmonic bases in $\mathbb{B}$.

Consider a *biharmonic plane* $\mu := \{\zeta = x\,e_1 + y\,e_2 : x, y \in \mathbb{R}\}$ which is a linear span of the elements e_1, e_2 of the biharmonic basis (2) over the field of real numbers $\mathbb{R}$. With a domain D of the Cartesian plane xOy we associate the congruent domain $D_\zeta := \{\zeta = xe_1 + ye_2 : (x, y) \in D\}$ in the biharmonic plane μ. In what follows, $\zeta = x\,e_1 + y\,e_2$ and $x, y \in \mathbb{R}$.

Inasmuch as divisors of zero don't belong to the biharmonic plane, one can define the derivative $\Phi'(\zeta)$ of function $\Phi : D_\zeta \longrightarrow \mathbb{B}$ in the same way as in the complex plane: $\Phi'(\zeta) := \lim\limits_{h \to 0,\, h \in \mu} \big(\Phi(\zeta + h) - \Phi(\zeta)\big)\, h^{-1}$. We say that a function $\Phi : D_\zeta \longrightarrow \mathbb{B}$ is *monogenic* in a domain D_ζ if the derivative $\Phi'(\zeta)$ exists in every point $\zeta \in D_\zeta$.

It is established in the paper [1] that a function $\Phi : D_\zeta \longrightarrow \mathbb{B}$ is monogenic in a domain D_ζ if and only if the following Cauchy–Riemann condition is satisfied

$$\frac{\partial\Phi(\zeta)}{\partial y} = \frac{\partial\Phi(\zeta)}{\partial x}\, e_2.$$

It is proved in the paper [1] that a function $\Phi(\zeta)$ having derivatives till fourth order in D_ζ satisfies the two-dimensional biharmonic equation

$$(\Delta_2)^2 U(x,y) := \left(\frac{\partial^4}{\partial x^4} + 2\,\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) U(x,y) = 0 \tag{3}$$

in the domain D owing to the relations (1) and $(\Delta_2)^2\Phi(\zeta) = \Phi^{(4)}(\zeta)\,(e_1^2 + e_2^2)^2$. Therefore, every component $U_k\colon D \longrightarrow \mathbb{R}$, $k = \overline{1,4}$, of the expansion

$$\Phi(\zeta) = U_1(x,y)\, e_1 + U_2(x,y)\, ie_1 + U_3(x,y)\, e_2 + U_4(x,y)\, ie_2 \tag{4}$$

satisfies also the equation (3), i.e. U_k is a *biharmonic function* in the domain D.

It is proved in the paper [3] that a monogenic function $\Phi : D_\zeta \longrightarrow \mathbb{B}$ has derivatives $\Phi^{(n)}(\zeta)$ of all orders in the domain D_ζ and, consequently, satisfies the two-dimensional biharmonic equation (3). In the paper [3] it was also proved such a fact that every biharmomic function $U_1(x,y)$ in a bounded simply connected domain D is the first component of the expansion (4) of monogenic function $\Phi : D_\zeta \longrightarrow \mathbb{B}$ determined in an explicit form.

Basic analytic properties of monogenic functions in a biharmonic plane are similar to properties of holomorphic functions of the complex variable. More exactly, analogues of the Cauchy integral theorem and integral formula, the Morera theorem, the uniqueness theorem, the Taylor and Laurent expansions are established in the paper [4].

Having an intention to reduce boundary problems for monogenic functions to integral equations, we study boundary properties of certain integral representations of monogenic functions.

2 Biharmonic Schwartz integral for a half-plane

Let a function $u : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and there exists a finite limit

$$u(\infty) := \lim_{t\to\infty} u(t)\,. \tag{5}$$

Under assumptions that the modulus of continuity

$$\omega_{\mathbb{R}}(u,\varepsilon)=\sup_{\tau_1,\tau_2\in\mathbb{R}:|\tau_1-\tau_2|\leqslant\varepsilon}|u(\tau_1)-u(\tau_2)|$$

and the local centered (with respect to the infinitely remote point) modulus of continuity

$$\omega_{\mathbb{R},\infty}(u,\varepsilon)=\sup_{\tau\in\mathbb{R}:|\tau|\geqslant 1/\varepsilon}|u(\tau)-u(\infty)|$$

of the function u satisfy the Dini conditions

$$\int\limits_0^1 \frac{\omega_{\mathbb{R}}(u,\eta)}{\eta}\,d\eta<\infty, \tag{6}$$

$$\int\limits_0^1 \frac{\omega_{\mathbb{R},\infty}(u,\eta)}{\eta}\,d\eta<\infty, \tag{7}$$

consider the *biharmonic Schwartz integral* for the half-plane $\Pi^+:=\{\zeta=xe_1+ye_2: y>0\}$ defined by the equality:

$$S_{\Pi^+}[u](\zeta)\equiv\frac{1}{\pi i}\int\limits_{-\infty}^{+\infty}\frac{u(t)(1+t\zeta)}{(t^2+1)}(t-\zeta)^{-1}\,dt\quad\forall\,\zeta\in\Pi^+. \tag{8}$$

Here and in what follows, all integrals along the real axis are understood in the sense of their Cauchy principal values, i.e.

$$\int\limits_{-\infty}^{+\infty} g(t,\cdot)\,dt:=\lim_{N\to+\infty}\int\limits_{-N}^{N} g(t,\cdot)\,dt,$$

$$\int\limits_{-\infty}^{+\infty}\frac{g(t,\cdot)}{t-\xi}\,dt:=\lim_{N\to+\infty}\ \lim_{\varepsilon\to 0+0}\left(\int\limits_{-N}^{\xi-\varepsilon}+\int\limits_{\xi+\varepsilon}^{N}\right)\frac{g(t,\cdot)}{t-\xi}\,dt\,,\qquad \xi\in\mathbb{R}\,.$$

The function $S_{\Pi^+}[u](\zeta)$ is the principal extension (see [5, p. 165]) into the half-plane Π^+ of the complex Schwartz integral

$$S[u](z) := \frac{1}{\pi i}\int\limits_{-\infty}^{+\infty} \frac{u(t)(1+tz)}{(t^2+1)(t-z)}\,dt\,,$$

which determines a holomorphic function in the half-plane $\{z = x+iy : y > 0\}$ of the complex plane $\mathbb{C}$ with the given boundary values $u(t)$ of real part on the real line $\mathbb{R}$. Furthermore, the equality

$$S_{\Pi^+}[u](\zeta) = S[u](z) - \frac{y}{2\pi}\,\rho\int\limits_{-\infty}^{\infty} \frac{u(t)}{(t-z)^2}dt \quad \forall\,\zeta = xe_1 + e_2 y \in \Pi^+ \tag{9}$$

holds, where $z = x + iy$ as well as in what follows.

We use the euclidian norm $\|a\| := \sqrt{|z_1|^2 + |z_2|^2}$ in the algebra $\mathbb{B}$, where $a = z_1e_1 + z_2e_2$ and $z_1, z_2 \in \mathbb{C}$.

The following theorem presents sufficient conditions for the existence of boundary values of the biharmonic Schwartz integral on the extended real line $\mathbb{R}\cup\{\infty\}$.

Theorem 1. *If a function $u : \mathbb{R} \longrightarrow \mathbb{R}$ has the finite limit* (5) *and the condition* (6) *is satisfied, then the equality*

$$\lim_{\zeta\to\xi,\,\zeta\in\Pi^+} S_{\Pi^+}[u](\zeta) = u(\xi) + \frac{1}{\pi i}\int\limits_{-\infty}^{\infty} \frac{u(t)}{t^2+1}\,\frac{1+t\xi}{t-\xi}\,dt \quad \forall\,\xi\in\mathbb{R} \tag{10}$$

is fulfilled. If, in addition, the function u satisfies the condition (7)*, then*

$$\lim_{\|\zeta\|\to\infty,\,\zeta\in\Pi^+} S_{\Pi^+}[u](\zeta) = u(\infty) - \frac{1}{\pi i}\int\limits_{-\infty}^{\infty} u(t)\frac{t}{t^2+1}\,dt. \tag{11}$$

Proof. In order to prove the equality (10) we use the expression (9) of the biharmonic Schwartz integral. The second summand in the right-hand part of equality (9) tends to zero with $\zeta \to \xi \in \mathbb{R}$. This statement follows from the equalities

$$y\int\limits_{-\infty}^{\infty}\frac{u(t)}{(t-z)^2}d\,t=y\int\limits_{-\infty}^{\infty}\frac{u(t)-u(x)}{(t-z)^2}d\,t=y\int\limits_{x-2|y|}^{x+2|y|}\frac{u(t)-u(x)}{(t-z)^2}\,dt+$$

$$+\,y\left(\int\limits_{-\infty}^{x-2|y|}+\int\limits_{x+2|y|}^{\infty}\right)\frac{u(t)-u(x)}{(t-z)^2}\,dt=:I_1+I_2$$

and the relations

$$|I_1|\leqslant|y|\int\limits_{x-2|y|}^{x+2|y|}\frac{|u(t)-u(x)|}{y^2}\,dt\leqslant 4\,\omega_{\mathbb{R}}(u,2|y|)\to 0,\quad z\to\xi\,,$$

$$|I_2|\leqslant|y|\left(\int\limits_{-\infty}^{x-2|y|}+\int\limits_{x+2|y|}^{\infty}\right)\frac{|u(t)-u(x)|}{|t-x|^2}\,dt\leqslant 2|y|\int\limits_{2|y|}^{\infty}\frac{\omega_{\mathbb{R}}(u,\eta)}{\eta^2}\,d\eta\to 0,\quad z\to\xi\,.$$

By virtue of the condition (6), the Schwartz integral $S[u](z)$ has limiting values on the real line (it follows, for example, from an appropriate result of the paper [6] for the Cauchy type integral), hence

$$S[u](z)\to u(\xi)+\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{u(t)}{t^2+1}\,\frac{1+t\xi}{t-\xi}\,dt\,,\quad z=x+iy\to\xi\in\mathbb{R},\quad y>0\,.$$

Thus, the equality (10) is proved.

In order to prove the equality (11) with using the change of variables $t=-1/t_1$, $z=-1/z_1$ (see., for example, [7, p. 36]) we rewrite the relation (9) in the form

$$S_{\Pi^+}[u](\zeta)=S[v](z_1)-\frac{1}{2\pi}\,\frac{z_1}{\bar z_1}\,\rho\,\mathrm{Im}\,z_1\int\limits_{-\infty}^{\infty}\frac{v(t_1)}{(t_1-z_1)^2}d\,t_1\,,\tag{12}$$

where $v(t_1):=u(-1/t)$ and $\bar z_1:=\mathrm{Re}\,z_1-i\,\mathrm{Im}\,z_1$. By virtue of Lemma 1 of the paper [8], the function v satisfies a condition of the form (6). Therefore, the equality (11) can be obtained from the equality (12) by passing to the limit when $z_1\to 0$, $\mathrm{Im}\,z_1>0$ by analogy with the proof of the equality (10). The theorem is proved.

It follows from Theorem 1 that under assumptions of this theorem the function

$$\Phi(\zeta)=S_{\Pi^+}[u](\zeta)$$

is a monogenic function in Π^+ for which boundary values on $\mathbb{R}$ of the first component U_1 of the expansion (4) are equal to the function u, i.e.

$$U_1(\xi, 0) = u(\xi) \quad \forall\, \xi \in \mathbb{R}\,. \tag{13}$$

Let us note that a monogenic function $\Phi : \Pi^+ \longrightarrow \mathbb{B}$ satisfying the boundary condition (13) is not unique. It is obviously that the function

$$\Phi(\zeta) = S_{\Pi^+}[u](\zeta) + S_{\Pi^+}[u_1](\zeta)\, e_2$$

satisfies also the condition (13) if the functions u_1 satisfies the same conditions as the function u in Theorem 1.

Thus, to find the unique monogenic function Φ satisfying the boundary condition (13) it is necessary to require a fulfilment of some additional conditions for the function Φ. Some statements of boundary value problems for monogenic functions are discussed in the paper [9].

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HYPERFUNCTIONS ON FRACTAL BOUNDARIES (II) (HOLOMORPHIC LINE BUNDLE)

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Abstract. In the previous paper [3], a concept of hyperfunction is introduced on a fractal boundary C on the unit circle in the complex plane and hyperfunction solutions for functions of $L^2(C, d\mu)$ are constructed by use of wavelet expansion theorem of Schauder type. Here μ is the Hausdorff measure on C. In this paper a description of hyperfunctions by holomorphic line bundle is proposed. Then a unified theory of hyperfunctions on the fractal boundary can be obtained.

1 Introduction

We can find many results for the boundary value problem on a domain in $\mathbf{C}$. One of the most important results is stated by Caratheodory theorem [1]. Also we have several methods for the problems. Here we state two important methods which have different characters:One is the most popular method by use of complex/real analysis. On the other hand we have a method of the theory of hyperfunction due to Prof. Sato which can describe a wide class of functions as a gap of two holomorphic functions on the inside and the outside of a domain [4]. This theory is described in terms of cohomology theory and its analysis is called algebraic analysis.

In the previous paper [3], we have given a method of a description of hyperfunction solution to the boundary value problem on a fractal set of circle type. There we have obtained complete descriptions only in the case of fractal sets of Cantor type. But we can not obtain satisfactory results for other cases.

In this paper we shall associate a holomorphic line bundle for a given boundary value and we can describe hyperfunctions for general circular fractal sets in a unified manner. Main results will be given in Section 4.

The basic idea of this paper can be stated in Propositions 5 and 6. The first proposition supplies a bridge from fractal geometry to complex analysis and the second proposition gives a possibiliy to the introduction of the theory of hyperfunctions to fractal geometry. In the final section we make comments on the possibilities in the generalization of our method of the the theory of hyperfunctions to more general fractal boundaries and the boundary value problems (see Propositions 7, 8).

By these observations we may expect to compare the both methods for boundary value problems.

2 Circular self similar fractal set

We introduce a concept of circuler fractal set and give its classification. Here we restrict ourselves only to a fractal set which is defined by a system of self similar contraction mappings $\{\sigma_j : j = 1, 2, \cdots, N\}$ from a compact set K_0 to itself. Namely there exist positive numbers $\{\lambda_i\, (0 < \lambda_i < 1)\}$ satisfying the conditions: $d(\sigma_i(x), \sigma_i(y)) = \lambda_i d(x, y)$ $(i = 1, 2, \cdots, N)$. We always assume the the separation condition: $\sigma_i(\overset{\circ}{K_0}) \cap \sigma_j(\overset{\circ}{K_0}) = \varphi$ $(i \neq j)$, where $\overset{\circ}{K_0}$ is the open kernel of K_0. Then we obtain self similar fractal set [2]: $K = \cap_{n=1}^{\infty} K_n$, where $K_n = \cup_{j=1}^{N} \sigma_j(K_{n-1})$. We notice that this condition does not restrict fractal sets to those of separable sets.

Some basic facts on self similar fractal set [2]: (1) We can calculate the Hausdorff dimension $D(= dim_H K)$ by the following formula: $\sum_{j=1}^{N} \lambda_j^D = 1$. (2) The Borel algebra is generated by $\{K_{j_n \cdots j_1}\}, K_{j_n \cdots j_1} = \sigma_{j_n} \circ \cdots \circ \sigma_{j_1}(K)$. (3)The Hausdorff measure μ is given by $\mu(K_{j_n \cdots j_1}) = \lambda_{j_n}^D \lambda_{j_{n-1}}^D ... \lambda_{j_1}^D$. The Hilbert space of L^2-functions with respect to this measure is denoted by $L^2(K, d\mu)$.

Definition 1 (Circular self similar fractal set)

A self similar fractal set C on S^1 which is generated by a system of self similar contractions $\sigma_i : S^1 \mapsto S^1$ $(j = 1, 2, .., N)$ is called circular self similar fractal set. Here $S^1 = \{z \in \mathbf{C} | |z| = 1\}$.

We can classify circular fractal sets into three families:

(1) Circular fractal set is called of *Cantor type* when $\sigma_i(S^1) \cap \sigma_j(S^1) = \varphi$ $(i \neq j)$ for any different pair i and j.

(2) Circular fractal set is called of *partial Cantor type* when $\sigma_i(S^1) \cap \sigma_j(S^1) = \varphi$ $(i \neq j)$ holds for some pair i and j and $\sigma_{i'}(S^1) \cap \sigma_{j'}(S^1) \neq \varphi (i' \neq j')$ holds for some other pair i' and j'.

(3) Circular fractal set is called of *Circle type* when there exists a sequence $1, 2, .., N$ satisfying $\sigma_i(S^1) \cap \sigma_{i+1}(S^1) \neq \varphi$ for each $i = 1, 2, ..., N - 1$. $\sigma_N(S^1) \cap \sigma_1(S^1) \neq \varphi$.

Examples: We give examples of three families:

$$\begin{cases} (1)\sigma_1(\vartheta) = \frac{1}{6}\vartheta, \sigma_2(\vartheta) = \frac{1}{6}\vartheta + \frac{2\pi}{3}, \sigma_3(\vartheta) = \frac{1}{6}\vartheta + \frac{4\pi}{3} \\ (2)\sigma_1(\vartheta) = \frac{1}{3}\vartheta, \sigma_2(\vartheta) = \frac{1}{6}\vartheta + \frac{2\pi}{3}, \sigma_3(\vartheta) = \frac{1}{6}\vartheta + \frac{4\pi}{3} \\ (3)\sigma_1(\vartheta) = \frac{1}{6}\vartheta, \sigma_2(\vartheta) = \frac{1}{3}\vartheta + \frac{2\pi}{3}, \sigma_3(\vartheta) = \frac{1}{2}\vartheta + \pi \end{cases} \tag{1}$$

In [3], we have already treated the first family. The main concern of this paper is to treat fractal sets of the second and third families by use of holomorphic line bundle.

3 Holomorphic line bundle

We introduce a new concept of (inductive limit of) holomorphic line bundle on a fractal set.

(I)Neighborhood system of fractal set. We take a circular fractal set C on S^1 which is generated by contractions $\sigma_j : S^1 \mapsto S^1$ $(j = 1, 2, .., N)$. Choosing neighborhoods U_j $(j = 1, 2, ., N)$ of $\sigma_j(S^1)$ in $\mathbf{C}$, we put $U_I = U_{i_n, i_{n-1}, .., i_1}(I = (i_n, i_{n-1}, .., i_1))$, where $U_{i_n \cdots i_1} = \sigma_{i_n} \circ \cdots \circ \sigma_{i_2}(U_{i_1})$. Here we have use the holomorphic extensions of $\sigma_j (j = 1, 2, ..N)$ and used the same notation. Putting $|I| = n$,we call $U^{(n)} = \{U_I | |I| = n\}$ neighborhood system C of degree n. We notice the following facts: (i) $\cap_{n=1}^{\infty} \overline{U^n} = C$. (ii)$U^{n+1} \subset U^n$ with the inclusions $\iota_{n+1} : U^{n+1} \subset U^n$.

We call a system $\mathcal{N}(C) = \{U^{(n)} : n = 1.2...\}$ *neighborhood system of C*.

(II) Hyperfunction on a fractal boundary.

Choosing a neighborhood system $\mathcal{N})(\mathcal{C})$, we make the following definition:

Definition 2 (Hyperfunction on fractal set)

For $f \in L^2(C, d\mu)$, a sequence of holomorphic functions $\{F_n^{(+)}\}$ (resp. $\{F_n^{(-)}\}$) on $U_n \cap \mathbf{D}$ (resp $U_n \cap \mathbf{W}$), where $\mathbf{D}$ is the open unit disc and $\mathbf{W} = \mathbf{C}\{\overline{\mathbf{D}}$, is called *hyperfunction*of f, if $f = \underline{lim}(F_n^{(+)} - F_n^{(-)})$,where we take the inductive limit with respect to $\{\iota_n\}$. We call the right side of (1) the *boundary value of f*.

Remark. We notice that we can find hyperfunction boundary values for elements in the wider class of functions. Here we restrict ourselves to only $L^2(C, d\mu)$ because of the use of wavelet expansion of Schauder type (See section 5).

(III) Holomorphic line bundle on neighborhood system. We choose an element $f \in L^2(C, d\mu)$. We choose a neighborhood U_I of degree n in $\mathcal{N}(C)$. We assume that we can find a system of boundary values $\{F_I^{(+)}\}$,$\{F_I^{(-)}\}$ of f on each componet U_I respectively.

Definition 3 (Holomorphic line bundle on fractal set)

(1) Making the quotient $\varphi_{I,J}{}^{(+)} = F_I^{(+)}/F_J^{(+)}$ on $U_I \cap U_J (\neq \varphi)$, where $I, J \in U^n$, we can define a holomorphic line bundle on $U^{(n)}$ which is denoted by $E_n^{(+)}$. Making the inductive limit, we can define a holomorphic line bundle of the boundary value f:$E^{(+)}(f) = \underline{lim} E_n^{(+)}$. Also we can define the line budle $E_n^{(-)}(f)$. (2) A system of functions $\psi = \{\psi_I\}$, where ψ_I is a function on U_I, is called a section of $E_n^{(+)}$ when $\psi_I^{(+)} = \varphi_{I,J}{}^{(+)} \psi_J^{(+)}$ on $U_I \cap U_J (\neq \varphi)$. Also we can define sections of the line

bundle $E_n^{(-)}$. Hence we see that $\{F_I^{(+)}\}$ and $\{F_I^{(-)}\}$ are sections of $E_n^{(+)}$ and $E_n^{(-)}$ respectively.

4 Main Theorem

Main Theorem. Let C be a self similar fractal set on S^1 which is generated by a system of self similar contractions $\sigma_j : S^1 \mapsto S^1$ $(j = 1, 2, .., N)$. Then we can find a neighborhood system $\mathcal{N}(C)$ so that we can obtain the following results on $U_{i_n,i_{n-1},..,i_1}(= U_I) \in \mathcal{N}(C)$:

(I) Putting $J_{i_n,i_{n-1},..,i_1}(z) = {\sigma_{i_n}}^* \circ {\sigma_{i_{n-1}}}^* \circ ... \circ {\sigma_{i_1}}^*(J_0)$ for $z \in U_{i_n,i_{n-1},..,i_1}$ (otherwise 0), by use of the *Jukovski function*:

$$J_0(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right) + 1, \tag{2}$$

we have a system of Schauder basis of $L^2(C, d\mu)$ and obtain the following wavelet expansion: For $f \in L^2(C, d\mu)$, we have

$$f = a_0 J_0 + \sum_{n=1}^{\infty} \sum_{j_1}^{N} \cdots \sum_{j_n}^{N} a_{j_n \cdots j_1} J_{j_n \cdots j_1}. \tag{3}$$

(II) For $f \in L^2(C, d\mu)$, we make holomorphic line bundles $E_n^{(+)}$ and $E_n^{(-)}$. Then we can find sections of both line bundles $\{F_I^{(+)}\}$ and $\{F_I^{(+)}\}$ so that we can describe the solution of the boundary value problem for f: $f = \underline{lim}(F_n^{(+)} - F_n^{(-)})$.

Remark. In the case where C is of Cantor type, since $U_I \cap U_J = \varphi$ for any pair of different I and J, we can find a hyperfunction solution for $f \in L^2(C, d\mu)$ without making the holomorphic line bundle. Hence we see that the use of the concept of line bundle is essential for the classes of partial Cantor type and those of circular fractal sets.

5 Wavelet expansion on a self similar fractal set on the interval

We recall basic facts on the wavelet expansion of Schauder type on a fractal set in $I(= [0, 1])$ [3]. We take a self similar fractal set C on I which is defined by contractions:σ_j $(j = 1, 2, .., M)$. We choose a non-negative continuous function G_0 on I which is positive except on the boundary: $G_0(0) = G_0(1) = 0$. It is called *base function*. We choose a base function G_0 on I. For an integer m, putting $G_{j_m \cdots j_1}(x) = \kappa_{j_m \cdots j_1} G_0(\sigma_{j_1}^{-1} \circ \cdots \circ \sigma_{j_m}^{-1}(x))$ $\quad (x \in C_{j_m \cdots j_1})$, otherwise 0, where

$\kappa_{j_m \cdots j_1}$ are normalization constants: $(G_{j_n,\cdots,j_1}, G_{j_n,\cdots,j_1}) = 1$. where $\kappa_{j_m \cdots j_1}$ are normalization constants: $(G_{j_n,\cdots,j_1}, G_{j_n,\cdots,j_1}) = 1$. Here the inner product is that of the Hilbert space $L^2(C, d\mu)$. In the following we make the orthonormalization of these basis by the following condition: $(G_{i_n,\cdots,i_1}, G_{j_m,\cdots,j_1})_* = 0 (n \neq m)$ which we call the *orthonormalized inner product*. The space endowed with the inner product is denoted by $L^2_*(C, d\mu)$. Then we have the following theorem.

Theorem 4 (Wavelet expansion on self similar fractal set on interval). Let C be a self similar fractal set which is defined by contractions σ_j $(j = 1, 2, .., N)$ on the interval I and let G_0 be a base function on I. Then we have the following assertions:

(1) The system $\{G_0, G_{j_m \cdots j_1}\}$ constitute a system of orthonormal basis of $L^2_*(C, d\mu)$.

(2) We have a wavelet expansion of *Schauder type*:For $f \in L^2(C, d\mu)$, we have

$$f = a_0 G_0 + \sum_{n=1}^{\infty} \sum_{j_1}^{N} \cdots \sum_{j_n}^{N} a_{j_n \cdots j_1} G_{j_n \cdots j_1}, \tag{4}$$

where $a_0 = (f, G_0)_*$ and $a_{j_n \cdots j_1} = (f, G_{j_n \cdots j_1})_*$.

6 Proof of Main Theorem

We prove the assertion (I). Taking Theorem 4 into account, it is enough to prove the following proposition.

Proposition 5. (1) The Jukowski function (2), which is a rational function gives a rise to a base function on S^1 by the restriction: $J(z) = -cos\vartheta + 1 (z = e^{i\vartheta})$ satisfying $J(e^{i\pi}) = J(e^{-i\pi}) = 0$. (2) We can generate a system of basis of Schauder type $\{J_{i_n, i_{n-1}, .., i_1}(z)\}$ on S^1.

Next we proceed to the proof of (II). We prove the following proposition.

Proposition 6. For $J_{j_n, j_{n-1}, .., j_1}$, we have holomorphic functions $F^{(+)}_{j_n, j_{n-1}, .., j_1}$ on $\mathbf{D}$ and $F^{(-)}_{j_n, j_{n-1}, .., j_1}$ on $\mathbf{W}$ such that $J_{j_n, j_{n-1}, .., j_1} = F^{(+)}_{j_n, j_{n-1}, .., j_1} - F^{(-)}_{j_n, j_{n-1}, .., j_1}$.

Proof. In order to make our idea clear, we prove the assertion in the case of $n = 1$ at first. We have a hyperfunction solution for the function $J(z)$:In fact, putting $F^{(+)}_{i_n, i_{n-1}, .., i_1}(z) = -\sigma_{i_n}{}^* \circ \sigma_{i_{n-1}}{}^* \circ ... \circ \sigma_{i_1}{}^*(\frac{1}{2} z)$ on $\mathbf{D}$, $F^{(-)}_{i_n, i_{n-1}, .., i_1}(z) = \sigma_{i_n}{}^* \circ \sigma_{i_{n-1}}{}^* \circ ... \circ \sigma_{i_1}{}^*(\frac{1}{2z} - 1)$ on $\mathbf{W}$.

Next we prove the assertion (II). We choose a neighborhood system $\mathcal{N}(C)$. Making holomorphic extensions of $F^{(+)}_J$ and $F^{(-)}_J$ on U_J, we introduce a holomorphic line bundle by $\varphi^{(+)}_{I,J} = F^{(+)}_I / F^{(+)}_J$ for $U_I \cap U_J (\neq \varphi)$. By use of the discussions in Section 3, we see that the both functions make sections and we see the following assertion: $f = \underline{lim} F_n$, where $F_n = F_n^{(+)} - F_n^{(-)}$, by use of the property (i) in (I), section 3.

7 Applications to boundary value problem on fractal boundary

We suggest possibilites of the method of hyperfunctions to the boundary value problem for the case of fractal boundary [3]. We consider a simply connected domain D with a fractal boundary ∂D. Here we are concerned with the following problem.

We make comments on the generalizations of our method to more general boundaries and to the boundary value problem [3]. We consider a simply connected domain D with a fractal boundary ∂D. At first we notice that we can generalize our results to the boudary by use of the Riemann mapping theorem. For example we can discuss the theory of hyperfunctions on polygons or Julia set by use of Schwarz-Christoffel mapping or Böttcher mapping. Then we can find base functions which are deformations of the Jukowski function. Especially for the closed Koch curve and Sierpinski Gasket we can expect to have their explict forms. Our paper supplies the very beginning to this field. This will be performed in the near furture. Next we are concerned with the following boundary value problem for hyperfunction.

Boundary value problem for hyperfunction

For a given $f \in L^2(\partial D, d\mu)$, can we find two globally defined holomorphic functions $F^+(U \cap D)$ and $F^-(U \cap \overline{D}^c)$ for some neighborhood U of ∂D such that $f = F^+ - F^-$ on ∂D?

We state some results concerning this problem without proofs.

(I) If the holomorphic line bundle is trivial, i.e., $\varphi_{I,J} = 1$ holds for any non-empty intersection $U_I \cap U_J$, then hyperfunction solution can be obtained by Main theorem.

(II) We introduce the following condition and discuss the solution of the problem.

Definition 7 (Painlevé continuation data)

The data $\{\sigma_j(j = 1, 2, .., N)\}$ is called *Painlevé continuation data*, if their holomorphic extensions constitute a complete set of a holomorphic covering $\tau : D \mapsto D$.

In order to describe the solutions, we have to introduce a concept of symmetric functions. We call function f in (3) *symmetric* when $a_{j_n, j_{n-1}, .., j_1}$ are permutation invariant for any integer n. We denote the space of symmetric functions in $L^2(C, d\mu)$ by $L^2_S(C, d\mu)$. Then we can prove the following proposition.

Proposition 8. If Painlevé continuation data $\{\sigma_j(j = 1, 2, .., N)\}$ are given, we can solve the hyperfunction solutions for symmetric functions of $L^2_S(C, d\mu)$.

(III) In the case of $\tau : D \mapsto D$ is a one-to-one covering, we see that τ gives a dynamical system of the fractal set. Hence we can treat Weierstrasse function and a filled Julia doamin by this method [3]. In the case where τ is a piecewise holomorphic mapping arising from a holomorphic properly finite covering, we can obtain more complicate functions as solutions. This will be discussed elsewhere.

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EVOLUTION MODELS FOR BUTTERFLY DESIGN PATTERNS BY ITERATION DYNAMICAL SYSTEMS OF DISCRETE LAPLACIANS ON THE PLANE LATTICE

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Key words: dynamical systems, discrete Laplacian

AMS Mathematics Subject Classification: 30G25,32H50

Abstract. In [1] and [3], we have given several computer simulations of iteration dynamical systems of discrete Laplacians. We may say that we can simulate body construction processes and evolutions quite well, especially in the time evolutions of extinct animals. In this paper, we simulate the design patterns of butterfly wings by this simulator. Then we make evolution models by this simulator and compare the results with those in evolutionary developmental biology.

1 Introduction: Iteration dynamical system of discrete Laplacian

We choose the plane lattice which is generated by two families of lines which are orthonormal each other. We identify a lattice point with a cell obtained by the lattice structure. We call a set of cells which are attached with the reference cell neighborhood. We call neighborhood even (or odd) if the number of the cells is even (resp. odd). We give several examples. Some of them are well known [1]:

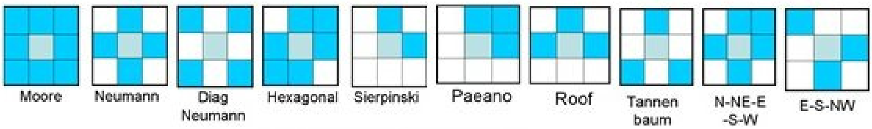

Figure 1 Examples of neighborhoods

We take the space F of 0,1 valued functions on the plane lattice. Choosing a neighborhood U_p, we define the discrte Lapalcian operation as follows:

$$\Delta_{U_p} f(p) = \sum_{q \in U_p} (f(q) - f(p)). \quad (1)$$

For an initial function $f_0 \in F$, we consider the dynamical system:

$$\{f_n\}, f_n = \Delta_U f_{n-1} \quad (n = 1, 2, ...). \quad (2)$$

We name the dynamical system following a given neighborhood, for example, for Moore neighborhood we call it the Moore dynamical system.

2 Computer simulation

Choosing sources and neighborhoods, we can realize a wide class of phenomena by these iteration dynamical systems. Here we call a cell Q source of the dynamical system $\{f_n\}$ when $f_n(Q) = 1$ for any step n. We give several examples.

(1) Crystals of water [1].

We can make crystals of water under suitable conditions. We can realize them choosing the hexagonal neighborhood quite well. We may expect to make its mathematical theory based on the discrete Lapalcian.

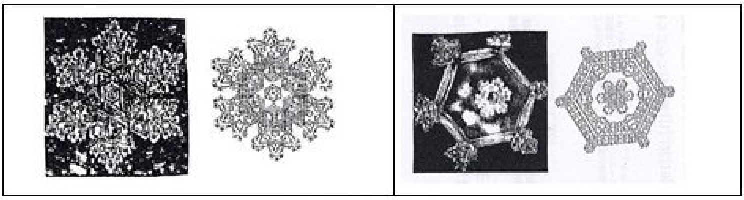

Figure 2 Crystals of water

(2) Evolution of extinct animals [4].

We present a computer simulation of echinoderm, one of extinct animals. The left side is the real deta given by Sepkoski [6] and the right side is a computer simulation by use of Moore neighborhood with a single source. We can observe a suprising hit. The reason for the well fitting is still not clarified till now.

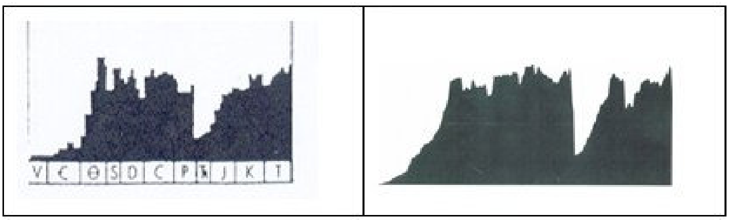

Figure 3 Evolution of an extinct animal

(3) Design patterns [3].

We can produce many kinds of design patterns including carpets and embroiders etc. We can make differences between european and japanese designs.

(4) Flower patterns [3].

We can show the possibilities of realizations of flower designs by use of our dynamical systems. This will be performed in near furture.

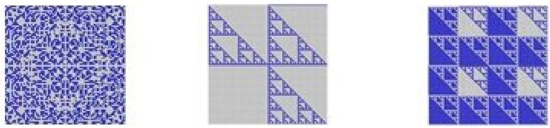

Figure 4 Generation of design patterns

Figure 5 Generation of design patterns

(5) Design patterns of butterfly wings.

We demonstrate simulations of design patterns of butterfly wings. The details of the realizations will be given in the remained part of this paper.

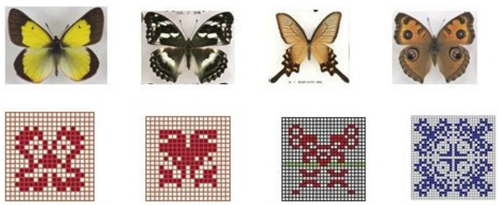

Figure 6 Generation of butterfly design patterns

3 Fundamental operations in evolutionary developmental biology

We recall some basic facts on operations of body constructions in evolutionary developmental biology [2]. Main operations constitute two basic operations:

(1) Tool kit operations. The tool kits supply a mechanism of body construction step by step. We can observe a fractal structure in the body constructions (Fig. 7).

(2) Switch mechanism. The varity in body constructions arises from the switch mechanism. We know that the tool kit itself has not so big differences between higher developed life things and primitive ones. The big difference comes from the switch mechanism. In DNA level, the body construction is determined by the on and off operations of the switchs. Prof. Carol has discovered that the eye pattarns of butterfly wings and the difference of the size in the forewings and hindwings arises by this mechanism (Fig. 8) [2].

Figure 7 Tool kits in body construction

Figure 8 Switch mechanism in DNA

4 Evolution model in butterfly design pattern by iteration dynamical system of discrete Laplacian

In this section we shall find the corresponding operations in our dynamical systems and expect to construct evolution models by these dynamical systems:

(I) The tool kit of our dynamical systems: Our dynamical systems are determined by the choices of (1) Neighborhoods and (2)Souces. Setting these tools, we can simulate the body constructions. This corresponds to the fractal structure of operations in life things. In fact, we can simulate many structures in generating bigger cells from smaller ones (Fig. 9).

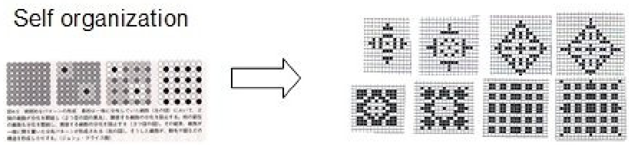

Figure 9 Tool kit mechanism in our dynamical system

(II) The switch mechanism: We prepare two candidate operators: (1) The change of neighborhoods, (2) Separation of diffusions by certain walls (in simulations it is indicated by dark color). Then we see that spot patterns are deformed into eye patterns or the tail patterns with long tails or those with several numbers of tails by the separations (Fig.10, Fig.14).

5 Realization I (Border type design and Spot design)

Here we realize patterns of spot type and border type. For both cases we choose the diagonal Neumann neighborhood. We begin with a setting of "standard type", which might be expected to be the most primitive one which might appear in the early stage of the evolution. We deform the standard source to symmetric or asymmetric sources. Then we can obtain designs which are different in the forewings

Figure 10 Switch mechanism in our dynamical system

and hindwings:We obtain border type designs for symmetric neighborhoods and spot type designs for asymmetric neighborhoods respectively (Fig. 11).

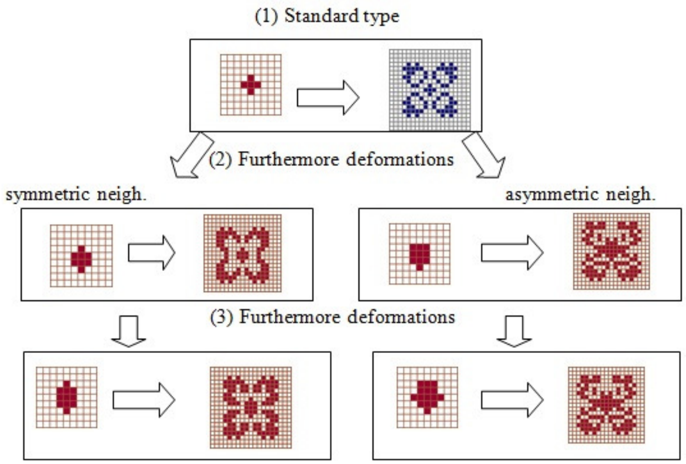

Figure 11 Description of evolutions by our simulations

Making the pattern fittings we can realize the both kinds of designs.

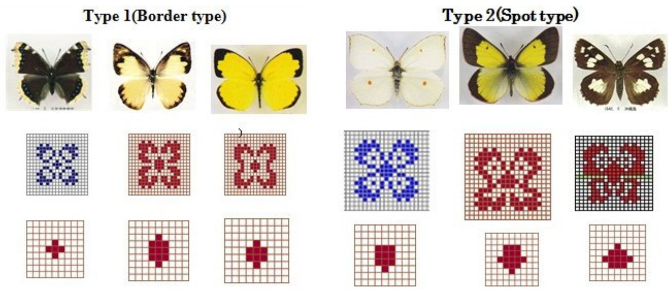

Figure 12 Realization of border type and spot designs

We notice that discocellar spots are clearly realized, just changing the sources.

6 Realization II (Tail design)

We give computer simulations of tail designs. We can show that the differences in tails can be obtained separating the diffusion effects by the choices of walls. For each case we choose Neumann neighborhood (Fig. 13).

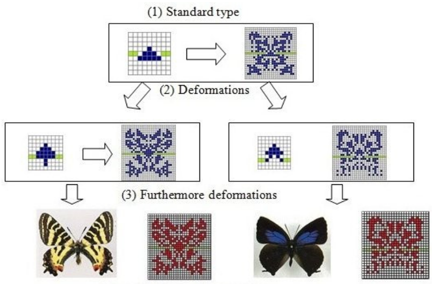

Figure 13 Realization of tail-type

We can find another sequence of patterns of tail type choosing different seeds and deformations. Without detail description of deformations, we state the results. We notice that we can realize long tails quite well (Fig. 14).

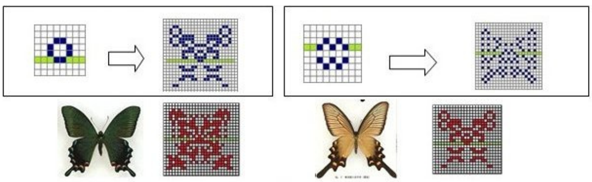

Figure 14 Realization of tail-type

We want to make comparisons of realization methods between our method and Turing pattern method. The Turing pattern method can realize the design patterns quite well. But it does not concern the shape of wings, for example, the difference in forewings and hindwings and those in long and short tails.

7 Realization III (Eye pattern designs)

We treat realization of eye patterns. This is just in progress. We give only three simulations. We want to notice that we can make eye patterns quite easily. We do not know whether the Turing pattern method can realize the eye patterns or not.

8 Conclusions and discussions)

We have simulated design patterns of butterfly wings by use of iteration dynamical systems systematically. We state the results and make some discussions.

Figure 15 Realization of eye-type

(1) We have done the simulations of the spot pattern, border pattern, tail pattern and eye patterns.

(2) We have observed that our dynamical systems have not only the structure of tool-kits but also switch mechanisms in evolutions. Hence we may expect to realize evolutions by our systems.

(3) We make the comparisons between our method and Turing pattern methods [5]. We know that the Turing methods can realize butterfly wing patterns well. We notice that their realization can be obtained as the stable states. Our realizations are obtained just controlling the steps by hand.This is one of the week points of our simulations. This defect should be conquered in near furtutre. Here we want to state the merit of use of our method. We have never seen the realization of eye patterns by the Turing pattern method. It concern only the design patterns and but not the shape of wings, for example, tails. At present we have a lot of knowledes on evolutionary developing biology and it suggest us that we should proceed to the realization of the body construction itself. These are future problems.

(4) We propose two problems. (i) The most interesting problem is to choose evolution models comparing the real evolutions in the nature and find the mechanism of evolutions. Also we may have interests in making the tree structure of our evolution models and comparing the real tree of that obtained by use of mitochondria and others. (ii) Finally we want to state the reason why we have proposed this topics in this section. B.Riemann developed the theory of complex analysis from electrodynamics. In our simulations we can find a new field of discrete complex anaysis. We wish this paper will open the door to the new field of complex analysis.

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ISOMETRIES OF BERGMAN-TYPE ZYGMUND F-ALGEBRAS OVER THE UNIT BALL

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Key words: Linear isometries, Bergman spaces, Zygmund F-algebras

AMS Mathematics Subject Classification: 32A36, 32A37, 32A38

Abstract. For each $\alpha > -1$ and $\beta > 0$, we consider the space $N\log^\beta N(\mathbb{B})$ of holomorphic functions on the unit ball $\mathbb{B}$ in the n-dimensional complex Euclidean space which these holomorphic functions f satisfy the condition

$$\|f\|_\beta = \int_{\mathbb{B}} \varphi_\beta(\log(1+|f(z)|))\,dV_\alpha(z) < \infty,$$

where $\varphi_\beta(x) = x\{\log(\gamma_\beta + x)\}^\beta$ for $x \in [0,\infty)$ and $\gamma_\beta = \max\{e, e^\beta\}$. This space $N\log^\beta N(\mathbb{B})$ is a complete metric space with respect to the translation-invariant metric $d_\beta(f,g) = \|f-g\|_\beta$. In this note we will characterize injective and surjective linear isometries of this space.

1 Introduction

Throughout this note, let $\mathbb{B}$ denote the open unit ball in the n-dimensional complex Euclidean space $\mathbb{C}^n$ and dV_n the Lebesgue measure on $\mathbb{C}^n$. Let $H(\mathbb{B})$ denote the space of all holomorphic functions on $\mathbb{B}$. For each $\alpha > -1$ we put $dV_\alpha(z) = c_{\alpha,n}(1-|z|^2)^\alpha dV_n(z)$ where $c_{\alpha,n}$ is a normalization constant.

Take an arbitrary $\beta > 0$ and consider the function $\varphi_\beta(x) = x\{\log(\gamma_\beta + x)\}^\beta$ for $x \in [0,\infty)$ where $\gamma_\beta = e$ if $0 < \beta < 1$ and $\gamma_\beta = e^\beta$ if $\beta \geqslant 1$. Then $\varphi_\beta(x)$ is strictly increasing and convex downward on $[0,\infty)$ and $\varphi_\beta(\log(1+x))$ is strictly increasing and convex upward on $[0,\infty)$ (see [1]).

The Bergman-type Zygmund F-algebra on $\mathbb{B}$ is defined as

$$N\log^\beta N(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) : \int_{\mathbb{B}} \varphi_\beta(\log(1+|f(z)|))\,dV_\alpha(z) < \infty \right\}.$$

Since the function $\varphi_\beta(\log(1+x))$ satisfies

$$\varphi_\beta(\log(1+x)) \leqslant (\log\gamma_\beta)^\beta x \tag{1}$$

for $x \in [0, \infty)$, we see that the space $N\log^{\beta} N(\mathbb{B})$ contains the weighted Bergman space $A^1(\mathbb{B}, dV_\alpha)$. The limiting case $\alpha \to -1$ for this space has been studied by O.M. Eminyan [1]. The quasi-norm $\|f\|_\beta$ on these spaces $N\log^{\beta} N(\mathbb{B})$ is defined by

$$\|f\|_\beta = \int_{\mathbb{B}} \varphi_\beta(\log(1 + |f(z)|))\, dV_\alpha(z).$$

Since this quasi-norm satisfies the triangle inequality, $d_\beta(f, g) = \|f - g\|_\beta$ defines a translation-invariant metric on $N\log^{\beta} N(\mathbb{B})$. By the same argument in [1], we see that $N\log^{\beta} N(\mathbb{B})$ is not only an F-space in the sense of Banach with respect to d_β but also a topological algebra.

The studies on linear isometries of holomorphic function spaces have been studied since the 1960s. For the Hardy space H^p $(0 < p \leqslant \infty,\ p \neq 2)$ on the unit disc, D. deLeeuw, W. Rudin and J. Wermer [7] $(p = 1, \infty)$ and F. Forelli [3] $(1 \leqslant p < \infty)$ characterized the linear isometries. For the details on these studies, we can also refer to the monograph [2]. For the several variables case, Forelli [4] and Rudin [8] have determined the injective and/or surjective isometries of H^p. For the weighted Bergman spaces $A^p(\mathbb{B}, dV_\alpha)$ $(0 < p < \infty,\ p \neq 2)$, the isometries was completely characterized in a sequence of papers by C. J. Kolaski [5, 6]. By these works we see that the isometries on these holomorphic function spaces are described as weighted composition operators defined by $\Psi C_\Phi(f) = \Psi \cdot (f \circ \Phi)$ for some holomorphic function Ψ and holomorphic self-map Φ of the unit ball, which is one of the reasons why these operators have been investigated so much recently in the settings of the unit ball. The Smirnov class N^* and the Privalov space N^p $(1 < p < \infty)$ which are contained in the Nevanlinna class N are F-spaces with respect to a suitable metric on them. K. Stephenson [9] and A. V. Subbotin [10] have studied linear isometries of these spaces. Their works showed that the injective isometries are weighted composition operators induced by some inner functions and inner maps of $\mathbb{B}$ whose radial limit maps satisfy a measure-preserving property.

The purpose of this note is to investigate a linear isometry of the space $N\log^{\beta} N(\mathbb{B})$. Recently, the author [11] have characterized the injective and surjective isometry for the case $\alpha \to -1$. By some modifications of proofs in [11] and an application of the result by C.J. Kolaski, we can get an analogous result for the case $\alpha > -1$.

2 Linear isometries on $N\log^{\beta} N(\mathbb{B})$

In order to prove main result, we need some lemmas. From now on, till the end of this note, we fix $\alpha > -1$ and $\beta > 0$.

Lemma 1. *Every* $f \in N\log^{\beta} N(\mathbb{B})$ *satisfies* $\lim_{r\to 1} \|f - f_r\|_\beta = 0$.

Proof. Take an $f \in N\log^{\beta} N(\mathbb{B})$ and $\varepsilon > 0$. Then we can choose an $r_0 \in (0,1)$ such that

$$\int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \varphi_\beta(\log(1+|f(z)|))dV_\alpha(z) < \frac{\varepsilon}{3}.$$

Since $\varphi_\beta(\log(1+|f(z)|))$ is a positive plurisubharmonic function in $\mathbb{B}$, we have

$$\int_{\partial\mathbb{B}} \varphi_\beta(\log(1+|f(rt\zeta)|))d\sigma(\zeta) \leqslant \int_{\partial\mathbb{B}} \varphi_\beta(\log(1+|f(t\zeta)|))d\sigma(\zeta),$$

for any r, $t \in (0,1)$. Here $d\sigma$ is the normalized Lebesgue measure on the boundary $\partial\mathbb{B}$ of $\mathbb{B}$. This inequality implies that

$$\int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \varphi_\beta(\log(1+|f_r(z)|))dV_\alpha(z) \leqslant \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \varphi_\beta(\log(1+|f(z)|))dV_\alpha(z) < \frac{\varepsilon}{3}, \quad (2)$$

for any $r \in (0,1)$.

Now we choose an $\varepsilon_0 > 0$ such that $\varphi_\beta(\log(1+\varepsilon_0)) = \varepsilon/3$. By the continuity of f on the compact subset $r_0\overline{\mathbb{B}}$, we see that there exists a $\delta \in (0,1)$ such that if $z, w \in r_0\overline{\mathbb{B}}$ with $|z-w| < \delta$, then $|f(z) - f(w)| < \varepsilon_0$. Set $r_1 = 1-\delta$. If $r_1 < r < 1$, then

$$\int_{r_0\overline{\mathbb{B}}} \varphi_\beta(\log(1+|f_r(z) - f(z)|))dV_\alpha(z) \leqslant \varphi_\beta(\log(1+\varepsilon_0)) = \frac{\varepsilon}{3}. \quad (3)$$

By (2) and (3), we obtain

$$\|f_r - f\|_\beta = \left(\int_{r_0\overline{\mathbb{B}}} + \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}}\right) \varphi_\beta(\log(1+|f_r(z)-f(z)|))dV_\alpha(z) \leqslant$$

$$\leqslant \int_{r_0\overline{\mathbb{B}}} \varphi_\beta(\log(1+|f_r(z)-f(z)|))dV_\alpha(z) +$$

$$+ \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \left\{\varphi_\beta(\log(1+|f_r(z)|)) + \varphi_\beta(\log(1+|f(z)|))\right\} dV_\alpha(z) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This completes the proof.

Lemma 2. *If T is a linear isometry of $N\log^{\beta}N(\mathbb{B})$, then the restriction of T to $A^1(\mathbb{B}, dV_\alpha)$ is also a linear isometry of $A^1(\mathbb{B}, dV_\alpha)$ into $A^1(\mathbb{B}, dV_\alpha)$.*

Proof. Take an $f \in A^1(\mathbb{B}, dV_\alpha)$ and put $g = Tf$. For each positive integer m we have $g/m = T(f/m)$, and so we obtain

$$\int_{\mathbb{B}} \varphi_\beta \left(\log\left(1 + \frac{|f|}{m}\right)\right) dV_\alpha = \int_{\mathbb{B}} \varphi_\beta \left(\log\left(1 + \frac{|g|}{m}\right)\right) dV_\alpha. \tag{4}$$

By the inequality (1), we have

$$m\varphi_\beta \left(\log\left(1 + \frac{|f|}{m}\right)\right) \leqslant (\log\gamma_\beta)^\beta |f| \quad \text{on } \mathbb{B}.$$

By the definition of $\varphi_\beta(x)$ we see that

$$\lim_{m\to\infty} m\varphi_\beta \left(\log\left(1 + \frac{|f|}{m}\right)\right) = (\log\gamma_\beta)^\beta |f| \quad \text{on } \mathbb{B}.$$

The Lebesgue dominated convergence theorem gives

$$\lim_{m\to\infty} \int_{\mathbb{B}} m\varphi_\beta \left(\log\left(1 + \frac{|f|}{m}\right)\right) dV_\alpha = (\log\gamma_\beta)^\beta \int_{\mathbb{B}} |f|\, dV_\alpha. \tag{5}$$

Combining this with (4), Fatou's lemma show that g is in $A^1(\mathbb{B}, dV_\alpha)$. By applying the Lebesgue dominated convergence theorem once again, we have

$$\lim_{m\to\infty} \int_{\mathbb{B}} m\varphi_\beta \left(\log\left(1 + \frac{|g|}{m}\right)\right) dV_\alpha = (\log\gamma_\beta)^\beta \int_{\mathbb{B}} |g|\, dV_\alpha. \tag{6}$$

By (4), (5) and (6), we see that T is a linear isometry of $A^1(\mathbb{B}, dV_\alpha)$.

Lemma 3. *There exist a bounded continuous function ϑ_β on $[0, \infty)$ and a positive constant K_β such that*

$$\varphi_\beta(\log(1 + x)) = (\log\gamma_\beta)^\beta x - K_\beta x^2 + x^3\vartheta_\beta(x) \quad \textit{for } x \in [0, \infty).$$

Proof. See [11].

Theorem 1. *Every linear isometry T of $N\log^{\beta}N(\mathbb{B})$ into itself is of the form $Tf = c(f \circ \Phi)$ for all $f \in N\log^{\beta}N(\mathbb{B})$, where c is a complex number with $|c| = 1$*

and Φ is a holomorphic self-map of $\mathbb{B}$ such that $\int_{\mathbb{B}}(h\circ\Phi)dV_\alpha = \int_{\mathbb{B}} h dV_\alpha$ for every bounded Borel function h in $\mathbb{B}$.

Proof. Assume that T is a linear isometry of $N\log^\beta N(\mathbb{B})$. Since T is also isometry of $A^1(\mathbb{B}, dV_\alpha)$ by Lemma 2, Kolaski's theorem ([6, Theorem 1]) implies that $Tf = g(f\circ\Phi)$ for every $f \in A^1(\mathbb{B}, dV_\alpha)$, where $g = T1 \in A^1(\mathbb{B}, dV_\alpha)$ and Φ is a holomorphic self-map of $\mathbb{B}$ which satisfies

$$\int_{\mathbb{B}} (h\circ\Phi)|g| dV_\alpha = \int_{\mathbb{B}} h dV_\alpha \tag{7}$$

for every bounded Borel function h in $\mathbb{B}$.

Fix an $f \in N\log^\beta N(\mathbb{B})$ and consider dilated functions $\{f_r\}_{0<r<1}$ of f. Since $f_r \in C(\overline{\mathbb{B}}) \cap H(\mathbb{B})$, and so $f_r \in A^1(\mathbb{B}, dV_\alpha)$, we have $T(f_r)(z) = g(z)\cdot f(r\Phi(z))$ for all $r \in (0,1)$ and $z \in \mathbb{B}$. On the other hand, by Lemma 1, we have

$$\lim_{r\to1} \|Tf - T(f_r)\|_\beta = \lim_{r\to1} \|f - f_r\|_\beta = 0.$$

Note that the convergence in $N\log^\beta N(\mathbb{B})$ implies the uniform convergence on compact subsets of $\mathbb{B}$. So we have that

$$Tf = \lim_{r\to1} T(f_r) = \lim_{r\to1} g\cdot(f_r\circ\Phi) = g\cdot(f\circ\Phi) \quad \text{in } \mathbb{B}.$$

Next we will prove that g is a constant c with $|c| = 1$. Since $g = T1 \in A^1(\mathbb{B}, dV_\alpha)$ and $V_\alpha(\mathbb{B}) = 1$, Hölder's inequality shows that

$$1 = \|1\|_{A^1} = \|g\|_{A^1} \leqslant \|g\|_{A^2}.$$

Since $\|\,\mathrm{tg}\,\|_{A^1} = t$ and $\|\,\mathrm{tg}\,\|_\beta = \varphi_\beta(\log(1+t))$ for any $t > 0$, it follows from Lemma 3 that

$$\int_{\mathbb{B}} \left\{K_\beta|\,\mathrm{tg}\,|^2 - |\,\mathrm{tg}\,|^3\vartheta_\beta(|\,\mathrm{tg}\,|)\right\} dV_\alpha = (\log\gamma_\beta)^\beta t - \varphi_\beta(\log(1+t)) = K_\beta t^2 - t^3\vartheta_\beta(t),$$

and so we have

$$\int_{\mathbb{B}} \left\{K_\beta|g|^2 - t|g|^3\vartheta_\beta(|\,\mathrm{tg}\,|)\right\} dV_\alpha = K_\beta - t\vartheta_\beta(t).$$

Also Lemma 3 gives that

$$K_\beta|g|^2 - t|g|^3\vartheta_\beta(|\operatorname{tg}|) = \{(\log\gamma_\beta)^\beta|\operatorname{tg}| - \varphi_\beta(\log(1+|\operatorname{tg}|))\}/t^2 \geqslant 0.$$

By the application of Fatou's lemma, we obtain that

$$\int_{\mathbb{B}} K_\beta|g|^2 dV_\alpha \leqslant \liminf_{t\to 0}\{K_\beta - t\vartheta_\beta(t)\} = K_\beta.$$

Thus we have $\|g\|_{A^2} \leqslant 1$ and $\|g\|_{A^1} = \|g\|_{A^2} = 1$. This implies that $|g| = 1$ in $\mathbb{B}$. Since $g \in H(\mathbb{B})$, $g \equiv c$ in $\mathbb{B}$ where c is a complex number with $|c| = 1$. Combining this with (7), we have the desired property of Φ.

Conversely, if T is a mapping of the form described in the statement of the present theorem, it is easily shown that T is a linear isometry of $N\log^\beta N(\mathbb{B})$ into itself. We accomplished the proof.

As a corollary of the above theorem, we can get the characterization for the surjective linear isometry of $N\log^\beta N(\mathbb{B})$.

Corollary. *A linear isometry T of $N\log^\beta N(\mathbb{B})$ is surjective if and only if $Tf = c(f\circ\mathcal{U})$ for all $f \in N\log^\beta N(\mathbb{B})$, where c is a complex number with $|c| = 1$ and $\mathcal{U}$ is a unitary transformation on $\mathbb{C}^n$.*

Proof. Let T be a surjective linear isometry of $N\log^\beta N(\mathbb{B})$. Then, by Theorem 1, there exists a complex number c with $|c| = 1$ and a holomorphic self-map Φ of $\mathbb{B}$ with $\int_{\mathbb{B}}(h\circ\Phi)dV_\alpha = \int_{\mathbb{B}} h dV_\alpha$ for every bounded Borel function h in $\mathbb{B}$ such that $Tf = c(f\circ\Phi)$ for all $f \in N\log^\beta N(\mathbb{B})$. The property $\int_{\mathbb{B}}(h\circ\Phi)dV_\alpha = \int_{\mathbb{B}} h dV_\alpha$ yields $\Phi(0) = 0$. Since the inverse T^{-1} of T is also a surjective isometry of $N\log^\beta N(\mathbb{B})$, it follows that Φ is biholomorphic. Hence Φ is a unitary operator on $\mathbb{C}^n$.

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I.2. Differential Equations: Complex and Functional Analytic Methods for Partial Differential Equations

(Sessions organizers: H. Begehr, J.Y. Du, A. Soldatov)

MATHEMATICAL MODELING FOR THE PURPOSE OF ANTI-REFLECTIVE OPTICAL SYSTEMS

I. Akhmedov, Yu. Hudak

Key words: mathematical modelling, optics, anti-reflective

AMS Mathematics Subject Classification: 93A30

Abstract. The article is devoted to the mathematical modeling of electromagnetic fields in layered media. A solution is given for a classical anti-reflective task (with a fixed frequency) at normal plane wave incidence on two-layered system.

1 Introduction

Between two semi-spaces with substance parameters ε_0, μ_0 and ε_3, μ_3 let there be located two flat magnetodielectric layers with the thickness of h_i and substance ε_i, μ_i, $i = 1, 2$.

If in the layered system there exists a flat electromagnetic field "parallel" to plane YOZ which changes in time according to the law $e^{-i\omega t}$, then the Maxwell equations are as follows:

$$\begin{aligned} E' &= i\omega\mu H \\ H' &= i\omega\mu E \end{aligned} \tag{1}$$

Solving system (1), we get $U \equiv E$ $V \equiv H$:

$$\begin{pmatrix} U \\ V \end{pmatrix} = C_0 \begin{pmatrix} 1 \\ p \end{pmatrix} e^{i\omega n(x-x_0)} + C_1 \begin{pmatrix} 1 \\ -p \end{pmatrix} e^{-i\omega n(x-x_0)};\ p = \sqrt{\frac{\varepsilon}{\mu}},\ n = \sqrt{\varepsilon\mu}.$$

The conditions of electromagnetic field components continuity on each plane of discontinuity of physical parameters of the medium lead to a system of transitions from one layer to another:

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ p_0 & -p_0 \end{pmatrix} \begin{pmatrix} C_0^{(0)} \\ C_1^{(0)} \end{pmatrix} &= \begin{pmatrix} e^{-i\nu_1\omega} & e^{i\nu_1\omega} \\ p_1 e^{-i\nu_1\omega} & -p_1 e^{i\nu_1\omega} \end{pmatrix} \begin{pmatrix} C_0^{(1)} \\ C_1^{(1)} \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ p_1 & -p_1 \end{pmatrix} \begin{pmatrix} C_0^{(1)} \\ C_1^{(1)} \end{pmatrix} &= \begin{pmatrix} e^{-i\nu_2\omega} & e^{i\nu_2\omega} \\ p_1 e^{-i\nu_2\omega} & -p_1 e^{i\nu_2\omega} \end{pmatrix} \begin{pmatrix} C_0^{(2)} \\ C_1^{(2)} \end{pmatrix} \end{aligned} \tag{2}$$

$$\begin{pmatrix} 1 & 1 \\ p_2 & -p_2 \end{pmatrix} \begin{pmatrix} C_0^{(2)} \\ C_1^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ p_3 & -p_3 \end{pmatrix} \begin{pmatrix} C_0^{(3)} \\ C_1^{(3)} \end{pmatrix}$$

In system (2) there are 6 equations and 8 unknown quantities.

Due to the structure of the system, its solution depends on two arbitrary constants.

Let one of them equal zero $C_1^{(3)} = 0$, which presupposes the absence of reflection at $+\infty$.

We discard the second one, having done a normalization of the previous wave: $C_0^{(3)} = 1$.

In transition from the left semi-space to the right one for the un-known quantities $C_k^{(j)},\ k = 0, 1;\ j = 0, 1, 2, 3$ the following identity is just:

$$p_0 \left(\left| C_0^{(0)} \right|^2 - \left| C_1^{(0)} \right|^2 \right) = p_1 \left(\left| C_0^{(1)} \right|^2 - \left| C_1^{(1)} \right|^2 \right) = \cdots = p_3 \left(\left| C_0^{(3)} \right|^2 - \left| C_1^{(3)} \right|^2 \right) \tag{3}$$

Choosing the first and last expressions from (3), we receive:

$$\left| C_0^{(0)} \right|^2 - \left| C_1^{(0)} \right|^2 = \Theta, \quad \text{where} \quad \Theta = \frac{p_3}{p_0}. \tag{4}$$

Let us rewrite (4) in the following form:

$$1 - \frac{\left| C_1^{(0)} \right|^2}{\left| C_0^{(0)} \right|^2} = \frac{\Theta}{\left| C_0^{(0)} \right|^2}. \tag{*}$$

Let us introduce the designations:

$$\mathbf{R}(\omega) = \frac{\left| C_1^{(0)}(\omega) \right|^2}{\left| C_0^{(0)}(\omega) \right|^2} \text{— energy reflection ratio;}$$

$$\mathbf{T}(\omega) = \frac{\Theta}{\left| C_0^{(0)}(\omega) \right|^2} \text{— energy transmission ratio.}$$

Then the ratio (*) can be recorded in the form of the law of energy conservation:

$$\mathbf{R}(\omega) + \mathbf{T}(\omega) = 1.$$

Let us consider the solution of the task of the best anti-reflective of a two-layer system with a fixed frequency ω_0.

$$\mathbf{R}(\omega_0) \overrightarrow{\vec{p}, \vec{\nu}} \min$$

Due to (4) the functions $\left|C_0^{(0)}(\omega)\right|^2$ and $\left|C_1^{(0)}(\omega)\right|^2$ have extremums of equivalent meaning in appropriate points.

That is why the function $\mathbf{R}(\omega)$ will have extremums of equivalent meaning in the same points, thus the task of anti-reflection is the following:

$$\left|C_1^{(0)}(\omega)\right|^2 \overrightarrow{\vec{p}, \vec{\nu}} \min$$

For a two-layered surface the function $\left|C_1^{(0)}(\omega)\right|^2$ can be recorded in the form:

$$\mathbf{F}_{p_1,p_2}(t_1, t_2) = (\alpha_0 \mathbf{x_1 x_2} - \alpha_3 \mathbf{y_1 y_2})^2 + (\alpha_1 \mathbf{x_1 y_2} + \alpha_2 \mathbf{y_1 x_2})^2 \tag{5}$$

and depending on the value of ν_1 and ν_2 is quasi-periodic per ω, which presupposes $t_1 = \nu_1\omega,\ t_2 = \nu_2\omega$ and $\mathbf{x_i} = \cos t_i, \mathbf{y_i} = \sin t_i,\ i = 1, 2,$

$$\alpha_0 = \frac{1}{2}(1 - \Theta), \quad \alpha_1 = \frac{1}{2}(\vartheta_1\vartheta_2 - \vartheta_3)$$
$$\alpha_2 = \frac{1}{2}(\vartheta_1 - \vartheta_2\vartheta_3), \quad \alpha_3 = \frac{1}{2}(\vartheta_2 - \vartheta_1\vartheta_3)$$

$$\vartheta_1 = \frac{p_1}{p_0}, \quad \vartheta_2 = \frac{p_2}{p_1}, \quad \vartheta_3 = \frac{p_3}{p_2}, \quad \Theta = \frac{p_3}{p_0}$$

The functional behavior of $\mathbf{F}_{p_1,p_2}(t_1, t_2)$ depends substantially on the parameters $\mathbf{P} = (p_1, p_2)$.

Meanwhile the parameter space $\mathcal{P} = \{\mathbf{P}\}$, $(p_1 > 0, p_2 > 0)$ is divided into several areas, limited by divergent un-limited quadratic curves (parabolas, hyperbolas) in a way that the functional behavior of $\mathbf{F}_{p_1,p_2}(t_1, t_2)$ changes qualitatively in transition from one area to another.

An exponential change of variables $(p_1, p_2) \to (s_1, s_2)$:

$$\vartheta_1 = \Theta^{s_1+\frac{1}{2}}, \quad \vartheta_3 = \Theta^{s_2-\frac{1}{2}}, \quad \vartheta_2 = \Theta^{s_2-s_1}$$

allows to straighten the limits of the above-mentioned areas.

In the coordinates (s_1, s_2) all the important areas for an analysis of function $\mathbf{F}_{p_1,p_2}(t_1, t_2)$ are limited by straight lines as shown on fig. 1.

Function $\mathbf{F}_{p_1,p_2}(t_1,t_2)$ zeros are the solutions to the system equations:

$$\alpha_0\mathbf{x_1x_2} - \alpha_3\mathbf{y_1y_2} = 0, \quad \alpha_1\mathbf{x_1y_2} + \alpha_2\mathbf{y_1x_2} = 0. \tag{6}$$

The solutions equations (6) exist then and only then, when:

$$\begin{cases} \frac{\alpha_3}{\alpha_0} > 0 \\ \frac{\alpha_1}{\alpha_2} < 0 \end{cases} \quad \text{or} \quad \begin{cases} \frac{\alpha_3}{\alpha_0} < 0 \\ \frac{\alpha_1}{\alpha_2} > 0. \end{cases} \tag{7}$$

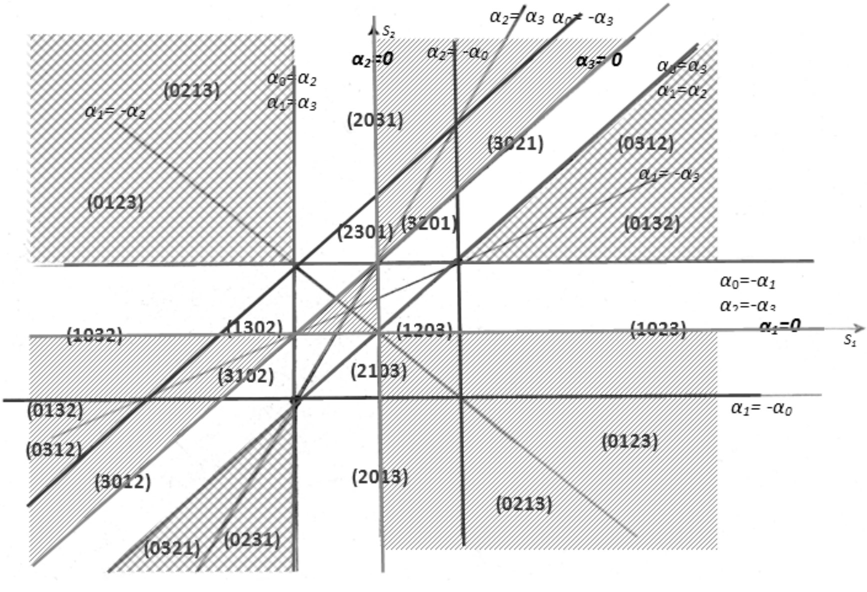

Figure 1. Parameter space s_1, s_2

On fig. 1: $t_1 = \arctan\sqrt{-\dfrac{\alpha_0\alpha_1}{\alpha_2\alpha_3}}$; $\quad t_2 = \arctan\sqrt{-\dfrac{\alpha_0\alpha_2}{\alpha_1\alpha_3}}$. Four triangular areas on space s_1, s_2 that solve inequalities (7) are shaded diagonally on fig.1.

For every point of (s_1, s_2) from these areas the function $\mathbf{F}_{p_1,p_2}(t_1,t_2)$ possesses in the above-mentioned pair of square function period points (t_1,t_2), which matches the parameters of task (s_1, s_2), the value of null.

After the fixation of parameters (p_1, p_2) the values of parameters (ν_1, ν_2) can be sought for each of the solutions (t_1, t_2) of system (6) using the following formulas:

$$\nu_1 = \frac{|t_1|}{\omega_0}; \quad \nu_2 = \frac{|t_2|}{\omega_0}.$$

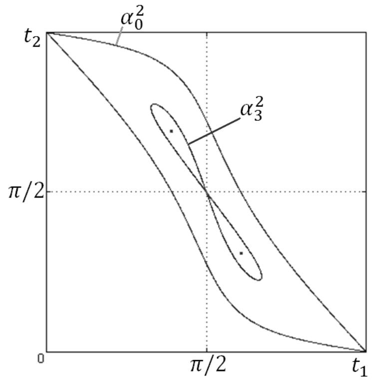

Figure 2. Three serial function $\mathbf{F}_{p_1,p_2}(t_1, t_2)$ level lines

Fig. 2 illustrates the sections of function $\mathbf{F}_{p_1,p_2}(t_1, t_2)$ with planes parallel to plane t_1, t_2 at the levels 0 (function $\mathbf{F}$ zeros), α_3^2, α_0^2.

Three angular areas on fig. 2, cross shaded, determine those values of parameters (s_1, s_2) in which anti-reflection is impossible for any frequency ω_0 of the spectrum $(0, +\infty)$.

Non-shaded areas determine those values of parameters (s_1, s_2), in which the anti-reflective task for the fixed frequency ω_0 can be solved in the same way as it is done above. But for each of these areas only a level of anti-reflection that corresponds to the least coefficient $\alpha_j^2 \neq 0$ of the area from function $\mathbf{F}_{p_1,p_2}(t_1, t_2)$ definition (5) can be achieved.

The map of function $\mathbf{F}_{p_1,p_2}(t_1, t_2)$ properties, shown on fig. 1, is also extremely useful in research and resolution of anti-reflective tasks of other formulations.

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THE ANTI-REFLECTIVE COATING FOR THE OBLIQUE INCIDENCE OF LIGHT

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Key words: anti-reflective, coating

AMS Mathematics Subject Classification: 93A30

Abstract. We consider an oblique incidence of light on the system of one anti-reflective coating.

1 Description of the mathematical model

Let us consider the propagation of classical electromagnetic waves in a medium that obeys Maxwell's equations:

$$\begin{cases} \operatorname{rot} \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \operatorname{rot} \vec{H} = \dfrac{\partial \vec{D}}{\partial t} + \vec{j} \end{cases}$$

and material equations of the simplest form:

$$\begin{cases} \vec{B} = \mu \vec{H} \\ \vec{D} = \varepsilon \vec{E} \\ \vec{j} = \vec{0} \end{cases} .$$

Here ε and μ are own for each of the mediums, and $\vec{j} = \bar{0}$ indicates the absence of currents, that is, we restrict ourselves to insulators. At the material boundaries assume the continuity of the tangential components of vectors $\vec{E}$ and $\vec{H}$.

We are interested only harmonic solutions, that is, having the form $\vec{E} = \vec{E}_r \cdot e^{-i\omega t}$ and $\vec{H} = \vec{H}_r \cdot e^{-i\omega t}$, where functions $\vec{E}_r$ and $\vec{H}_r$ depend only on spatial coordinates, but not on time. Substituting them into equations, and omitting the index r, transforming, we obtain:

$$\begin{cases} \operatorname{rot} \vec{E} = i\omega\mu\vec{H} \\ \operatorname{rot} \vec{H} = -i\omega\varepsilon\vec{E} \end{cases} \tag{1}$$

In the task of the minimization of reflection we consider a system of optical mediums. The mediums are the two half-spaces with the flat anti-reflective layers between them. In this study, we restrict ourselves to only one anti-reflective layer thickness d. We assume that the environment in a half-space $x < 0$ with the number 0 has the characteristics ε_0, μ_0, the layer $0 < x < d$ number 1 – the values ε_1, μ_1, and $x > d$ – with the number 2 values ε_2, μ_2. We also set $\varepsilon = \varepsilon_1, \mu = \mu_1$.

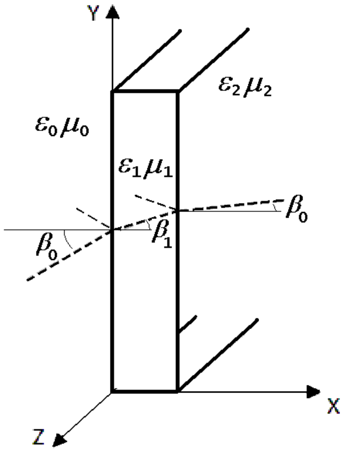

Figure 1. The choise of the coordinate axes

We choose the coordinate axes (see Fig. 1) to OX axis is directed along the normal to the layers (all layers are parallel to the boundary plane YOZ), the deviation of the incident wave at normal incidence angle $\beta = \beta_0$ will let the plane XOY, and the shift in the axis OZ field does not change. Moreover, all the layers are located in the half-space $x \geq 0$, and the boundary of the first layer coincides with the plane $x = 0$.

In the case of oblique incidence of light the wave propagation problem is divided into two independent tasks, which differ in the polarization of the incident wave:

TE-wave ($\perp$): $\vec{E} = (0, 0, E_z)$, $\vec{H} = (H_x, H_y, 0)$.

TM-wave (//): $\vec{E} = (E_x, E_y, 0)$, $\vec{H} = (0, 0, H_z)$.

Substituting in (1), we obtain the following differential equations:

TE-wave:

$$\operatorname{rot} \vec{E} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & E_z \end{vmatrix} = \mathbf{i}\frac{\partial E_z}{\partial y} - \mathbf{j}\frac{\partial E_z}{\partial x} = i\omega\mu\left(\mathbf{i}H_x + \mathbf{j}H_y\right) = i\omega\mu\vec{H};$$

$$\operatorname{rot}\vec{H}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ H_x & H_y & 0\end{vmatrix}=\mathbf{k}\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)=i\omega\varepsilon\cdot\mathbf{k}E_z=i\omega\varepsilon\vec{E}$$

We find:

$$\begin{cases}\dfrac{\partial E_z}{\partial y}=i\omega\mu H_x\\ \dfrac{\partial E_z}{\partial x}=-i\omega\mu H_y\\ \dfrac{\partial H_x}{\partial y}-\dfrac{\partial H_y}{\partial x}=i\omega\varepsilon\cdot E_z\end{cases} \tag{2}$$

Next, we transform:

$$\frac{\partial^2 E_z}{\partial y^2}=i\omega\mu\frac{\partial H_x}{\partial y}$$

$$\frac{\partial^2 E_z}{\partial x^2}=-i\omega\mu\frac{\partial H_y}{\partial x}$$

Adds:

$$\frac{\partial^2 E_z}{\partial y^2}+\frac{\partial^2 E_z}{\partial x^2}=i\omega\mu\left(\frac{\partial H_x}{\partial y}-\frac{\partial H_y}{\partial x}\right)$$

Substituting, we obtain the Helmholtz equation:

$$\frac{\partial^2 E_z}{\partial y^2}+\frac{\partial^2 E_z}{\partial x^2}=-\omega^2\varepsilon\mu E_z=-\omega^2 n^2 E_z$$

We use the method of separation of variables:

$$E_z=f(x)\cdot g(y)$$

$$f(x)\cdot g''_{yy}(y)+f''_{xx}(x)\cdot g(y)=-\omega^2\varepsilon\mu\cdot f(x)\,g(y)$$

or

$$\frac{g''_{yy}(y)}{g(y)}+\frac{f''_{xx}(x)}{f(x)}=-\omega^2 n^2$$

i.e.

$$\frac{g''_{yy}(y)}{g(y)}=-\frac{f''_{xx}(x)}{f(x)}-\omega^2 n^2$$

Since the left side is a function only of y, and the right only of x, they are constant, which is denoted by $\omega^2\gamma^2$:

$$\frac{g''_{yy}(y)}{g(y)} = -\frac{f''_{xx}(x)}{f(x)} - \omega^2 n^2 = \omega^2\gamma^2,$$

whence it is clear that: $g(y) = g_0 e^{\pm\omega\gamma y}$.

Given (2), we obtain the form of solutions:

$$\begin{cases} E_z = A(x)\cdot e^{i\omega\gamma y} \\ H_y = B(x)\cdot e^{i\omega\gamma y} \\ H_x = C(x)\cdot e^{i\omega\gamma y} \end{cases}$$

Substituting into (2) and setting $E_z(x,0,0) = E(x)$ and $H_y(x,0,0) = H(x)$, transforming we obtain:

$$\begin{pmatrix} E \\ H \end{pmatrix}' = \begin{pmatrix} 0 & i\omega\mu \\ i\omega\varepsilon\cos^2\beta & 0 \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix}$$

The characteristic polynomial:

$$|A-\lambda E| = \begin{vmatrix} -\lambda & i\omega\mu \\ i\omega\varepsilon\cos^2\beta & -\lambda \end{vmatrix} = \lambda^2 + \omega^2\varepsilon\mu\cos^2\beta$$

roots: $\lambda_{1,2} = \pm i\omega\sqrt{\varepsilon\mu}\cos\beta$ general solution:

$$\begin{pmatrix} E \\ H \end{pmatrix} = C_0\begin{pmatrix} 1 \\ \overset{\vee}{p} \end{pmatrix} e^{i\omega\overset{\vee}{n}(x-x_0)} + C_1\begin{pmatrix} 1 \\ -\overset{\vee}{p} \end{pmatrix} e^{-i\omega\overset{\vee}{n}(x-x_0)}$$

where:

$$n = \sqrt{\varepsilon\mu};\ p = \sqrt{\frac{\varepsilon}{\mu}},\quad \overset{\vee}{n} = n\cos\beta;\ \overset{\vee}{p} = p\cos\beta$$

<u>TM-wave:</u>

Similarly to the case of TE, we obtain:

$$\begin{pmatrix} E \\ H \end{pmatrix}' = \begin{pmatrix} 0 & i\omega\mu\cos^2\beta \\ i\omega\varepsilon & 0 \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix}$$

The characteristic polynomial:

$$|A-\lambda E| = \begin{vmatrix} -\lambda & i\omega\mu\cos^2\beta \\ i\omega\varepsilon & -\lambda \end{vmatrix} = \lambda^2+\omega^2\varepsilon\mu\cos^2\beta$$

roots: $\lambda_{1,2} = \pm i\omega\sqrt{\varepsilon\mu}\cos\beta$ general solution:

$$\begin{pmatrix} E \\ H \end{pmatrix} = C_0\begin{pmatrix} 1 \\ q \end{pmatrix} e^{i\omega\overset{\vee}{n}(x-x_0)} + C_1\begin{pmatrix} 1 \\ -q \end{pmatrix} e^{-i\omega\overset{\vee}{n}(x-x_0)}$$

where:

$$n=\sqrt{\varepsilon\mu};\ p=\sqrt{\frac{\varepsilon}{\mu}},\quad \overset{\vee}{n}=n\cos\beta;\ q=\frac{p}{\cos\beta}.$$

Let the plane separates the mediums j and $j+1$. Matching the solutions at interfaces is provided by Snell's law:

$$n_0\sin\beta_0 = n_1\sin\beta_1 = n_2\sin\beta_2,$$

which is obtained by additional study of the system (2), and the condition of continuity of the tangential component of the field:

$$\begin{cases} E_\tau^{(j)}(a_j-0) = E_\tau^{(j+1)}(a_j+0) \\ H_\tau^{(j)}(a_j-0) = H_\tau^{(j+1)}(a_j+0) \end{cases}$$

For TE-wave:

$$\begin{pmatrix} E^{(j)} \\ H^{(j)} \end{pmatrix} = C_0^{(j)}\begin{pmatrix} 1 \\ p^{(j)}\cdot\cos\beta_j \end{pmatrix} e^{i\omega n^{(j)}(x-x_0)\cos\beta_j} +$$
$$+C_1^{(j)}\begin{pmatrix} 1 \\ -p^{(j)}\cdot\cos\beta_j \end{pmatrix} e^{-i\omega n^{(j)}(x-x_0)\cos\beta_j},$$

If $x=a_j$, we have:

$$M_j^+\begin{pmatrix} C_0^{(j)} \\ C_1^{(j)} \end{pmatrix} = M_{j+1}^-\begin{pmatrix} C_0^{(j+1)} \\ C_1^{(j+1)} \end{pmatrix},$$

where

$$M_j^+ = \begin{pmatrix} 1 & 1 \\ p^{(j)}\cdot\cos\beta_j & -p^{(j)}\cdot\cos\beta_j \end{pmatrix},$$

and

$$M_{j+1}^{-} = \begin{pmatrix} 1 & 1 \\ p^{(j+1)} \cdot \cos\beta_j & -p^{(j+1)} \cdot \cos\beta_j \end{pmatrix}.$$

When passing through the layer thickness d_1 variation of the field provides a matrix:

$$M_j^{-} = e^{-i\omega n_1 d_1 \cos\beta_1} \cdot M_j^{+}$$

that leads to the relation:

$$\begin{pmatrix} C_0^{(0)} \\ C_1^{(0)} \end{pmatrix} = P_1 \begin{pmatrix} C_0^{(1)} \\ C_1^{(1)} \end{pmatrix} = P_1 P_2 \begin{pmatrix} C_0^{(2)} \\ C_1^{(2)} \end{pmatrix},$$

where the superscripts in parentheses indicate number of medium and

$$P_j = \left(M_{j+1}^{+}\right)^{-1} M_j^{-}.$$

We introduce the following notation:

$$\vartheta = \frac{p_2}{p_0};\ \vartheta_1 = \frac{p_1}{p_0};\ \vartheta_2 = \frac{p_2}{p_1}$$

$$h = \frac{\cos\beta_2}{\cos\beta};\ h_1 = \frac{\cos\beta_1}{\cos\beta};\ h_2 = \frac{\cos\beta_2}{\cos\beta_1}$$

and $\nu = \nu_1 = n_1 d$ and $\nu_2 = 0$.

Let $C_0^{(2)} = 1$, and $C_1^{(2)} = 0$.

TE-wave:

$$P_j = \frac{1}{2} \begin{pmatrix} (1+\vartheta_j h_j) & (1-\vartheta_j h_j) \\ (1-\vartheta_j h_j) & (1+\vartheta_j h_j) \end{pmatrix} \begin{pmatrix} e^{-i\omega\,\nu_j \cos\beta_j} & 0 \\ 0 & e^{i\omega\,\nu_j \cos\beta_j} \end{pmatrix}$$

where $j = 1, 2$.

TM-wave:

$$P_j = \frac{1}{2} \begin{pmatrix} \left(1+\frac{\vartheta_j}{h_j}\right) & \left(1-\frac{\vartheta_j}{h_j}\right) \\ \left(1-\frac{\vartheta_j}{h_j}\right) & \left(1+\frac{\vartheta_j}{h_j}\right) \end{pmatrix} \begin{pmatrix} e^{-i\omega\nu_j \cos\beta_j} & 0 \\ 0 & e^{i\omega\,\nu_j \cos\beta_j} \end{pmatrix},$$

where $j = 1,\ 2,$

Noting the important fact: $P_j^* J P_j = \overset{\vee}{\vartheta}_j J$, where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we have:

$$\begin{pmatrix} \bar{C}_0^{(j+1)} & \bar{C}_1^{(j+1)} \end{pmatrix} P_j^* J P_j \begin{pmatrix} C_0^{(j+1)} \\ C_1^{(j+1)} \end{pmatrix} = \overset{\vee}{\vartheta}_j J,$$

where

$$\overset{\vee}{\vartheta}_j = \frac{\overset{\vee}{p}_{j+1}}{\overset{\vee}{p}_j}$$

taking into account

$$\begin{pmatrix} C_0^{(j)} \\ C_1^{(j)} \end{pmatrix} = P_{j+1} \begin{pmatrix} C_0^{(j+1)} \\ C_1^{(j+1)} \end{pmatrix}$$

also

$$\begin{pmatrix} \bar{C}_0^{(j)} & \bar{C}_1^{(j)} \end{pmatrix} J \begin{pmatrix} C_0^{(j)} \\ C_1^{(j)} \end{pmatrix} = \overset{\vee}{\vartheta}_j J$$

we have: $\overset{\vee}{p}_j \left(\left| C_0^{(j)} \right|^2 - \left| C_1^{(j)} \right|^2 \right) = \overset{\vee}{p}_k \left(\left| C_0^{(k)} \right|^2 - \left| C_1^{(k)} \right|^2 \right)$, for all j and k. Then we find:

$$\left| C_1^{(0)} \right|^2 = \alpha_0^2 \cos^2 t_* + \alpha_1^2 \sin^2 t_*,$$

where

<u>TE-wave:</u> $\overset{\vee}{\vartheta} = \vartheta h$; $\alpha_0^{\perp} = \frac{1}{2}\left(1 - \vartheta h\right)$; $\alpha_1^{\perp} = \frac{1}{2}\left(\vartheta_2 h_2 - \vartheta_1 h_1\right)$

<u>TM-wave:</u> $\overset{\vee}{\vartheta} = \frac{\vartheta}{h}$; $\alpha_0^{//} = \frac{1}{2}\left(1 - \frac{\vartheta}{h}\right)$; $\alpha_1^{//} = \frac{1}{2}\left(\frac{\vartheta_2}{h_2} - \frac{\vartheta_1}{h_1}\right)$

and $\nu = n_1 d, \quad t = \nu\omega; \quad t_* = t\cos\beta_1 = \omega d\sqrt{n_1^2 - n_0^2 \sin^2\beta}$.

2. The problem of best uniform anti-reflective effect in a range of frequencies $[\Omega_1, \Omega_2]$ at a constant angle of incidence.

The energy reflection coefficient be the quantity:

$$R = \frac{\left| C_1^{(0)} \right|^2}{\left| C_0^{(0)} \right|^2} = \frac{\left| C_1^{(0)} \right|^2}{\left| C_1^{(0)} \right|^2 + \overset{\vee}{\vartheta}_j}.$$

The task: by selecting a values of p and ν minimize the functional:

$$\max_{\omega \in [\Omega_1, \Omega_2]} R(\omega) \tag{3}$$

We can establish that the problem reduces to the problem of minimizing the easier functional:

$$\max_{\omega\in[\Omega_1,\Omega_2]}\left|C_1^{(0)}\right|^2 \tag{4}$$

Indeed, the expression

$$\frac{\left|C_1^{(0)}(\omega_1)\right|^2}{\left|C_1^{(0)}(\omega_1)\right|^2+\overset{\vee}{\vartheta}_j}<\frac{\left|C_1^{(0)}(\omega_2)\right|^2}{\left|C_1^{(0)}(\omega_2)\right|^2+\overset{\vee}{\vartheta}_j}$$

is equivalent to

$$\left|C_1^{(0)}(\omega_1)\right|^2\left|C_1^{(0)}(\omega_2)\right|^2+\left|C_1^{(0)}(\omega_1)\right|^2\overset{\vee}{\vartheta}_j<\left|C_1^{(0)}(\omega_2)\right|^2\left|C_1^{(0)}(\omega_1)\right|^2+\left|C_1^{(0)}(\omega_2)\right|^2\overset{\vee}{\vartheta}_j$$

since the denominators are positive.

Cancelling terms and factors, we have

$$\left|C_1^{(0)}(\omega_1)\right|^2<\left|C_1^{(0)}(\omega_2)\right|^2.$$

Theorem 1. *1. For any value of parameter p from the range $[p_0,p_2]$ there is a finite amount of values ν at which the functional $\max\limits_{\omega\in[\Omega_1,\Omega_2]}R(\omega)$ has a local minimum.*

2. For any $p\in[p_0,p_2]$ the points of minimum values of the functional $\max\limits_{\omega\in[\Omega_1,\Omega_2]}R(\omega)$ are strictly in order of magnitude, whereby there is always a single global minimum which is reached when $\overset{\vee}{\nu}=\frac{\pi}{\Omega_1+\Omega_2}$, where $\overset{\vee}{\nu}=\nu\cos\beta_1$.

3. The global minimum of functional $\max\limits_{\omega\in[\Omega_1,\Omega_2]}R(\omega)$ will be the best of possible when $\overset{\vee}{p}=\sqrt{\overset{\vee}{p}_0\overset{\vee}{p}_2}$.

Proof. Using the previously noticed property will be in place (3) use (4).

$$\left|C_1^{(0)}\right|^2=\alpha_0^2\cos^2 t_*+\alpha_1^2\sin^2 t_*$$

Select the areas in which the *condition of enlightenment*:

$$\alpha_0^2\cos t_*+\alpha_1^2\sin t_*<\alpha_0^2$$

is satisfied for t_*, then we obtain:

$$\left(\alpha_0^2 - \alpha_1^2\right)\sin^2 t_* > 0$$

Now do we get: 1°. $\sin^2 t_* \neq 0$, i.e. $t_* \neq k\pi$, $k = 0, 1, 2\ldots$

2°. $\left(\alpha_0^2 - \alpha_1^2\right) > 0$

Given that $\vartheta_1\vartheta_2 = \vartheta$ and $h_1 h_2 = h$ we get:

I. ($\perp$): $\left(\alpha_0^{\perp}\right)^2 > \left(\alpha_1^{\perp}\right)^2$

$$(1 - \vartheta h)^2 > (\vartheta_2 h_2 - \vartheta_1 h_1)^2\,,$$

or

$$\left(1 - \vartheta_1^2 h_1^2\right)\left(1 - \vartheta_2^2 h_2^2\right) > 0$$

II. (//): $\left(\alpha_0^{//}\right)^2 > \left(\alpha_1^{//}\right)^2$

$$\left(1 - \frac{\vartheta}{h}\right)^2 > \left(\frac{\vartheta_2}{h_2} - \frac{\vartheta_1}{h_1}\right)^2,$$

or

$$\left(1 - \frac{\vartheta_1^2}{h_1^2}\right)\left(1 - \frac{\vartheta_2^2}{h_2^2}\right) > 0$$

The condition of enlightenment will be satisfied in whole range of frequencies $[\Omega_1, \Omega_2]$ if and only if:

$$\frac{k\pi}{\overset{\vee}{\nu}} < \Omega_1 < \Omega_2 < \frac{(k+1)\,\pi}{\overset{\vee}{\nu}}$$

or

$$\frac{k\pi}{\Omega_1} < \overset{\vee}{\nu} < \frac{(k+1)\,\pi}{\Omega_2}$$

whence

$$\frac{\Omega_2}{\Omega_1} < 1 + \frac{1}{k}.$$

It is easy to notice an increase in the functional (4) with growth k.

The best antireflective effect obtained when $\alpha_1^2 = 0$, i.e.

$$\overset{\vee}{p} = \sqrt{\overset{\vee}{p}_0 \overset{\vee}{p}_2}$$

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INCREASING SMOOTHNESS PROPERTY OF SOLUTIONS TO MIXED HYPERBOLIC PROBLEMS ON THE PLANE

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Key words: First-order hyperbolic systems in two variables x, t, wave equation, initial-boundary problems, increasing smoothness of the solutions.

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Abstract. The initial-boundary problems for first-order hyperbolic systems and for the wave equation are considered in the half-strip $\Pi = \{(x,t) : 0 < x < 1, t > 0\}$. Boundary conditions which guarantee the increasing of smoothness of the solutions to problems under consideration as t grows are formulated.

1 Introduction

Mixed problems for hyperbolic systems in two independent variables arise in the mathematical modeling of physical and chemical processes in connection with the phenomena of warm-and mass-transfer. An extensive literature (see the references in [1]– [5]) is devoted to studying the qualitative properties of solutions to these problems (existence of global (in t) solutions, existence of periodic solutions, propagation of discontinuities, bifurcation of solutions, stability of solutions, fredholmness for linear hyperbolic periodic-Dirichlet problems and so on). This article is a survey of the results established by the author, concerning the increasing smoothness property of solutions to some mixed problems for hyperbolic systems in two variables x, t. Full proofs of the results stated below one can found in the references to this article.

It is well-known that the smoothness of solutions to the Cauchy problem for the simplest hyperbolic system of two equations with constant coefficients

$$u_t + u_x = au + bv, \quad v_t - v_x = cu + dv,$$
$$u|_{t=0} = u_0(x), \quad v|_{t=0} = v_0(x)$$

and for the wave equation

$$u_{tt} - u_{xx} = 0, \quad u|_{t=0} = u_0(x),\, u_t|_{t=0} = u_1(x)$$

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is not higher than the smoothness of the initial data. If the point $x = x_0$ is the jump discontinuity of the initial data, this discontinuity propagates along both characteristics $t = x - x_0$ and $t = -x + x_0$ of these problems and will exist at any time $t > 0$.

In this work we consider the well-posed initial-boundary value problems for first-order hyperbolic systems and for the wave equation in the half-strip $\Pi = \{(x,t) : 0 < x < 1, t > 0\}$. We formulate the boundary conditions which guarantee the increasing smoothness property of the solutions to problems under consideration as t grows.

Henceforth, the containment of $F(x,t)$ in $C^{k,m}_{x,t}(\bar{\Pi})$ is understood as follows: $F(x,t) \in C^{k,m}_{x,t}(\bar{\Pi}_T)$ for any $T > 0$, where $\bar{\Pi}_T = \{(x,t) : 0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant T\}$. We denote by K and A the constants depending on the coefficients of problem and independent of t and $U_0(x)$.

2 Mixed problems for hyperbolic systems

In the half-strip $\Pi = \{(x,t) : 0 < x < 1, t > 0\}$ we consider the following problem:

$$U_t - L_{\mathcal{A}} U = F(x,t), \quad (x,t) \in \Pi, \tag{1}$$

$$I_0 U(0,t) + I_1 U(1,t) = 0, \quad U(x,0) = U_0(x). \tag{2}$$

Here $U(x,t)$ is an n-dimensional column-vector of unknown functions, $F(x,t)$ is the n-dimensional vector of right hand part.

$$L_{\mathcal{A}} U = -\mathcal{K}(x) U_x + \mathcal{A}(x) U, \quad \mathcal{A}(x) = (a_{ij}(x))_{i,j=1,\dots,n},$$

$\mathcal{K}(x)$ is the diagonal matrix with entries $k_i(x) \neq k_j(x)$ $(i \neq j)$, the first p of them are positive, and the rest $n-p$ of them are negative, moreover $1 \leqslant p < n$, $n \geqslant 2$.

The boundary conditions are reflection boundary conditions, i.e.

$$I_0 = \begin{pmatrix} 1 & . & 0 & \alpha_{1,p+1} & . & \alpha_{1,n} \\ . & . & . & . & . & . \\ 0 & . & 1 & \alpha_{p,p+1} & . & \alpha_{p,n} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad I_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{p+1,1} & . & \beta_{p+1,p} & 1 & . & 0 \\ . & . & . & . & . & . \\ \beta_{n,1} & . & \beta_{n,p} & 0 & . & 1 \end{pmatrix}.$$

In [1] the question of well-posedness of problem (1), (2) in spaces of continuous, continuously differentiable and integrable functions is investigated. Let $\mathcal{A}(x)$, $\mathcal{K}(x)$, $U_0(x) \in C^1[0,1]$, $F(x,t) \in C^{1,0}_{x,t}(\bar{\Pi})$, then one can see that the continuously differentiable function $U(x,t)$ is a classical solution of the problem if and only if it is a solution of integral system, resulting from (1) by integrating along the corresponding chsrscteristics. A necessary condition for the differentiability of a solution in the half-strip Π is the fulfillment of the zero-order compatibility conditions

$$I_0 U_0(0) + I_1 U_0(1) = 0 \tag{3}$$

and the first-order compatibility conditions

$$I_0 U_1(0) + I_1 U_1(1) = 0, \quad \text{where} \quad U_1(x) = L_{\mathcal{A}} U_0(x) + F(x,0). \tag{4}$$

Theorem 1 (see [1]). *$\mathcal{A}(x)$, $\mathcal{K}(x) \in C^1[0,1]$, $F(x,t) \in C^{1,0}_{x,t}(\bar{\Pi})$ and the initial data $U_0(x) \in C^1[0,1]$ satisfies the compatibility conditions* (3), (4)*. Then problem* (1), (2) *has a unique continuously differentiable solution $U(x,t)$ in the half-strip Π; moreover, for $t > 0$ it satisfies the estimate*

$$||U(x,t)||_{C^1[0,1]} \leqslant K e^{At} \left(||U_0||_{C^1[0,1]} + \max_{0 \leqslant \tau \leqslant t} ||F(x,\tau)||_{C^1[0,1]} \right).$$

In [6] the notion of piecewise smooth solution (PSS) of problem (1), (2) was introduced by the author. This solution is a solution of corresponding integral system if $\mathcal{A}(x), U_0(x)$, $\mathcal{K}(x) \in C^1[0,1]$, $F(x,t) \in C^{1,0}_{x,t}(\bar{\Pi})$. If the compatibility conditions (3), (4) are not fulfilled, then PSS $U(x,t)$ of problem and its derivatives are discontinuous on the set of characteristics of system (1) which is not more then countable. At the points where the derivatives exist, the function $U(x,t)$ satisfies system (1). If the compatibility conditions (3), (4) are fulfilled, then the piecewise smooth solution is a classical solution of problem (1), (2).

Definition 1. We say that problem (1), (2) possesses the increasing smoothness property up to the order k if there exists a number $T(k) > 0$ such that every PSS $U(x,t)$ to problem (1), (2) is k times continuously differentiable for $t > T(k)$.

In other words, problem (1), (2) possesses the increasing smoothness property up to order k if there exist such numbers $T(0)$, $T(1)$,. . . ,$T(k)$,.. that for any function $U_0(x) \in C^1[0,1]$ PSS $U(x,t)$ becomes continuous for $t > T(0)$, it becomes continuously differentiable for $t > T(1)$ and becomes k-times continuously differentiable for $t > T(k)$.

In [6] a class of the so-called B-regular boundary conditions was described. The mentioned conditions are necessary and sufficient for linear homogeneous problem (1), (2) to possess the increasing smoothness property up to the order k, which is determined by the smoothness of the matrices $\mathcal{K}(x)$ and $\mathcal{A}(x)$ and does not depend on the smoothness of the initial data.

We consider the diagonal matrix

$$\mathcal{T}(x,\lambda) = (e^{\lambda \mathcal{T}_j(x)+\mathcal{B}_j(x)}\delta_{ij})_{i,j=1,\ldots,n},$$

where

$$\mathcal{T}_j(x) = \int\limits_0^x \frac{-1}{k_j(\xi)}\,d\xi, \quad \mathcal{B}_j(x) = \int\limits_0^x \frac{a_{jj}(\xi)}{k_j(\xi)}\,d\xi,$$

δ_{ij} is the Kronecker symbol, λ is a complex parameter, and introduce the expression $X(\lambda) = I_0 + I_1\mathcal{T}(1,\lambda)$. We have that

$$\det X(\lambda) = e^{\lambda \sum\limits_{i=p+1}^{n} \mathcal{T}_i + \sum\limits_{i=p+1}^{n} \mathcal{B}_i}\,\Delta(\lambda),$$

where $\mathcal{T}_i = \mathcal{T}_i(1)$, $\mathcal{B}_i = \mathcal{B}_i(1), i = 1,\ldots,n$,

$$\Delta(\lambda) = 1 + \sum_{k=1}^{M} E_k e^{-\lambda\beta_k}. \tag{5}$$

Here E_k are real numbers determined by entries of the matrices I_0 and I_1 and numbers $\mathcal{B}_i$; numbers $0<\beta_1<\ldots<\beta_M$ are determined via $\mathcal{T}_i$; $i = 1,\ldots,n$.

Definition 2. Boundary conditions (2) for problem (1) are called B-regular if $E_k = 0$ $(k = 1,\ldots,M)$ in (5), i.e. $\Delta(\lambda) \equiv 1$.

Theorem 2 (see [6]). *Let $F(x,t) \equiv 0$, $\mathcal{A}(x)$, $\mathcal{K}(x) \in C^{k+2}[0,1]$. Then B–regularity of boundary conditions* (2) *is the necessary and sufficient condition for problem* (1), (2) *to possess the increasing smoothness property of PSS $U(x,t)$ up to the order k for arbitrary initial data $U_0(x) \in C^1[0,1]$; $k = 0,1,\ldots.$ The solution $U(x,t)$ for $t > T(k)$ satisfies the estimate*

$$\|D^{\alpha,\beta}_{x,t}U(x,t)\|_{C[0,1]} \leqslant K(t)\|U_0(x)\|_{L_2[0,1]}, \tag{6}$$

where $\alpha+\beta \leqslant k$, the constant $K(t)$ is independent of the initial data and depends on the coefficients of the problem and t.

The proof of this theorem is based on a study for $|\lambda| \to \infty$ asymptotic properties of function $\tilde{U}(x,\lambda) = \int_0^\infty U(x,t)e^{-\lambda t}\,dt$, which is the Laplace transform of solution $U(x,t)$ to the original problem. A similar approach to problems with reflection boundary conditions was used earlier in [7, 8].

Let's designate the operator $L_A : L_2(0,1) \to L_2(0,1)$,

$$L_A Y = L_{\mathcal{A}} Y, \quad D(L_A) = \{Y \in W_2^1(0,) : I_0 Y(0) + I_1 Y(1) = 0\}.$$

The spectrum of this operator consists only of eigenvalues, which can have only infinity as their limit point. We'll call these eigenvalues the eigenvalues of hyperbolic problem (1), (2). We should point out that the increasing smoothness property of the linear problems depends on the behaviuor of eigenvalues of these problems in infinity.

If conditions of the theorem 1 are fulfilled, then we obtain the following presentation for $\tilde{U}(x,\lambda)$ in the domain of its analyticity by $|\lambda| \to \infty$:

$$\tilde{U}(x,\lambda) = \sum_{n=1}^{k+1} \frac{\tilde{U}_n(x,\lambda)}{\lambda^n} + \frac{\tilde{U}_{k+2}(x,\lambda)}{\lambda^{k+2}}, \quad |U_{k+2}(x,\lambda)| \leqslant K. \tag{7}$$

Due to the smoothness of coefficients $\mathcal{A}(x)$ and $\mathcal{K}(x)$, the inverse Laplace transform of the last summand in (7) is k times differentiable function by t for $t \geqslant 0$.

We consider the inverse Laplace transform of the main part of the asymptotic of function $\tilde{U}(x,\lambda)$, i.e of the first summand $\frac{\tilde{U}_1(x,\lambda)}{\lambda}$ in (7). The function $\tilde{U}_1(x,\lambda)$ is the sum of the following expressions $\dfrac{e^{\lambda\varphi(x)}\psi(x)}{\Delta(\lambda)}$, where $\varphi(x)$, $\psi(x)$ are smooth functions and for $\varphi(x)$ inequality $-T(0) \leqslant \varphi(x) \leqslant 0$ is valid, $T(0) \geqslant 0$; $\Delta(\lambda)$ - is either the Dirichlet polynomial or its square.

The reasoning to foolow is based on the well-known formula taken from Laplace transform theory

$$\frac{1}{2\pi i} \int\limits_{a-i\infty}^{a+i\infty} \frac{e^{\lambda(t-\tau)}}{\lambda}\,d\lambda = \begin{cases} 1, & t > \tau, \\ 0, & t < \tau, \end{cases}$$

$a > 0$. Two situations are possible.

1. If $\Delta(\lambda) \not\equiv 1$, then an infinite number of eigenvalues of the problem lie in the strip parallel to the imaginary axis. If $|\lambda| \to \infty$, these eigenvalues tend to the roots of the Dirichlet polynomial $\Delta(\lambda)$. In this case $\tilde{U}_1(x,\lambda)$ is meromorphic in λ function. If the zero-order compatibility conditions for the initial data are not fulfilled, the

inverse Laplace transform of the first summand in (7) has discontinuities on the infinite number of characteristic lines of the system (1). Moreover, the number of such characteristic lines is infinite on every set $\Pi \setminus \{[0,1] \times [0,t]\}$, $t > 0$. Really, this exspression

$$\frac{1}{\Delta(\lambda)} = \sum_{k=1}^{\infty} D_k e^{-\lambda d_k}, \quad 0 = d_1 < d_2 < \ldots < d_k < \ldots, \quad d_k \to \infty,$$

is the Dirichlet series. Therefore the inverse Laplace transform of $\tilde{U}_1(x, \lambda)$ is a function discontinuous on curves $t + \varphi(x) - d_k = 0, \quad k = 1, 2, \ldots$, because

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{\lambda(t+\varphi(x))}\psi(x)}{\lambda} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \sum_{k=1}^{\infty} D_k \frac{e^{\lambda(t+\varphi(x)-d_k)}\psi(x)}{\lambda} \, d\lambda.$$

2. If $\Delta(\lambda) \equiv 1$, then the following estimate is true for eigenvalues λ of the problem under consideration as $|\lambda| \to \infty$:

$$Re\lambda \leqslant -q \ln |Im\lambda|, \quad q > 0.$$

In this case $\tilde{U}_1(x, \lambda)$ is entire in λ function of the exponential type. The inverse Laplace transform of the first summand in (7) is infinitely differentiable by t function for $t > T(0)$, since it is the sum of such expressions

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{\lambda(t+\varphi(x))}\psi(x)}{\lambda} \, d\lambda = \psi(x), \quad t > T(0).$$

So, in case of B-regular conditions it is proved that there is such a number $T(k) \geqslant 0$ that the inverse Laplace transform of the first summand (of the whole sum) in (7) is infinitely differentiable by t function for large $t > T(k)$. The inverse Laplace transform of function $\tilde{U}(x, \lambda)$ is the PSS $U(x, t)$ to the original homogeneous problem, which satisfies the differential system (1) a.e. in Π. Consequently, if $t > T(k)$, the function $U(x, t)$ is k-times differentiable.

We note that for the system of two equations

$$\begin{gathered} u_t + k_1(x)u_x = a(x)u + b(x)v, \quad v_t - k_2(x)v_x = c(x)u + d(x)v, \\ u|_{x=0} = \alpha v|_{x=0}, \quad v|_{x=1} = \beta u|_{x=1}, \\ u|_{t=0} = u_0(x), \quad v|_{t=0} = v_0(x), \end{gathered}$$

boundary conditions are B-regular if and only if $\alpha\beta = 0$.

It is shown in [9] that the increasing smoothness property also takes place for nonhomogeneous linear problem (1), (2) and some nonlinear hyperbolic systems as well.

The boundary conditions with time delay

$$\sum_{k=0}^{\mathfrak{m}} (A_k U(0, t-\tau_k) + B_k U(1, t-\tau_k)) + \sum_{k=1}^{\mathfrak{m}} \left(\int_0^{\tau_k} \sum_{r=0,1} \Phi_k^r(\xi) U(r, t-\xi)\, d\xi \right) = 0, \tag{8}$$

for linear problem (1) was considered in [10]. The delay times τ_k in (8) are fixed real numbers: $0 = \tau_0 < \tau_1 < \ldots < \tau_{\mathfrak{m}}$, $\mathfrak{m} \geqslant 0$. A_k and B_k are $n \times n$ real matrices, $k = 0, 1, \ldots, \mathfrak{m}$. The entries of matrices $\Phi_k^r(\xi)$ are the smooth functions on the corresponding intervals $[0, \tau_k]$ ($r = 0, 1$; $k = 1, \ldots, \mathfrak{m}$). The initial data $\bar{U}(x, t)$ is given on the set Γ

$$U(x, t)|_{\Gamma} = \bar{U}(x, t), \tag{9}$$

which guarantees the well-posedness of the initial-boundary value problem.

In [10] the existence of classical solution to (1), (8), (9) in the half-strip Π is proved. In [11] a class of the so-called P-regular boundary conditions (8) was described, for which the corresponding homogeneous linear problem (1), (8), (9) possesses the increasing smoothness property.

3 Mixed problems for wave equation

The above results for hyperbolic systems allow to define a class of boundary conditions given on the lateral sides of Π for the wave equation. These conditions are necessary and sufficient conditions for every solution of

$$\begin{gathered} u_{tt} - a^2 u_{xx} = f(x, t) \quad (x, t) \in \Pi, \quad (a > 0), \\ u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x) \end{gathered} \tag{10}$$

to be a function of $C^k[0, 1]$ as t grows, if $u_0(x) \in C^3[0, 1]$, $u_1(x) \in C^2[0, 1]$, $f(x, t) \in C^\infty(\overline{\Pi})$, where k is an arbitrary natural number [12].

For the wave equation (10) on the lateral sides of Π we set the boundary conditions

$$u_t - \alpha(u_t - a u_x)|_{x=1} = 0, \quad u_t - a u_x - \beta u|_{x=0} = 0 \tag{11}$$

or

$$u_t - \alpha(u_t + a u_x)|_{x=0} = 0, \quad u_t + a u_x - \beta u|_{x=1} = 0. \tag{12}$$

Theorem 3. *Let $f(x,t)$ be $k+2$ times continuously differentiable in $\overline{\Pi}$ function; $k \geqslant 0$. Then $\alpha\beta = 0$ is the necessary and sufficient condition for problems* (10), (11) *and* (10), (12) *to possess the increasing smoothness property of PSS $u(x,t)$ up to the order k for arbitrary initial data $u_0(x) \in C^3[0,1]$, $u_1(x) \in C^2[0,1]$. The solution $u(x,t)$ for $t > T(k)$ satisfies the estimate*

$$\|D^{\alpha,\beta}_{x,t} u(x,t)\|_{C[0,1]} \leqslant K(t)(\max(\|u_0(x)\|_{W^1_2(0,1)}, \|u_1(x)\|_{L_2(0,1)})+ \\ + ||f(x,t)||_{C^k([0,1]\times[0,t])}),$$

where $\alpha + \beta \leqslant k$, the constant $K(t)$ is independent of u_0, u_1, f and depends on the coefficients of the problem and t.

More complete review of the author's results on this topic can be found in [13].

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ON NUMERICAL SOLUTION OF SOME CONVOLUTION EQUATIONS

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Key words: convolution equation, Calderon-Zygmund kernel

AMS Mathematics Subject Classification: 44A35

Abstract. Digitization problem and numerical solution for convolution equations (including Calderon- Zygmund operators) are considered. In contrast to projection methods one suggests to use the fast Fourier transform. It permits to obtain a theoretical basis for such approximation and serious prospects concerning the time of calculations. The last is essentially actual for large dimensions. Test examples are considered, some illustrations are given.

1 Discrete Singular Operators

Let's consider multidimensional singular integral equation

$$au(x) + \int\limits_{R^m} K(x, x-y)u(y)dy = v(x), \quad x \in R^m \tag{1}$$

where $K(x,y)$ is Calderon-Zygmund kernel, i.e. the function defined and differentiable on $\dot{R}^m \times (R^m \backslash \{0\})$, with the following properties:

$$1)\ K(x,ty) = t^{-m}K(x,y), \quad \forall x \in \dot{R}^m, \quad \forall t > 0;$$

$$2) \int\limits_{S^{m-1}} K(x,\omega)d\omega = 0, \quad \forall x \in \dot{R}^m;$$

$\dot{R}^m$ is compactification of R^m, S^{m-1} denotes the unit sphere in R^m. Solvability theory for such equations is established sufficiently [1,2]. We stop here on simplest discrete variant of such equation, namely, the kernel $K(x,y)$ doesn't depend on pole x, and integral is over whole-space R^m or half-space$R^m_+ = \{x \in R^m \colon x = (x_1, \dots, x_m), x_m > 0\}$

$$(Ku)(x) \equiv au(x) + \int\limits_{R^m} K(x-y)u(y)dy = v(x), \quad x \in R^m. \tag{2}$$

We set $K(0) = 0$, and write the discrete equation (see also [6])

$$au_d^h(\tilde{x}) + \sum_{\tilde{y} \in Z_h^m} K(\tilde{x} - \tilde{y}) u_d^h(\tilde{y}) h^m = v_d^h(\tilde{x}), \quad \tilde{x} \in Z_h^m, \tag{3}$$

where Z_h^m is integer lattice (mod h) in m-dimensional space R^m.

Theorem 1. *The equations* (2) *and* (3) *are solvable at the same time for arbitrary right-hand side* $v(x) \in L_2(R^m)$, $v_d^h(\tilde{x}) \in L_2(Z_h^m)$.

Let's denote $Z_{h,+}^m = \{\tilde{x} \in Z_h^m : \tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_m) \tilde{x}_m > 0\}$, $v_d^h \in L_2(Z_{h,+}^m)$.

The symbol of operator

$$K_d^h \colon u_d^h(\tilde{x}) \mapsto au_d^h(\tilde{x}) + \sum_{\tilde{y} \in Z_h^m} K(\tilde{x} - \tilde{y}) u_d^h(\tilde{x}) h^m, \quad \tilde{x} \in Z_h^m,$$

is called the function $\sigma_h(\xi) = a + \sigma'_h(\xi)$ defined on $[-h^{-1}\pi,\, h^{-1}\pi]^m$:

$$\sigma'_h(\xi) = \lim_{N \to \infty} \sum_{\tilde{y} \in Q_N} K(\tilde{y}) e^{i\tilde{y} \cdot \xi} h^m. \tag{4}$$

If we define a projector $P_{h,+}$ by formula

$$(P_{h,+} u_d^h)(\tilde{x}) = \begin{cases} u_d^h(\tilde{x}), & \tilde{x} \in Z_{h,+}^m \\ 0, \ \tilde{x} \notin Z_{h,+}^m \end{cases},$$

then the equation (3) can be rewritten in operator form

$$P_{h,+} K_d^h u_{d,+}^h(\tilde{y}) = v_{d,+}^h(\tilde{y}) \tag{5}$$

where $v_{d,+}^h \in L_2(Z_{h,+}^m)$, and the solution $u_{d,+}^h$ is sought in the space $L_2(Z_{h,+}^m)$.

Formally, the equation

$$au(x) + \int_{R_+^m} K(x - y) u(y) dy = v(x), x \in R_+^m \tag{6}$$

in the space $L_2(R_+^m)$ corresponds to equation (5) in the space $L_2(Z_+^m)$ under $h \to 0$.

It is proved (see result below) the unique solvability of the equation (6) implies unique solvability of the equation (5) for $h > 0$.

The equation (5) from solvability point of view is equivalent to solvability of so-called paired equation

$$(K_d^h P_{h,+} + IP_{h,-})U_h = V_h \tag{7}$$

in the space $L_2(Z_h^m)$, where $P_{h,-}$ is analogous projector on $Z_{h,-}^m = \{\tilde{x} \in Z_h^m{:}\tilde{\mathrm{x}} = (\tilde{\mathrm{x}}_1, \ldots, \tilde{\mathrm{x}}_{\mathrm{m}}), \tilde{\mathrm{x}}_{\mathrm{m}} < 0\}$, I is identity operator in $L_2(Z_h^m)$.

The discrete Fourier transform can be applied to the equation (7), and it is reduced to one-dimensional singular integral equation with Hilbert kernel on variable ξ_m under fixed $\xi' = (\xi_1, \ldots, \xi_{m-1})$:

$$\frac{1-\sigma_h(\xi',\xi_m)}{2}\tilde{U}_h(\xi) + \frac{1+\sigma_h(\xi',\xi_m)}{4\pi}\int\limits_{-\pi h^{-1}}^{\pi h^{-1}} \cot\frac{\xi_m - t}{2}\tilde{U}_h(t)dt = \tilde{v}_h(\xi), \tag{8}$$

where the sign "~" denotes multi-variable discrete Fourier transform.

It was shown such singular integral equation closely related to Riemann problem for a strip [5], and its solvability description is fully determined by index of its symbol $\sigma_h(\xi', \xi_m)$ on variable ξ_m. This index doesn't depend on h, ξ', and coincides with the index $\sigma(\xi)$ on variable ξ_m [2, 7].

Theorem 2. *Equations* (5) *and* (6) *are solvable at the same time for arbitrary right-hand side* $v(x) \in L_2(R_+^m)$, $v_{d,+}^h(\tilde{x}) \in L_2(Z_{h,+}^m)$.

2 Finite-Dimensional Approximation

To obtain good finite dimensional approximation for (7) we need to choice such finite dimensional approximation instead of infinite system of linear algebraic equations that this finite system of linear algebraic equations is like (in its properties) generating infinite system.

The authors sure that more applicable variant is so-called cyclic convolution. Such convolutions are used widely in the theory of digital signal processing [**?**, 4, 8], and, it seems, this point reflects the fact that infinite signal (in time or in space) is impossible for human sensors.

Briefly our point of view using cyclic convolution one can explain by the following scheme. Applying the discrete Fourier transform to discrete kernel transforms it to periodic function. The inverse Fourier transform maps periodic function to the function of discrete variable. If we denote $K_N(x)$ the truncated kernel $K(x)$ periodically extended on whole R^m, then applying the standard Fourier transform leads to the function of discrete argument (Fourier coefficients). Under large N

the distance between lattice points (on which the function of discrete argument is defined) will be small, and as a limit it will be zero. If now we take discrete approximation for periodic kernel $K_N(x)$, construct cyclic convolution and apply discrete Fourier transform, then as a result we obtain a product of two discrete (and finite valued) functions.

In our point of view this approximation is more convenient than [3] at least from computational point of view because it permits to use fast Fourier transform. The simplest numerical experiments with test-function $\exp(-|x|^2)$ gives good approximation immediately.

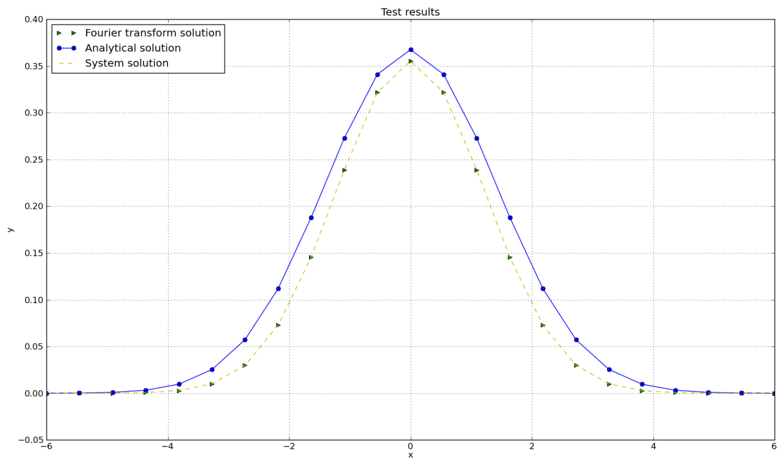

Figure 1

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I.5. Clifford and Quaternionic Analysis

(Sessions organizers: S. Bernstein, I. Sabadini, F. Sommen)

DIFFERENTIAL ALGEBRA OF BIQUATERNIONS. DIRAC EQUATION AND ITS GENERALIZED SOLUTIONS

L. A. Alexeyeva

Key words: biquaternion, bigradient, Dirac equation, generalized solution, KGFSh-equation, scalar potential, spinor, harmonic spinor, ω-spinor.

AMS Mathematics Subject Classification: 81Q05P

Abstract. The equation of quantum mechanics – Dirac equation is researched and its generalized solutions are constructed in biquaternionic form by using scalar potentials. The one equation for scalar potential has been built (*KGFSh-equation*) which unite the two known equations of quantum mechanics: Klein-Gordon-Fock and Schrodinger equations. The nonstationary, steady-state and harmonic on time scalar potentials and generated by them spinors and spinors fields are defined in biquaternionic representation.

1 Biquaternions, mutual bigradients and biwave equation

The biquaternions space $\mathbf{B}$ is the space of hypercomplex numbers, which are presented in Hamilton form as $\mathbf{B} = \{\mathbf{F} = f + F\}$, where f is complex number ($z \in \boldsymbol{C}$), F is three dimensional vector with complex components : $F = F_1 e_1 + F_2 e_2 + F_3 e_3$, e_1, e_2, e_3 are the basis vectors of Cartesian coordinate system in R^3 (hereinafter always scalar part of biquaternion (Bq.) we mark small letter, but vector part of Bq. with the same name capital).

$\mathbf{B}$ is linear space with addition (+):

$$a\mathbf{F} + b\mathbf{G} = a(f + F) + b(g + G) = (af + bg) + (aF + bG), \ \forall a, b \in \boldsymbol{C},$$

and known operation of quaternionic multiplication ($\circ$):

$$\mathbf{F} \circ \mathbf{G} = (f + F) \circ (g + G) = fg - (F, G) + fG + gF + [F, G].$$

Here $(F, G) = \sum\limits_{j=1}^{3} F_j G_j$ is usual scalar product F and G, $[F, G] = \sum\limits_{j=1}^{3} \varepsilon_{jkl} F_j G_k e_l$ is their vector product, ε_{jkl} is Levi–Civita symbol.

Biquaternions algebra is not commutative: $\mathbf{F} \circ \mathbf{G} - \mathbf{G} \circ \mathbf{F} = 2\,[F, G]$, but associative: $\mathbf{F} \circ \mathbf{G} \circ \mathbf{H} = (\mathbf{F} \circ \mathbf{G}) \circ \mathbf{H} = \mathbf{F} \circ (\mathbf{G} \circ \mathbf{H})$.

Definitions. Bq. $\mathbf{F}^- = f - F$ we name *mutual Bq.* for $\mathbf{F} = f + F$.

The *complex conjugate* to $\mathbf{F}$ is $\bar{\mathbf{F}} = \bar{f} + \bar{F}$ (upper line marks conjugate complex number). Bq. $\mathbf{F}^* = \bar{\mathbf{F}}^- = \bar{f} - \bar{F}$ we name *conjugate* to $\mathbf{F}$.

Scalar product of Bqs. is bilinear operation: $(\mathbf{F}_1, \mathbf{F}_2) = f_1 f_2 + (F_1, F_2)$.

The norm of $\mathbf{F}$ is real scalar value $\|\mathbf{F}\| = \sqrt{(\mathbf{F}, \bar{\mathbf{F}})} = \sqrt{f \cdot \bar{f} + (F, \bar{F})} = \sqrt{|f|^2 + \|F\|^2}$.

The pseudonorma of $\mathbf{F}$ is the value $\langle \mathbf{F} \rangle = \sqrt{f \cdot \bar{f} - (F, \bar{F})} = \sqrt{|f|^2 - \|F\|^2}$.

The Bq. $\mathbf{F}^{-1}$ is *inverse* to $\mathbf{F}$ if $\mathbf{F}^{-1} \circ \mathbf{F} = \mathbf{F} \circ \mathbf{F}^{-1} = 1$.

It's easy to prove the theorem [1].

Theorem 1. *If* $(\mathbf{F}, \mathbf{F}) \neq 0$ *then* $\mathbf{F}^{-1} = \mathbf{F}^- / (\mathbf{F}, \mathbf{F})$. *Bilinear equations:* $\mathbf{F} \circ \mathbf{G} = \mathbf{B}$ *and* $\mathbf{G} \circ \mathbf{F} = \mathbf{B}$ *have the unique solution* $\mathbf{G} = \mathbf{F}^{-1} \circ \mathbf{B}$ *and* $\mathbf{G} = \mathbf{B} \circ \mathbf{F}^{-1}$ *accordingly.*

We will consider the functional space of biquaternions on Minkovskiy space M: $\mathsf{B}(\mathsf{M}) = \{\mathbf{F} = f(\tau, x) + F(\tau, x)\}$, where f and F are complex generalized functions and vector-functions on M.

The *convolution of biquaternions* has the form:

$$\mathbf{A}(\tau, x) * \mathbf{B}(\tau, x) = a * b - \sum_{i,j,l=1}^{3} (A_j * B_j) + (a * B_j)\, e_j + (b * A_j)\, e_j + \varepsilon_{ijl}\,(A_i * B_j)\, e_l,$$

where parenthetically there are usual convolutions of generalized functions [1]. It's easy to see that here the two operation of quaternionic multiplication and convolution are united.

Mutual bigradients are the differential operators of type [3]: $\nabla^+ = \partial_\tau + i\nabla$, $\nabla^- = \partial_\tau - i\nabla$, where $\nabla = grad$. Their action on $\mathsf{B}(\mathsf{M})$ is defined as

$$\nabla^\pm \mathbf{F} = (\partial_\tau \pm i\nabla) \circ (f + F) = (\partial_\tau f \mp i\,(\nabla, F) \pm i\nabla f \pm \partial_\tau F \pm i[\nabla, F].$$

Here $(\nabla, F) = \operatorname{div} F$, $[\nabla, F] = \operatorname{rot} F$.

Their superposition possesses the remarkable property:

$$\nabla^- \left(\nabla^+ \mathbf{F}\right) = \nabla^+ \left(\nabla^- \mathbf{F}\right) = \left(\nabla^- \circ \nabla^+\right) \mathbf{F} = \Box \mathbf{F}, \tag{1}$$

where $\Box$ is classic wave operator: $\Box = \frac{\partial^2}{\partial \tau^2} - \Delta$, Δ is Laplace operator.

Using this property it's easy to solve the differential equations of the type:

$$\nabla^\pm \mathbf{B} = (\partial_\tau b \mp i \operatorname{div} B) + \partial_\tau B \pm i \operatorname{grad} b \pm i \operatorname{rot} B = \mathbf{G}(\tau, x). \tag{2}$$

We name equation (2) *the biwave equation* (from biquaternionic wave equation). Its solutions and property of invariance for Lorentz transformations are considered in detail in the paper [3].

2 Bigradients and Dirac matrixes

Biwave Eq. (2) may be written in matrix form:

$$\sum_{j=0}^{3} D^{\pm}_{m\,j} b_j = g_m, \quad m, j = 0, 1, 2, 3 \tag{3}$$

where $b_0 = b,\ g_0 = g,\ b_j = B_j,\ g_j = G_j,\ j = 1, 2, 3;$ and $D^{\pm}_{m\,j}$ are components of matrix $D^{\pm}$, which have the form:

$$D^+ = \begin{Bmatrix} \partial_\tau & -i\partial_1 & -i\partial_2 & -i\partial_3 \\ i\partial_1 & \partial_\tau & -i\partial_3 & i\partial_2 \\ i\partial_2 & i\partial_3 & \partial_\tau & -i\partial_1 \\ i\partial_3 & -i\partial_2 & i\partial_1 & \partial_\tau \end{Bmatrix}, \ D^- = \begin{Bmatrix} \partial_\tau & i\partial_1 & i\partial_2 & i\partial_3 \\ -i\partial_1 & \partial_\tau & i\partial_3 & -i\partial_2 \\ -i\partial_2 & -i\partial_3 & \partial_\tau & i\partial_1 \\ -i\partial_3 & i\partial_2 & -i\partial_1 & \partial_\tau \end{Bmatrix} \tag{4}$$

It's easy to check, that

$$\sum_{j=0}^{3} D_{m\,j} D_{jl} = \delta_{m\,l} \Box, \quad j, m, l = 0, 1, 2, 3 \tag{5}$$

where δ_{ml} is the Kronecker symbol.

We shall show that (4) are the differential matrix Dirac operator, which possess such property [4]. For this we present them in matrix form: $D = \sum_{j=0}^{3} D^j \partial_j$, where, as follow from (4), matrix D^j has have such components: $D^0 = I$ is unit matrix,

$$D^1 = \begin{Bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{Bmatrix}, \ D^2 = \begin{Bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{Bmatrix}, \ D^3 = \begin{Bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{Bmatrix}. \tag{6}$$

These are 4-dimensional unitary Dirac matrixes. Their property (5) has have simple for calculation biquaternionic form:

$$\nabla^{\mp}\nabla^{\pm} \equiv \nabla^{\mp} \circ \nabla^{\pm} = \Box. \tag{7}$$

This property of mutual bigradients allows easy to build the solutions of the some equations of electrodynamics and quantum mechanics, which possible to write in such biquaternionic form. In particular the system of Maxwell equations for electro-magnetic field can be written in form of the one biwave equation [3, 5].

3 Biquaternionic form of Dirac equation and KGFSh-equation

Let consider the differential biquaternionic equation:

$$\mathbf{D}_m^{\pm}\mathbf{B} \equiv \left(\nabla^{\pm} + m\right) \circ \mathbf{B} = \mathbf{F}, \quad m \in \boldsymbol{C}. \tag{8}$$

Under (5)–(6), this Eq. may be named as *generalized Dirac equation* in biquaternionic form, and differential operators $\mathbf{D}_m^{+} = \nabla^{+} + m$, $\mathbf{D}_m^{-} = \nabla^{-} + m$ we name *bigradiental representation* of Dirac matrix operators.

It'easy to show that their superposition possesses the very useful property

$$\mathbf{D}_m^{+}\mathbf{D}_m^{-} = \mathbf{D}_m^{-}\mathbf{D}_m^{+} = \Box + m^2 + 2m\partial_\tau, \quad \mathbf{D}_{im}^{+}\mathbf{D}_{im}^{-} = \Box - m^2 + 2im\partial_\tau. \tag{9}$$

Theorem 2. *Solutions of generalized Dirac Eq. (8) are Bqs. of the type*

$$\mathbf{B} = \mathbf{B}^0 + \mathbf{D}_m^{\mp}\left(\psi * \mathbf{F}\right), \tag{10}$$

where $\mathbf{B}^0(\tau, x)$ *is a solution of Dirac Eq.* $\mathbf{D}_m^{\pm}\mathbf{B}^0 = 0$, $\psi(\tau, x)$ *is a fundamental solution of Eq.*

$$\Box\psi + m^2\psi + 2m\partial_\tau\psi = \delta(\tau)\delta(x). \tag{11}$$

If $m = i\rho$, $Im\rho = 0$, *then* ψ *is a solution of the Eq.*

$$\Box\psi - \rho^2\psi + 2i\rho\,\partial_\tau\psi = \delta(\tau)\delta(x). \tag{12}$$

Here the left part of Eq. (12) contains the Klein-Gordon-Fock operator $(\Box - \rho^2)$, and Schrodinger operator $(\triangle + 2i\rho\partial_\tau)$. By this cause we name

$$\Box u + 2m\partial_\tau u + m^2 u = f(\tau, x) \tag{13}$$

as *Klein–Gordon–Fock–Schrodinger equation* (*KGFSh-equation*) .

It is interesting that appearance of the additional member $(2m\partial_\tau u)$ in the KGFSh-equation essentially simplifies its fundamental solutions in comparison with fundamental solution of KGF-equation, which was constructed by Vladimirov V.S. (see [6]).

4 Generalized solutions of KGFSh-equation. Scalar potentials

Using Fourier transform on τ in Eq. (11) we get Helmholtz equation

$$\left\{\Delta - k^2\right\} F_\tau[\psi](\omega, x) + \delta(x) = 0, \quad k = i\omega - m.$$

Its fundamental solutions are well known [1]:

$$F_\tau[\psi] = \frac{1}{4\pi \|x\|} \left(a e^{(i\omega - m)\|x\|} + (1-a) e^{-(i\omega - m)\|x\|} \right).$$

Using inverse Fourier transform we obtain the solution of Eq. (11).

Theorem 3. *Fundamental solution of KGFSh-equation (11) is*

$$\psi = \frac{1}{4\pi \|x\|} \left(a\, e^{-m\|x\|} \delta(\tau - \|x\|) + (1-a)\delta(\tau + \|x\|) e^{m\|x\|} \right) + \psi_0, \ \forall a \in \mathbf{C}$$

where $\delta(\tau \pm \|x\|)$ is the simple layers on the cones $\|x\| = \mp|\tau|$, $\psi_0(\tau, x)$ is a solution of uniform Eq.(by $f = 0$).

In particular, the function $\psi = \frac{e^{-i\rho\|x\|}}{4\pi\|x\|}\delta(\tau - \|x\|)$ is fundamental solution for $m = i\rho$. It's interesting that here the density of simple layer on cone is fundamental solution of Helmholtz Eq. with wave number ρ.

General solutions of Eq. (13) have the form: $u = f * \psi^m + u_0$, where $u_0(\tau, x)$ is a solution of uniform KGFSh-Eq.

$$\Box u + m^2 u + 2m\partial_\tau u = 0. \tag{14}$$

In space of Fourier transforms we obtain from here

$$\left(\|\xi\|^2 - (\omega + im)^2 \right) u^*(\omega, \xi) = 0, \tag{15}$$

where $u^*(\omega, \xi) = F_{\omega,\xi}\left[u(\tau, x)\right]$ is full Fourier transform on τ, x.

If $\operatorname{Re} m \neq 0$, then $\|\xi\|^2 - (\omega + im)^2 \neq 0$ for $\forall \xi \in R^3$. In this case this equation has only trivial zero decision: $u^* = 0$.

However under purely imaginary $m = i\rho$ Eq. (15) has an uncountable ensemble of the decisions:

$$u^*(\omega,\xi) = \varphi(\omega,\xi)\delta\left(\|\xi\|^2 - (\omega-\rho)^2\right). \qquad (16)$$

Here $\varphi(\omega,\xi)$ is arbitrary given function on the cones $\|\xi\| = |\omega-\rho|$.

By calculating its original we get (by $\tau \geqslant 0$)

$$u(\tau,x) = \frac{1}{(2\pi)^4}\int\limits_{-\infty}^{\infty} d\omega \int\limits_{\|\xi\|=|\omega-\rho|} \varphi(\omega,\xi)\exp\left(-i(\xi,x) - i\omega\tau\right) dS(\xi) =$$

$$= \frac{e^{-i\rho\tau}}{(2\pi)^4}\int\limits_{R^3}\left\{\varphi(\rho+\|\xi\|,\xi)e^{-i\|\xi\|\tau} - \varphi(\rho-\|\xi\|,\ \xi)e^{i\|\xi\|\tau}\right\}\exp\left(-i(\xi,x)\right) dV(\xi),$$

where $dV(\xi) = d\xi_1 d\xi_2 d\xi_3$, $dS(\xi)$ is differential of surfaces area of the sphere of the radius, specified under sign corresponding to integral. Thence, on the strength of randomness φ, the theorem follows.

Theorem 4. *If* $\mathrm{Re}\, m \neq 0$, *then Eq .(14) has only single zero decision. But if* $\mathrm{Re}\, m = 0$, $m = i\rho$, *decisions exist and they can be presented in the form*

$$\psi_0(\tau,x) = e^{-i\rho\tau}\int\limits_{R^3}\varphi(\xi)\exp\left(i\left((\xi,x) \pm \|\xi\|\,\tau\right)\right) dV(\xi), \quad \forall\varphi(\xi) \in L_1(R^3), \qquad (17)$$

or in the manner of a sum of such decisions.

5 Generalized solutions of Dirac equation. Biquaternionic representation of spinors fields

We shall consider biquaternionic decisions of the uniform Dirac equation

$$\left(\nabla^{\pm} + i\rho\right)\mathbf{Sp} = 0, \qquad \mathrm{Re}\rho = 0. \qquad (18)$$

In quantum mechanics they are named *spinors* [4, 7].

Theorem 5. *The solutions of Dirac equation (18) may be presented so:*

$$\mathbf{Sp} = \mathbf{D}^{\mp}_{i\rho}\left(\psi_0 * \mathbf{C}(\tau,x)\right) = \mathbf{\Psi}^{\mp}_0 * \mathbf{C}(\tau,x), \qquad (19)$$

$$\mathbf{\Psi}^{\mp}_0 = \left(\nabla^{\mp} + i\rho\right)\psi_0 = i\rho\psi_0 + \partial_\tau\psi_0 \mp i\, grad\psi_0.$$

Here ψ_0 is a solution of uniform Eq. (14), $\mathbf{C}$ *is arbitrary biquaternion, for which such convolutions exist, or in the manner of amounts of the such decisions.*

Proof. If $\mathbf{Sp}$ is equal to (19) then

$$\mathbf{D}_m^{\pm}\mathbf{Sp} = \mathbf{D}_m^{\pm}\mathbf{D}_m^{\mp}\left(\psi_0 * \mathbf{C}\right) = \left(\Box\psi_0 + 2m\partial_\tau\psi_0 + m^2\psi_0\right) * \mathbf{C} = 0, \quad m = i\rho.$$

Inversely, if $\mathbf{Sp}$ is the solution of Eq. (18) then $\left(\Box + 2m\partial_\tau + m^2\right)\mathbf{Sp} = \mathbf{D}_m^{\mp}\mathbf{D}_m^{\pm}\mathbf{Sp} = \mathbf{D}_m^{\mp}0 = 0$. I.e. scalar part and components of the vector part $\mathbf{Sp}$ are decisions of uniform KGFSH-equations. Consequently, $\mathbf{Sp}$ is possible to present in the manner of amounts of the decisions of the type (19).

As the spinor (19) contains scalar-vector field $\mathbf{C}(\tau, x)$ and scalar potential ψ_0, it may be named ψ_0-*spinor of* $\boldsymbol{C}$*-field.* $\Box$

6 Harmonic scalar potentials of spinors

Let consider formula (17) where two plane harmonic waves stand under integral, which are also the solutions of uniform Eq. (14):

$$\varphi_\xi^{\pm}(\tau, x) = \exp\left(i\left((\xi, x) - \rho\tau \pm \|\xi\|\,\tau\right)\right), \tag{20}$$

Wave vector ξ defines the direction of wave motion, *wave length* $\lambda = 2\pi/\|\xi\|$, *frequency* $\omega = |\rho \pm \|\xi\||$, *period* $T = 2\pi/|\rho \pm \|\xi\||$, *phase velocities* $V = 1 \pm \frac{\rho}{\|\xi\|}$.

If $\|\xi\| \to \infty$ then $\omega \to \infty$, and $V \to 1 \pm 0$. Ff $\|\xi\| \to |\rho|$ then $V \to 1; 0$, $\omega \to \frac{\pi}{\rho}; \infty$ accordingly. Generated by these waves spinors have the form:

$$\left(\nabla^{\mp} + i\rho\right)\varphi_\xi^{\pm}(\tau, x) = \pm\left(i\,\|\xi\| + \xi\right)\varphi_\xi^{\pm}.$$

Definition. *Harmonic spinors* are named spinors of type

$$\mathbf{Sp}_\xi^{\pm} = \frac{\exp\left(i\left((\xi, x) - \rho\tau \pm \|\xi\|\,\tau\right)\right)}{\sqrt{2}}\left(i + \frac{\xi}{\|\xi\|}\right), \quad \left\|\mathbf{Sp}_\xi^{\pm}\right\| = 1, \quad \left\langle\mathbf{Sp}_\xi^{\pm}\right\rangle = 0.$$

Harmonic spinor $\mathbf{C}$*-field* is the spinor $\mathbf{Sp} = \mathbf{C}(\tau, x) * \mathbf{Sp}_\xi^{\pm}(\tau, x)$.

Theorem 6. *Spinor* $\boldsymbol{C}$*-field can be presented in the form:*

$$\mathbf{Sp} = \mathbf{C}(\tau, x) * \int\limits_{R^3}\varphi(\xi)\mathbf{Sp}_\xi^{\pm}(\tau, x)dV(\xi),$$

where $\varphi(\xi) \in L_1(R^3)$, or in the manner of linear combination of like spinor fields.

7 Harmonic vibrations and static solutions of Dirac equation

In the case of stationary harmonic vibration with constant frequency ω: $\mathbf{B} = \mathbf{B}(x)e^{-i\omega\tau}$ (right part of (9) has the like form). Then the complex amplitude satisfies to Eq.:

$$\left(\nabla_{\omega}^{\pm} + \rho\right) \mathbf{B}(x) = \mathbf{F}(x). \tag{21}$$

The solutions of uniform Eq. (21) are named ω-*spinors* and are designated $\mathbf{Sp}^{\omega}$.

ω-*bigradients* are operators $\nabla_{\omega}^{\pm} = \omega \pm \nabla$. They possess useful property: $(\nabla_{\omega}^{\pm} + \rho) \circ (\nabla_{\omega}^{\mp} + \rho) = (\omega + \rho)^2 + \Delta$. Using it the next theorem has been proved.

Theorem 7. *Solution of Eq. (21) can be presented as*

$$\mathbf{B} = \left(\nabla_{\omega}^{\mp} + \rho\right) \circ (\chi * \mathbf{F}) + \mathbf{Sp}^{\omega},$$

$$\mathbf{Sp}^{\omega}(x) = \mathbf{C}(x) * \int\limits_{\|\mathbf{e}\|=1} p(\mathbf{e}) \mathbf{\Psi}_0^{\omega}(x, \mathbf{e}) dS(\mathbf{e}), \quad \forall p(\mathbf{e}) \in L_1(Sp_{\mathbf{e}}).$$

$$\mathbf{\Psi}_0^{\omega}(x, \mathbf{e}) = \frac{1}{k\sqrt{2}} \left(\nabla_{\omega}^{\mp} + \rho\right) e^{-ik(\mathbf{e},x)} = \frac{1}{k\sqrt{2}} \left(\omega + \rho \pm ik\mathbf{e}\right) e^{-ik(\mathbf{e},x)},$$

$$\chi = -\frac{a e^{ik\|x\|}}{4\pi \|x\|} - \frac{(1-a)e^{-ik\|x\|}}{4\pi \|x\|}, \quad k = |\omega + \rho| \neq 0, \ \forall a,$$

$\mathbf{C}(x)$ *is arbitrary Bq., admissive this convolution,* $Sp_{\mathbf{e}} = \{\mathbf{e} \in R^3 : \|\mathbf{e}\| = 1\}$.

Here we introduced *harmonic* $\omega - \boldsymbol{e}$ *spinor* $\mathbf{\Psi}_0^{\omega}(x, \mathbf{e})$: $\|\mathbf{\Psi}_0^{\omega}\| = 1$, $\langle \mathbf{\Psi}_0^{\omega} \rangle = 0$, vector $\mathbf{e}$ defines its direction, $k = |\omega + \rho|$ is its wave number.

Static spinors are obtained for $\omega = 0$. Formulae of theorem 7 herewith save the type as $k = |\rho| \neq 0$.

For $m = 0$ generalized Dirac equation (for $\mathbf{D}_0^{+}$) are equivalent to system of Maxwell equation and all these solutions gives its generalized solutions [2, 5].

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MODULAR COMPUTATIONS IN THE ALGEBRA OF QUADRATIC MATRIXES OF ORDER N

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Key words: Modular computations, quadratic matrixes, algebra, computer arithmetic

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Abstract. The principal topic of this paper is the analysis of matrix algebras from the positions of computer arithmetic.

1 Introduction

We consider the algebra of matrixes over the field of real numbers R here. The basic acceptance is given onto the matrix analog of modular arithmetic as a basis for parallelizing of matrixes algebra operations [1] and logarithmetic of finite field $GF(p)$ as a basis of modular computations [2, 3]. Multiple applications of the hypercomplex numbers [4] in the practice and theory of BINS — board inertial navigate systems [5] are known. It's observed that computer data processing differ substantially depending on the type of their presentation. Complex numbers and quaternions form the significant class of hypercomplex numbers. Methods of computations parallelizing by means of modular arithmetic in the complex numbers domain are studied in the book [6], in the domain of quaternions — in the book [7]. Subsequent propagation of modular arithmetic onto the computations in finite- dimensional algebras is regulated by theorems of G. Frobenius and D. Wedderbarn [8].

Frobenius theorem. Arbitrary associative algebra with division is isomorphic to one of three algebras or field of real numbers (R) or field of complex numbers (C) or field of quaternions (Q_{nu}).

Wedderbarn theorem. All simple associative algebras over field K — this is exactly all complete matrix algebras with elements from associative algebra with division over K [9].

According to the mentioned theorems, computations in the class of simple associative algebras are reduced to the computations in the following matrix algebras:

a) in algebra of matrixes over R (with elements — real numbers);

b) in algebra of matrixes over C (with elements — complex numbers);

c) in algebra of matrixes over Q_ν (with elements — quaternions).

All computations are based on the Archimedes axiom here. Let's consider the case $K = R$ below. Matrix computations construct on the hierarchy of linear algebra operations. The scalar multiplication of vectors is one of the basic operations of linear algebra consisting from scalar operations of summing and multiplication: multiplication of matrix onto vector is conducted by means of scalar multiplications; multiplication of matrixes is reduced to the set of multiplications of matrixes onto vector. All these operations may be described in algorithmic form. The stylized version of Matlab language is used for the writing of algorithms.

2 Archimedes forming of matrix algebra $M_n(Z)$ on scalar modulus (coordinate-wise method)

The action of Archimedes axiom in the matrix algebra $M_n(R)$ of size n over field R is defined as Archimedes forming. The following proposition reveals the essence of Archimedes forming in $M_n(R)$.

Proposition 1. $\forall A = (a_{ij}) \in M_n(R)$

$$A = [A] + \partial(A), \tag{1}$$

where $[A] := ([a_{ij}]), \partial(A) := \partial(a_{ij})$; $[a_{ij}]$ — integer part of number $a_{ij}; \partial(a_{ij})$ — fractional part of number a_{ij}. Really let's apply Archimedes identity for every element $a_{ij} = [a_{ij}] + \partial(a_{ij}), \forall(i, j)$. Then we'll obtain (1) according to the definition of sum of matrixes. The relation (1) in the matrixes algebra $M_n(R)$ is the analog of Archimedes identity in the field R.

Proposition 2. $\forall A, B \in M_n(R)$ the following equalities are valid
(μ_1) $[A + B] = [A] + [B] + [\partial(A) + \partial(B)]$
(μ_2) $\partial(A + B) = \partial(\partial(A) + \partial(B))$
(μ_3) $[AB] = [A][B] + [\partial(A)[B] + [A]\partial(B) + \partial(A)\partial(B)]$
(μ_4) $\partial(AB) = \partial(\partial(A)[B] + [A]\partial(B) + \partial(A)\partial(B))$
(μ_5) If $A \in M_n([0, 1])$, then $[A] = 0, \partial(A) = A$; if $A \in M_n(Z)$, then $[A] = A$ and $\partial(A) = 0$.

From here it follows in particular: $\forall A \in M_n(R) : [\partial(A)] = 0$, $\partial([A]) = 0$, $[[A] = A]$, $\partial(\partial(A)) = \partial(A)$.

(μ_6) If $A \in M_n(Z)$, then $\forall B \in M_n(R)$, $\partial(AB) = \partial(A\partial(B))$.

The proof of (μ_6) follows immediately from (μ_4) and from the remark that according equality (μ_4) the following equalities take place $\partial(A) = 0$ and $[A] = A$.

Proposition 3 (Euclidian theorem over $M_n(Z)$). For arbitrary matrix $A = (a_{ij}) \in M_n(Z)$ and arbitrary $m \in N(m \neq 0, 1)$ the following equality is valid $A = [\frac{A}{m}]m + |A|_m$, where $\left[\frac{A}{m}\right] = ([a_{ij}/m])$, $|A|_m = (|a_{ij}|_m)$, $\forall i, j$.

Conclusion. So the modular arithmetic of ring Z by means of Archimedes format arises to the level of modular arithmetic of matrixes ring $M_n(Z)$.

We'll refer on matrixes elements under detailed writing of algorithm according to Fortran rules: $A(i, j)$.

The following propositions are valid:

(ν_1) $\forall A, B \in M_n(Z)$ and $\forall m \in N(m \neq 0, 1)$ the following equalities take place $|A \pm B|_m = ||A|_m \pm |B|_m|_m$, $|AB|_m = ||A|_m|B|_m|_m$, $|A * B|_m = ||A|_m * |B|_m|_m$, where $*$ — operation of tensor multiplication.

(ν_2) Function $y = |x|_m$ defined on $M_n(Z)$ homomorphly maps the ring $M_n(Z)$ into the ring $M_n(Z_m)$.

(ν_3) CTR (Chinese theorem on residues): let's $p_1, p_2, ..., p_s$ - pairwise prime natural numbers and $P = p_1 p_2 ... p_s$, $P_i = P/p_i$ $(1 \leqslant i \leqslant r)$. Then arbitrary matrix $X \in M_n(Z_p)$ represents by unique way in the form: $X = \left|\sum\limits_{k=1}^{r} \left||X|_{p_k} P_k^{-1}\right|_{p_k} P_k\right|_p$.

3 Modular arithmetic

The modular arithmetic is the basis for parallerizing of the matrix algebra operations as it was mentioned before. The modular arithmetic allows to remove the problem of speed loss under the arithmetic computations what attracts it in the most cases of intensive computations problems. But the problem of the "overhead" reduction on the realization of modular operations exists. These expedintures are caused by fact that arithmetic operation $*(i.e. +, -, x)$ over residues $x, y mod m$, as over integer numbers may lead to the result of the operation $x * y$ out the range Z_m and then the correction of the result became necessary, i.e. taking the residue from the number $x * y$ on $mod m |x * y|_p$. The operation of taking the residue $|x * y|_p$ is expressed by formula: $|x * y|_p = x * y - \left[\frac{x*y}{p}\right] p$. Technically the realization of operation by this scheme demands the fulfilment of four actions what lead to additional expenditures in the connection with the incommensurability of modulo p with the degree of two. Algorithms of matrix computations contain big number of multiplication operations in most cases what lead to big time expenditures if You use traditional approach for the realization of modular operations. Let's show that logcomputations in the finite field $GF(p)$ open new perspectives for modular computations. Let's consider the discrete logarithm over field $GF(p)$ from the known scheme suggested in the beginning of nineteenth century in the Gauss and Jakobi works.

Definition 1. Let's w — generator element of field $GF(p)$. The discrete logarithm on the basis w over $GF(p)$ is called the function of argument x ($x \in Z$), prescribed by formula: $\lg_w |x|_p = \lambda_p \delta(|x|_p) + ind_w |x|_p \hat{\delta}(|x|_p)$, where $|x|_p$ — residue of number x $mod(p)$; $\delta(|x|_p)$ — Dirac function; $\hat{\delta}(|x|_p)$ - Dirac cofunction, i.e. $\hat{\delta}(|x|_p) = 1 - \delta(|x|_p)$; $ind_w |x|_p$ — index of the residue $|x|_p; \lambda_p$ — symbol which is not the element of the ring Z_{p-1}.

Let's assume $\lambda_p = 2^t - 1$. The technology of this selection is caused by the following: if p — t-bit number i.e. $2^{t-1} < p < 2^t$, then it's appropriate to use t -bit binary notation of the number $2^t - 1$; in the role of the symbol λ_p; the inequality $p \leqslant 2^t - 1$ is valid since p — prime. Thus the number $2^t - 1$ isn't a symbol representative the element of ring Z_{p-1} for every prime $p \geqslant 3$. All values of the function $y = \lg_w |x|_p$ are different and are described by t — bit binary code for this λ_p selection.

The set $J_p = 0, 1, 2, ..., p-2, \lambda_p$ is the range of values of discrete logarithm (1). The characteristic points from $GF(p)$ of the mapping $\lg_w : GF(p) \to J_p$ for all p and arbitrary selection of w are the points $0, 1, w, p-1$; they maps into the points of the set $J_p : \lambda_p, 0, 1, \frac{p-1}{2}$.

The logarithmic function maps finite field $GF(p)$ bijective onto J_p by construction. Thus the structure of finite field isomorphic to the structure of field $GF(p)$ generates on J_p. This proposition serves as basis for the formation of logarithmetic over field $GF(p)$.

Let's consider the corresponding component-wise operations $\lg_w |x_1 x_2|_p$; $\lg_w |x_1 + x_2|_p$ in detail. Let's denote $\alpha = \lg_w |a|_p$, $\beta = \lg_w |b|_p$, then

$$\alpha \boxplus \beta = \begin{cases} 2^t - 1, \text{ if } \delta\left(\alpha - \left(2^t - 1\right)\right) \wedge \delta\left(\beta - \left(2^t - 1\right)\right) \vee \delta\left[|\beta - \alpha|_{p-1} - \dfrac{p-1}{2}\right] = 1; \\ \alpha, \text{ if } \hat{\delta}\left(\alpha - \left(2^t - 1\right)\right) \wedge \delta\left(\beta - \left(2^t - 1\right)\right) = 1; \\ \beta, \text{ if } \delta\left(\alpha - \left(2^t - 1\right)\right) \wedge \hat{\delta}\left(\beta - \left(2^t - 1\right)\right) = 1; \\ \left|\alpha + J_w(|\beta - \alpha|_{p-1})\right|_{p-1}, \text{ otherwise,} \end{cases}$$

where $J_w(|u|_{p-1}) = \lg_w |1 + w^{|u|_{p-1}}|_p$ — Jacobi logfunction.

$$\alpha \boxtimes \beta = \begin{cases} 2^{t-1}, \text{ if } \delta(\alpha - (2^t - 1)) \vee \delta\left(\beta - \left(2^t - 1\right)\right) = 1; \\ |\alpha + \beta|_{p-1}, \text{ otherwise.} \end{cases}$$

The immediate computation of Jacobi logarithm values has complexity of discrete logarithm. The tabular realization of Jacobi logarithm is restricted by the quantity of digits of base modulus p Galua field $GF(p)$ from above by virtue of technical limitations on admissible volume of used tables. It's written about the reduction of Jacobi table's volume under the necessity in the work [10].

Let's consider the basic operations of matrix computations in terms of modular arithmetic in the frames of this paper.

1. *Scalar vector multiplication.* It's necessary to compute $c = x^T y$ on vectors $x, y \in V_n(Z_m)$.

The problem of determination of scalar vector multiplication in the logarithmetics basis reduces to the problem of Gauss logarithm from N variables finding. Let's denote it as $G(z_1, z_2, ..., z_N) = \lg_w \left| \sum_{i=1}^{N} w^{|Zi|_{p-1}} \right|_p$, where $|z_i|_{p-1} = |\log_w |x_i|_p + \log_w |y_i|_p|_{p-1}$. The methods of Gauss logarithm computation and proper technical solutions are described in [11] in detail. The complexity of that algorithm — $O(n)$ (volume of performance depends linear from the vectors dimension).

2. *External multiplication.* It's necesssary to compute $c = xy^T$ on vectors $x, y \in V_n(Z_m)$.

The problem is reduced to the evaluation of additive operation of modular logarithmetic: $\log_w |c_{ij}|_{p-1} = |\log_w |x_i|_p + \log_w |y_j|_p|_{p-1}$.

3. *Saxpy.* This algorithm computs $z = ax + y$ by vectors $x, y \in V_n(Z_m)$ and scalar $a \in Z_m$.

It's necessary to conduct the calculations by scheme:

$$\log_w |z_i|_{p-1} = \Big| |\log_w |\alpha|_p + \log_w |x_i|_p|_{p-1} + \\ + \log_w \left(1 + w^{|\log_w |y_i| - |\log_w |\alpha|_p + \log_w |x_i|_p|_{p-1}|_{p-1}}\right)\Big|_{p-1}.$$

The complexity of saxpy has the same order $O(n)$. Its difference consists in the fact that it returns not a scalar but a vector.

Let's consider $A \in M_n(Z_m)$, it's necessary to compute the multiplication $z = Ax$, where $x \in V_n(Z_m)$. The standard method of computation consists in consecutive calculation of scalar multiplications: $z_i = \left| \sum_{j=1}^{n} a_{ij} x_j \right|_m$.

Let's consider the modification of matrix by external multiplication: $A := A + xy^T$, $A \in M_n(Z_m)$, $x, y \in V_n(Z_m)$. The matrix modification by external multiplication take big place in traditional formulations of many important matrix algorithms, it's possible to reformulate the most of them so that the operation gaxpy: $|z|_m = |y + Ax|_m$ became dominant.

4. *Gaxpy.* The algorithm computs $|z|_m = |y + Ax|_m$ by means of $x, y \in V_n(Z_m)$ and $A \in M_n(Z_m)$.

$$\begin{aligned}
&z = y; \\
&\text{for } \; j = 1 : n \\
&\qquad |z|_m = |z + |x(j)A(:, j)|_m|_m \\
&\text{end}
\end{aligned}$$

So the sum is accumulated in vector z which value is renewed by the sequence of saxpy operations.

The corresponding technical realizations were elaborated on the basis of considered algorithms. The input dates were taken by 16 bit digits, vectors were taken of size $n = 10$. The structure synthesis was conducted by means of SCAD Synopsys synplify in the basis PLIS Altera Stratix II EP2S15F484C3. The simulation and verification of Verilog projects were conducted by means of ModelSim Mentor Graphics. The speed of scheme is defined by the time frequency, the realization complexity is measured by the number of adaptive logic blocks of taular types. The results of the numerical experiment are adduced in table 1.

Table 1

Apparatus and time expenditures on matrix operations

	Scalar multiplication		External multiplication		saxpy	
	MHz	ALUT	MHz	ALUT	MHz	ALUT
BNS	168	387	500	1	290	33
RNS	375	314	500	21	385	68
RLNS	437	325	500	25	452	75

Proposition 4 (on polyadic expansion). Any matrix $X \in M_n(Z_P)$, where $P = p_1 p_2 ... p_n$, is decomposable in polyadic series by unique way

(ν_4) $X = A_1 + a_2 p_1 + A_3 p_1 p_2 + ... + A_r p_1 ... p_{r-1}$, where $A_1 = [X]_{p_1}$, $A_k = \left\| \frac{x}{p_1 ... p_{k-1}} \right\|_{p_k}$, $2 \leqslant k \leqslant r$.

All nonmodular operations of modular arithmetic have corresponding matrix interpretation. So the modular arithmetic became the effective tool of parallelizing of matrix computations over R.

The results of synthesis of described algorithms show that the logarithmetics may be used succesfully for the efficiency rise of modular computations realization under the solution of matrix algebra problems with the respect to the significant simplification of multiplication operation.

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MEAN VALUE PROPERTIES FOR K-HYPERMONOGENIC FUNCTIONS

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Abstract. Let $C\ell_{0,n}$ be the (universal) Clifford algebra generated by $e_1, ..., e_n$ satisfying $e_i e_j + e_j e_i = -2\delta_{ij}$, $i, j = 1, ..., n$. The modified $k-$Dirac operator is introduced by $M_k f = Df + kx_n^{-1}Q'f$, where $'$ is the main involution and Qf is given by the decomposition $f(x) = Pf(x) + Qf(x)\,e_n$ with $Pf(x)$ and $Qf(x)$ in $C\ell_{0,n-1}$. A continuously differentiable function f is called k-hypermonogenic, if $x_n M_k f(x) = 0$. Note that 0-hypermonogenic are monogenic and $n-1$-hypermonogenic functions are hypermonogenic defined by H. Leutwiler and the author. The function $|x|^{k-n+1} x^{-1}$ is k-hypermonogenic. Hypermonogenic functions are related to harmonic functions with respect to the Riemannian metric $ds^2 = x_n^{2k/(1-n)}\left(dx_0^2 + ... + dx_n^2\right)$. We present the mean value property for k-hypermonogenic functions and related results. Earlier the mean value properties has been proved for hypermonogenic functions. The kea idea is to transform functions to the eigenfunctions of the Laplace Beltrami-operator of Poincare upper half space model.

1 Introduction

We consider generalized holomorphic functions in the upper half space

$$\mathbb{R}_+^{n+1} = \{(x_0, ..., x_n) \mid x_i \in \mathbb{R}, x_n > 0\}$$

related to the Riemannian metric $ds^2 = x_n^{-\frac{2k}{n-1}}\left(dx_0^2 + ... + dx_n^2\right)$. The Laplace-Beltrami operator with respect to this metric is $\triangle_k f = x_n^{\frac{2k}{n-1}}\left(\triangle f - kx_n \frac{\partial f}{\partial x_n}\right)$. When $k = n-1$ the metric is the hyperbolic metric of the Poincare upper half space model and solutions of the Laplace-Beltrami operator are hyperbolic harmonic functions. In I992 Heinz Leutwiler noticed in [11] and [12] that the power function $x^m\,(m \in \mathbb{N}_0)$, calculated using Clifford algebras, is a conjugate gradient of a hyperbolic harmonic function with respect to the hyperbolic metric $ds^2 = x_n^{-2}\sum_{i=0}^{n} dx_i^2$ The modified Dirac operator M and hypermonogenic functions in this case were introduced by H. Leutwiler and the author in [5]. An introduction to theory is given in [6]. A Cauchy-type formula for hypermonogenic functions was proved in [4]. The main idea in the proof was $k-$hypermonogenic functions, introduced in [2].

In this paper we prove the mean value properties for k-hypermonogenic functions. For hypermonogenic and hyperbolic harmonic functions they were proved in [7, 9] and the special case by H. Leutwiler in [13] with different methods. New results are based on deeper understanding of the interplay between different metrics and Laplaced-Beltrami operators connected to them. The key idea for dealing with solutions of $\triangle_k f = 0$ is to consider them as transformed eigenfunctions of the hyperbolic Laplace operator $\triangle_{n-1}$. The invariance properties of the hyperbolic metric are important. The results have also connections the Weinstein equation

$$\Delta u - \frac{k}{x_n}\frac{\partial u}{\partial x_n} + \frac{\ell}{x_n^2}u = 0 \tag{1}$$

on the upper half space $\mathbb{R}^n_+$ for $\ell \leqslant (k+1)^2/4$. In the future work we shall study more applications of mean value properties for example Maximum modulus theorem.

We review the main notations and concepts. Denote by $C\ell_{0,n}$ the universal real Clifford algebra generated by $e_1, ..., e_n$ satisfying the relation $e_ie_j + e_je_i = -2\delta_{ij}$, where δ_{ij} is the usual Kronecker delta. An element $x = x_0 + x_1e_1 + ... + x_ne_n$ for $x_0, ..., x_n \in \mathbb{R}$ is called a *paravectors.* The set $\mathbb{R}^{n+1}$ is identified with the real vector space of paravectors.

We use several common involutions. The *main involution* is the mapping $a \to a'$ defined by $e_i' = -e_i$ for $i = 1, ..., n$ and extended to the total algebra by linearity and the product rule $(ab)' = a'b'$. Similarly the *reversion* is the mapping $a \to a^*$ defined by $e_i' = -e_i$ for $i = 1, ..., n$ and extended to the total algebra by linearity and the product rule $(ab)^* = b^*a^*$. The *conjugation* is the mapping $a \to \overline{a}$ defined by $\overline{a} = (a')^* = (a^*)'$.

We recall that any element w in $C\ell_{0,n}$ may be written as $w = \sum\limits_{\nu\subset\{1,...,n\}} a_\nu e_\nu$, where a_ν is real and $e_\emptyset = 1$ and $e_\nu = e_{\nu_1}..e_{\nu_k}$ if $1 \leqslant \nu_1 < \nu_2 < ... < \nu_k \leqslant n$. The norm of w is defined by $|w|^2 = \sum a_\nu^2$. We also use a simple way to generalize real and imaginary parts of complex numbers to the Clifford algebra $C\ell_{0,n}$. Since any element $a \in C\ell_{0,n}$ may be uniquely decomposed as $a = b + ce_n$ for $b, c \in C\ell_{0,n-1}$ (the Clifford algebra generated by $e_1, ..., e_{n-1}$) we define the mappings $P : C\ell_{0,n} \to C\ell_{0,n-1}$ and $Q : C\ell_{0,n} \to C\ell_{0,n-1}$ by $Pa = b$ and $Qa = c$ (see [5]) In order to compute the P- and Q- parts we define the involution $a \to \widehat{a}$ by $\hat{e}_i = (-1)^{\delta_{in}} e_i$ for $i = 1, ..., n$ and extend it to the total algebra by linearity and the product rule $\widehat{ab} = \widehat{a}\widehat{b}$. Then we obtain the formulas

$$Pa = \frac{1}{2}(a + \widehat{a}), \tag{2}$$

$$Qa = -\frac{1}{2}(a - \widehat{a})\, e_n. \tag{3}$$

We consider functions $f : \Omega \to C\ell_{0,n}$, defined on an open subset Ω of $\mathbb{R}^{n+1}$, and assume that their components are continuously differentiable. The left Dirac operator and the right Dirac operator in $C\ell_{0,n}$ are defined by $D_l f = \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i}$ and $D_r f = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} e_i$. Their conjugate operators $\overline{D_l}$ and $\overline{D_r}$ are introduced by $\overline{D_l} f = \sum_{i=0}^{n} \overline{e_i} \frac{\partial f}{\partial x_i}$ and $\overline{D_r} f = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} \overline{e_i}$.

The modified Dirac operators M_k^l, $\overline{M}_k^l$, M_k^r and $\overline{M}_k^r$ are introduced in the upper half space $\mathbb{R}_+^{n+1} = \left\{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \mid x_n > 0\right\}$ in [5] and [3] by

$$M_k^l f(x) = D_l f(x) + k\frac{Q' f}{x_n}, \quad M_k^r f(x) = D_r f(x) + k\frac{Qf}{x_n},$$
$$\overline{M}_k^l f(x) = \overline{D}_l f(x) - k\frac{Q' f}{x_n}, \quad \overline{M}_k^r f(x) = \overline{D}_r f(x) + k\frac{\partial Qf}{\partial x_n}(x).$$

where $(Qf)' = Q'f$. The operator M_{n-1}^l is also denoted briefly by M.

Definition 1. Let $\Omega \subset \mathbb{R}^{n+1}$ be open. A function $f : \Omega \to C\ell_{0,n}$ is called **left k-hypermonogenic**, if $f \in \mathcal{C}^1(\Omega)$ and $x_n M_k^l f(x) = 0$ for any $x \in \Omega$. The **right k-hypermonogenic functions** are defined similarly. The $(n-1)$ –left hypermonogenic functions are called **hypermonogenic functions.** A twice continuously differentiable function $f : \Omega \to C\ell_{0,n}$ is called **k-hyperbolic harmonic** if $\overline{M}_k^l M_k^l f = 0$.

We recall that $k-$hypermonogenic functions form a right $C\ell_{0,n-1}$-module in an open subset Ω of $\mathbb{R}^{n+1}$ (see [2]). Moreover the set of k-hyperbolic harmonic functions in an open subset Ω of $\mathbb{R}^{n+1}$ is a left and right $C\ell_{0,n-1}$-module in Ω (see [2]).

The operator M_{n-1} decomposes the hyperbolic Laplace operator $\triangle_{n-1} f = x_n^2 \triangle f - (n-1)\, x_n \frac{\partial f}{\partial x_n}$ for $C\ell_{0,n-1}$-valued functions. The general decomposition result is the following.

Proposition 1 (see [2]). Let $f : \Omega \to C\ell_{0,n}$ be twice continuously differentiable. Then

$$M_k \overline{M}_k f = \overline{M}_k M_k f = \triangle f - \frac{k}{x_n}\frac{\partial f}{\partial x_n} + k\frac{Qf}{x_n^2} e_n.$$

A similar characterization of k-hypermonogenic functions as complex holomorphic function is stated next.

Theorem 1 (see [2]). *Let $f : \Omega \to C\ell_{0,n}$ be twice continuously differentiable. Then f is k-hypermonogenic if and only if f and xf are k-hyperbolic harmonic functions.*

There is an important relation between k- and $-k$-hypermonogenic functions.

Theorem 2 (see [3]). *Let Ω be an open subset of $\mathbb{R}^{n+1} \backslash \{x_n = 0\}$ and $f : \Omega \to C\ell_{0,n}$ be a $\mathcal{C}^1(\Omega, C\ell_n)$ function. A function $f : \Omega \to C\ell_n$ is k-hypermonogenic if and only if the function $x_n^{-k} f e_n$ is $-k$-hypermonogenic.*

2 Mean value properties

The key idea for proving mean value properties is a relation between k-hyperbolic harmonic functions and eigenfunctions of the Laplace-Beltrami equation of the Poincaré upper half plane, stated as follows.

Lemma 1. *Let Ω be an open set contained in $\mathbb{R}^{n+1}_+$. A function $f : \Omega \to C\ell_{0,n}$ is k-hyperbolic harmonic if and only if the function $g(x) = x_n^{\frac{n-k-1}{2}} f(x)$ satisfies the following system of equations*

$$\triangle Pg - \frac{n-1}{x_n}\frac{\partial Pg}{\partial x_n} + \frac{1}{4}\left(n^2 - (k+1)^2\right)\frac{Pg}{x_n^2} = 0, \tag{4}$$

$$\triangle Qg - \frac{n-1}{x_n}\frac{\partial Qg}{\partial x_n} + \frac{1}{4}\left(n^2 - (k-1)^2\right)\frac{Qg}{x_n^2} = 0. \tag{5}$$

Proof of the result is just simple computations. Note that in the case $k = 0$ we obtain the following corollary.

Corollary 1. Let Ω be an open set contained in $\mathbb{R}^{n+1}_+$. A function $f : \Omega \to C\ell_{0,n}$ is harmonic if and only if the function $g(x) = x_n^{\frac{n-1}{2}} f(x)$ satisfies the equation

$$\triangle g - \frac{n-1}{x_n}\frac{\partial g}{\partial x_n} + \frac{1}{4}\left(n^2 - 1\right)\frac{g}{x_n^2} = 0. \tag{6}$$

The preceding equations are important, since the hyperbolic Laplace is invariant under the Möbius transformations mapping the upper half space onto itself. We also use an idea to consider k-hyperbolic harmonic functions as transformed eigenfunctions of the hyperbolic Laplace operator. In addition to the equation on the upper half space, we also need their models on the unit ball, denoted by $B_{n+1} = \left\{x \in \mathbb{R}^{n+1} \mid |x| < 1\right\}$. The surface area of B_{n+1} is denoted by ω_n. We recall that the fundamental isometry $T : B_{n+1} \to \mathbb{R}^{n+1}_+$ between the upper half plane and the unit ball is induced by $T(y) = (y + e_n)(e_n y + 1)^{-1}$. Moreover we have $PT(y) = 2Py|y - e_n|^{-2}$ and $QT(y) = \left(1 - |y|^2\right)|y - e_n|^{-2}$. Note also that

$$y = T^{-1}(x) = (x - e_n)(-e_n x + 1)^{-1} \quad \text{and} \quad |y| = \frac{|x - e_n|}{|x + e_n|} = \tanh \frac{d_h(x, e_n)}{2}$$

where $d_h(x, e_n)$ is the hypeerbolic distance given by the metric $ds^2 = x_n^2\left(dx_0^2 + ... + dx_n^2\right)$.

Applying (see [1, p. 20]), we obtain the model of our equation for $C\ell_{0,n}$-valued k-hyperbolic harmonic functions in the unit ball as follows.

Proposition 2. Let Ω be an open set contained in $\mathbb{R}^{n+1}_+$. If a function $u : \Omega \to C\ell_{0,n}$ is k-hyperbolic then the function

$$v(y) = \frac{|y - e_n|^{1-n+k}}{\left(1 - |y|^2\right)^{\frac{1-n+k}{2}}} u(T(y))$$

satisfies the equation

$$\triangle v + \frac{2(n-1)}{1 - |y|^2} \sum_{i=0}^{n} y_i \frac{\partial v}{\partial y_i} + \frac{n^2 - (k+1)^2}{\left(1 - |y|^2\right)^2} Pv + \frac{n^2 - (k-1)^2}{\left(1 - |y|^2\right)^2} Qve_n = 0 \tag{7}$$

on $\Phi^{-1}(\Omega) \subset B_{n+1}$. Conversely, if the function $v : T^{-1}(\Omega) \to \mathbb{R}$ satisfies the equation (7) then the function $u(x) = x_n^{\frac{n-k-1}{2}} v\left(T^{-1}(x)\right)$ is k-hyperbolic in Ω.

Applying the previous proposition and [1, p.24] we obtain an important solution in terms of hypergeometric functions, defined by

$${}_2F_1(a, b; c; x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!}.$$

.

Theorem 3. *Let $n \in \mathbb{N}$ and $k \in \mathbb{R}$. Denote by σ_e the Euclidean surface measure of B_{n+1}. The function*

$$\mu_{n,k}(r_h) = \left(\cosh \frac{r_h}{2}\right)^{-(n+|k+1|)} \frac{1}{\omega_n} \int_{\partial B_{n+1}} \frac{d\sigma_e(y)}{\left|\tanh \frac{r_h}{2} e_n - y\right|^{n+|k+1|}} =$$

$$= \frac{1}{e^{\frac{n+|k+1|}{2}r_h}} {}_2F_1\left(\frac{n+|k+1|}{2}, \frac{n}{2}; n; 1-e^{-2r_h}\right).$$

is a unique eigenfunction of the hyperbolic Laplace $\triangle_h$ *corresponding to the eigenvalue* $\frac{1}{4}\left((k+1)^2 - n^2\right)$ *depending only on the hyperbolic distance* $r_h = d_h(x, e_n)$ *in* $\mathbb{R}^{n+1}_+$*and satisfying* $\mu_{n,k}(0) = 1$*. Moreover the function* $x_n^{\frac{1-n+k}{2}} \mu_{n,k}(r_h)$ *is k-hyperbolic.*

Proof. Applying [1] we infer that there exists a unique solution $u : B_{n+1} \to \mathbb{R}$ of (7) depending only on the radius and satisfying $u(0) = 1$ given by

$$u(r) = (1-r^2)^{\frac{1}{2}(n+|k+1|)} \frac{1}{\omega_n} \int_{\partial B_{n+1}} \frac{d\sigma_e(y)}{|re_n - y|^{n+|k+1|}}.$$

Applying [4] we infer the equality

$$\int_{\partial B_{n+1}} \frac{d\sigma(y)}{|y-re_n|^{2s}} = 2^n \omega_{n-1} \int_0^\infty \frac{t^{n-1}}{(1+t^2)^{n-s}((1+r)^2+(1-r)^2t^2)^s} dt =$$

$$= (1+r)^{-2s} 2^n \omega_{n-1} \int_0^\infty \frac{t^{n-1}}{(1+t^2)^{n-s}\left(1+\frac{(1-r)^2}{(1+r)^2}t^2\right)^s} dt$$

where ω_{n-1} is a surface area of $n-$dimensional unit ball B_nHence we have

$$\int_0^\infty \frac{t^{n-1}}{(1+t^2)^{n-s}\left(1+\frac{(1-r)^2}{(1+r)^2}t^2\right)^s} dt = \frac{1}{2} B\left(\frac{n}{2}, \frac{n}{2}\right) {}_2F_1\left(s, \frac{n}{2}; n; 1-\frac{(1-r)^2}{(1+r)^2}\right).$$

where

$$B\left(\frac{n}{2}, \frac{n}{2}\right) = \int_0^1 t^{\frac{n}{2}-1}(1-t)^{\frac{n}{2}-1} dt = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+1}{2}\right)} = \frac{\omega_n}{2^{n-1}\omega_{n-1}}$$

and therefore

$$u(r) = (1-r^2)^{\frac{1}{2}(n+|k+1|)} \frac{1}{\omega_n} \int_{\partial B_{n+1}} \frac{d\sigma_e(y)}{|re_n - y|^{n+|k+1|}} =$$

$$= \left(\frac{1-r}{1+r}\right)^{\frac{1}{2}(n+|k+1|)} F_1\left(\frac{n+|k+1|}{2}, \frac{n}{2}; n; 1-e^{-2r_h}\right) =$$

$$= \frac{1}{e^{\frac{n+|k+1|}{2}r_h}} {}_2F_1\left(\frac{n+|k+1|}{2}, \frac{n}{2}; n, 1-e^{-2r_h}\right).$$

Since $r = \tanh \frac{r_h}{2}$ we just simplify

$$\frac{(1-r)^2}{(1+r)^2} = \left(\frac{\cosh \frac{r_h}{2} - \sinh \frac{r_h}{2}}{\cosh \frac{r_h}{2} + \sinh \frac{r_h}{2}}\right)^2 = e^{-2r_h}.$$

Hence we obtain the function

$$\mu_{n,k}(r_h) = \frac{1}{e^{\frac{n+|k+1|}{2}r_h}} {}_2F_1\left(\frac{n+|k+1|}{2}, \frac{n}{2}; n, 1-e^{-2r_h}\right).$$

The mean value property for eigenfunctions of the Laplace operator, stated next, is verified in [10].

Theorem 4. *Let $U \subset \mathbb{R}_+^{n+1}$ be open. If $f : U \to \mathbb{R}$ satisfies the equation (4) then*

$$f(a) = \frac{1}{\omega_n \sinh^n r_h \mu_{n,k}(r_h)} \int_{\partial B_h(a,R_h)} f(x) d\sigma_h(x)$$

for all hyperbolic balls satisfying $\overline{B_h(a, r_h)} \subset U$, where σ_h is the hyperbolic surface measure given by $d\sigma_h = \frac{d\sigma_e}{x_n^n}$.

Using Lemma 1 we obtain the following mean value property.

Theorem 5. *If $f : U \to \mathbb{R}$ is k-hyperbolic harmonic then*

$$f(a) = \frac{a_n^{\frac{1-n+k}{2}}}{\omega_n \sinh^n r_h \mu_{n,k}(r_h)} \int_{\partial B_h(a,r_h)} x_n^{\frac{n-k-1}{2}} f(x) \, d\sigma_h(x)$$

for all hyperbolic balls satisfying $\overline{B_h(a, r_h)} \subset U$.

Theorem 6. *If $f : U \to C\ell_{0,n-1}$ is k-hyperbolic harmonic then*

$$f(a) = \frac{a_n^{\frac{1-n+k}{2}}}{\sigma_n \sinh^n r_h \mu_{n,k}(r_h)} \int_{\partial B_h(a,r_h)} x_n^{\frac{n-k-1}{2}} f(x) \, d\sigma_h(x)$$

for all hyperbolic balls satisfying $\overline{B_h(a, r_h)} \subset U$.

Proof. We may apply Theorem 5 for all components of f_ν where $\nu \subset \{0, ..., n-1\}$, since they are real k-hyperbolic harmonic. Then multiplying with e_ν and summing the equations we obtain

$$f(a) = \sum_{\nu \subset \{0,...,n-1\}} f_\nu(a) e_\nu =$$

$$= \sum_{\nu \subset \{0,...,n-1\}} \frac{a_n^{\frac{1-n+k}{2}}}{\sigma_n \sinh^n r_h \mu_{n,k}(r_h)} \int_{\partial B_h(a,R_h)} x_n^{\frac{n-k-1}{2}} (Pf)_\nu(x) e_\nu d\sigma_h(x) =$$

$$= \frac{a_n^{\frac{1-n+k}{2}}}{\sigma_n \sinh^n r_h \mu_{n,k}(r_h)} \int_{\partial B_h(a,r_h)} x_n^{\frac{n-k-1}{2}} Pf(x) d\sigma_h(x),$$

completing the proof. □

If f is k-hypermonogenic functions then using Proposition 1 we infer that Pf is $C\ell_{0,n-1}$-valued k-hyperbolic harmonic function. Applying the previous result we conclude.

Theorem 7. *If* $f : U \to C\ell_{0,n}$ *is* k*-hypermonogenic then*

$$Pf(a) = \frac{a_n^{\frac{1-n+k}{2}}}{\sigma_n \sinh^n R_h \mu_{n,k}(R_h)} \int_{\partial B_h(a,R_h)} x_n^{\frac{n-k-1}{2}} Pf(x) d\sigma_h(x)$$

for all hyperbolic balls satisfying $\overline{B_h(a, R_h)} \subset U$.

Corollary 2.

$$\mu_{n,k}(r_h) = \frac{1}{\sigma_n \sinh^n r_h} \int_{\partial B_h(e_n,r_h)} x_n^{\frac{n-k-1}{2}} d\sigma_h(x) =$$

$$= \frac{1}{\sigma_n \sinh^n r_h} \int_{\partial B(\cosh r_h, \sin r_h)} x_n^{-\frac{n+k+1}{2}} d\sigma_e(x).$$

If we apply the preceding theorem for xf we obtain the mean value property for the Q-part as follows.

Theorem 8. *If $f : U \to C\ell_{0,n}$ is k-hypermonogenic then*

$$Qf(a) = \frac{a_n^{\frac{1-n+k}{2}}}{\omega_n \sinh^n r_h \mu_{n,k}(r_h)} \int\limits_{\partial B_h(a,r_h)} -x_n^{\frac{n-k-1}{2}} P'\left(\frac{x - Pa}{a_n} f(x)\right) d\sigma_h(x) =$$

$$= \frac{a_n^{\frac{1-n+k}{2}}}{\omega_n \sinh^n r_h \mu_{n,k}(r_h)} \int\limits_{\partial B_h(a,R_h)} x_n^{\frac{n-k-1}{2}} Q\left(e_n \frac{x - Pa}{a_n} f(x)\right) d\sigma_h(x)$$

for all hyperbolic balls satisfying $\overline{B_h(a, R_h)} \subset U$.

Proof. Assume that f is k-hypermonogenic. Then by virtue of Theorem 1 the function xf is also k-hyperbolic harmonic. Since k-hyperbolic functions are left and right $C\ell_{0,n-1}$ module we note that $\frac{Pa}{a_n} f$ is also $k-$hyperbolic harmonic. Hence $\frac{x-Pa}{a_n} f(x)$ is k-hyperbolic harmonic and $P\left(\frac{a-Pa}{a_n} f(a)\right) = -Q' f(a)$. Consequently the result follows from the previous theorem. □

Combining the last two theorems we obtain the mean value property for the k-hypermonogenic functions.

Theorem 9. *If $f : U \to C\ell_{0,n}$ is k-hypermonogenic and $\overline{B_h(a, r_h)} \subset U$*

$$f(a) = C \int\limits_{\partial B_h(a,r_h)} x_n^{\frac{n-k-1}{2}} \left(e_n (x - a) f(x) + e_n (\widehat{x} - a) \widehat{f(x)}\right) d\sigma_h(x).$$

for all hyperbolic balls satisfying $\overline{B_h(a, R_h)} \subset U$, where $C = \frac{a_n^{\frac{k-n-1}{2}}}{\omega_n \sinh^n r_h \mu_{n,k}(r_h)}$.

Proof. Assume that f is k-hypermonogenic. Since $f(x) = -e_n a_n e_n f(x) / a_n$ and $\widehat{f(x)} = -e_n a_n e_n \widehat{f(x)} / a_n$, applying the previous theorems and properties (2) and (3) we obtain the final formula. □

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SLICE REGULARITY IN SEVERAL VARIABLES

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Abstract. We introduce a class of slice regular functions of several variables on a real alternative algebra. In the quaternionic case, several variables have been considered recently by Colombo, Sabadini and Struppa [1]. Our approach to the definition of slice functions, which is based on the concept of stem functions, is the same as the one adopted by these authors. However, the condition of regularity is different, and allows to include in our class, in particular, the family of ordered polynomials in several variables. We prove some basic properties and results of slice and slice regular functions and give examples to illustrate this function theory.

1 Introduction

The theory of power series and more generally of *slice regular* functions of *one variable* in a real alternative algebra is now fairly developed. It was introduced firstly for functions of one quaternionic variable by Gentili and Struppa in [3, 4].

A related theory, concerning *slice monogenic* functions on Clifford algebras, was introduced by Colombo, Sabadini and Struppa in [2]. In [5] and [6], a new approach to slice regularity, based on the concept of *stem function*, allowed to extend the theory to any real alternative algebra A of finite dimension.

In the present paper, we propose a possible generalization of the theory to *several variables* in A. Our function theory includes, in particular, the class of (ordered) polynomials in several variables. For $A = \mathbb{H}$, several variables have been studied recently by Colombo, Sabadini and Struppa [1]. The approach via stem functions is similar, but the definition of regularity is different, as we will see in Section 4.3.

After having given the basic definitions, we state some results which show the richness of this function theory. We give a Cauchy integral formula and some of its fundamental consequences, and we show that some results about the removability of singularities, which are true for several complex variables, continue to hold in our setting.

2 The quadratic cone

Let A be a *finite-dimensional, alternative real algebra with identity* with a fixed $\mathbb{R}$-linear *antiinvolution*. Define the *trace* $t(x) := x + x^c \in A$ and the *norm* $n(x) := xx^c \in A$.

Definition 1. The *quadratic cone* of A is the subset

$$\mathcal{Q}_A := \mathbb{R} \cup \{x \in A \mid t(x) \in \mathbb{R},\ n(x) \in \mathbb{R},\ 4n(x) > t(x)^2\}.$$

The square roots of -1 in the algebra are the elements of $\mathbb{S}_A := \{J \in \mathcal{Q}_A \mid J^2 = -1\}$.

Examples

1. $\mathbb{H}$ and $\mathbb{O}$ with the usual conjugation ($\mathcal{Q}_\mathbb{H} = \mathbb{H}$ and $\mathcal{Q}_\mathbb{O} = \mathbb{O}$).
2. The real Clifford algebra $Cl_{0,n} = \mathbb{R}_n$ with Clifford conjugation. The quadratic cone $\mathcal{Q}_n$ of $\mathbb{R}_n$ is the real algebraic subset defined by $x_K = 0$, $x \cdot (xe_K) = 0\ \forall e_K \neq 1$ such that $e_K^2 = 1$. It contains the subspace of *paravectors*.
3. In $\mathbb{R}_3$, $\mathcal{Q}_A = \{x \in \mathbb{R}_3 \mid x_{123} = 0, x_1x_{23} - x_2x_{13} + x_3x_{12} = 0\}$.

The algebras with $\mathcal{Q}_A = A$ are *only* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ with the usual conjugation (cf. Frobenius–Zorn's Theorem)

Proposition 1. Let $\mathrm{Im}\,(A) := \{x \in A \mid x^2 \in \mathbb{R}, x \notin \mathbb{R} \setminus \{0\}\}$. For every $x \in \mathcal{Q}_A$, there exist unique elements $x_0 \in \mathbb{R}$, $y \in \mathrm{Im}\,(A) \cap \mathcal{Q}_A$ with $t(y) = 0$, such that $x = x_0 + y$. For $J \in \mathbb{S}_A$, let $\mathbb{C}_J := \langle 1, J \rangle \simeq \mathbb{C}$ be the subalgebra generated by J. Then $\mathcal{Q}_A = \bigcup_{J \in \mathbb{S}_A} \mathbb{C}_J$ and $\mathbb{C}_I \cap \mathbb{C}_J = \mathbb{R}$ for every $I, J \in \mathbb{S}_A$, $I \neq \pm J$.

3 Slice regular functions: one variable

We recall some definitions from [5,6], where the slice regular functions of one variable in A were introduced. Let $A \otimes \mathbb{C} = A \otimes_\mathbb{R} \mathbb{C}$ be the complexified algebra.

Definition 2. Let $D \subseteq \mathbb{C}$. If a function $F : D \to A \otimes \mathbb{C}$ is complex intrinsic, i.e. $F(\overline{z}) = \overline{F(z)}$ for every $z \in D$ such that $\overline{z} \in D$, then F is called a *stem function* on D. Let $\Omega_D := \{x = \alpha + \beta J \in \mathbb{C}_J \mid \alpha + i\beta \in D,\ J \in \mathbb{S}_A\}$ be a circular set in the quadratic cone $\mathcal{Q}_A$. Any stem function $F = F_1 + iF_2 : D \to A \otimes \mathbb{C}$ induces a *(left) slice function* $f = \mathcal{I}(F) : \Omega_D \to A$. If $x = \alpha + \beta J \in D_J := \Omega_D \cap \mathbb{C}_J$, we set

$$f(x) := F_1(z) + JF_2(z) \quad (z = \alpha + i\beta).$$

Definition 3 (see [5, 6]). A slice function is *slice regular* if its stem function F is holomorphic. $\mathcal{SR}(\Omega_D) := \{f \in \mathcal{S}(\Omega_D) \mid f = \mathcal{I}(F), F \text{ holomorphic}\}$.

Examples

1. *Polynomials* $p(x) = \sum_{j=0}^{m} x^j a_j$ with right coefficients in A are slice regular functions on $\mathcal{Q}_A$.
2. *Convergent power series* $\sum_k x^k a_k$ are slice regular functions on the intersection of $\mathcal{Q}_A$ with a ball centered in the origin.
3. If $A = \mathbb{H}$ and $D \cap \mathbb{R} \neq \emptyset$, then $f \in \mathcal{SR}(\Omega_D)$ if and only if it is Cullen regular [3,4].
4. If $A = \mathbb{R}_n$, $n > 2$, and $D \cap \mathbb{R} \neq \emptyset$, then $f \in \mathcal{SR}(\Omega_D)$ if and only if the restriction of f to the paravectors is a *slice monogenic function* [2].

4 Slice regular functions: several variables

4.1 Stem functions and slice functions

Let D be an open subset of $\mathbb{C}^n$, invariant w.r.t. complex conjugation in every variable $z_1, \ldots, z_n$.

Definition 4. Given a function $F : D \to A \otimes \mathbb{R}_n$, with $F = \sum_{K \in \mathcal{P}(n)} e_K F_K$, we say that F is *Clifford-intrinsic* if, for each $K \in \mathcal{P}(n)$, $h \in \{1, \ldots, n\}$ and $z = (z_1, \ldots, z_n) \in D$, the components $F_K : D \to A$ satisfy:

$$F_K(z_1, \ldots, z_{h-1}, \overline{z}_h, z_{h+1}, \ldots, z_n) = \begin{cases} F_K(z) & \text{if } h \notin K \\ -F_K(z) & \text{if } h \in K \end{cases}$$

Definition 5. A continuous Clifford–intrinsic function is a *stem function* on D.

Define the (ordered) product $\prod_{h \in H}^{\rightarrow} x_h$ of $x_{h_1}, \ldots, x_{h_p} \in A$ as $\prod_{h \in H}^{\rightarrow} x_h := (\cdots((x_{h_1} x_{h_2}) x_{h_3}) \cdots) x_{h_p}$. Let Ω_D be the *circular* subset of $\mathcal{Q}_A^n$ associated to the open set $D \subseteq \mathbb{C}^n$:

$$\Omega_D = \{x \in \mathcal{Q}_A^n : x_h = \alpha_h + \beta_h J_h \in \mathbb{C}_J, J_h \in \mathbb{S}_A, (\alpha_1 + i\beta_1, \ldots, \alpha_n + i\beta_n) \in D\}.$$

Definition 6. Given a stem function $F : D \to A \otimes \mathbb{R}_n$ with $F = \sum_{K \in \mathcal{P}(n)} e_K F_K$, we define the *(left) slice function* $\mathcal{I}(F) : \Omega_D \to A$ induced by F as follows $\mathcal{I}(F)(x) := \sum_{K \in \mathcal{P}(n)} J_K F_K(\alpha_1 + i\beta_1, \ldots, \alpha_n + i\beta_n)$ for each $x = (x_1, \ldots, x_n) = (\alpha_1 + J_1\beta_1, \ldots, \alpha_n + J_n\beta_n)$, where $J_K := \prod_{k \in K}^{\rightarrow} J_k$.

Examples

1. For each $h = 1, \ldots, n$, the coordinate function x_h is a slice function on $\mathcal{Q}_A^n$: if $x_h = \alpha_h + J_h\beta_h$, x_h is induced by the stem function $\zeta_h(z) := \alpha_h + e_h\beta_h$.

2. For each $\ell \in \mathbb{N}^n$ and $a \in A$, the stem function $\zeta_1^{\ell_1}(z)\cdots\zeta_n^{\ell_n}(z)a := (\alpha_1 + e_1\beta_1)^{\ell_1}\cdots(\alpha_n + e_n\beta_n)^{\ell_n}a$ induces the *monomial slice function* $x^\ell a = (\prod^{\rightarrow}_{h\in\{1,\ldots,n\}} x_h^{\ell_h})a$.
3. Let $L \subset \mathbb{N}^n$ and $a_\ell \in A$. Then the *polynomial function* from $\mathcal{Q}_A^n$ to A, sending x into $p(x) = \sum_{\ell\in L} x^\ell a_\ell$, is a slice function.
4. *Convergent power series* $\sum_{\ell\in\mathbb{N}^n} x^\ell a_\ell$ are slice functions on the intersection of $\mathcal{Q}_A^n$ with a product of balls centered in the origin.

Proposition 2 (Smoothness). Let $f = \mathcal{I}(F) : \Omega_D \to A$ be a slice function. The following statements hold:

1. If $F \in C^0(D, A\otimes\mathbb{R}_n)$, then $f \in C^0(\Omega_D, A)$.
2. Let $s_1 = 2^n(s+1)-1$. If $F \in C^{s_1}(D, A\otimes\mathbb{R}_n)$ for some $s \in \mathbb{N}^*\cup\{\infty\}$, then $f \in C^s(\Omega_D, A)$.
3. If $F \in C^\omega(D, A\otimes\mathbb{R}_n)$, then $f \in C^\omega(\Omega_D, A)$.

Proposition 3 (Identity principle). Let $f, g : \Omega_D \to A$ be slice functions and let $I \in \mathbb{S}_A$ such that $f = g$ on $\Omega_D \cap (\mathbb{C}_I)^n$. Then $f = g$ on the whole Ω_D.

4.2 Complex structures on $\mathbb{R}_n$

Let us introduce some complex structures on $\mathbb{R}_n$.

Definition 7. For each $h = 1,\ldots,n$, define the *complex structure* $\mathcal{J}_h$ on $\mathbb{R}_n$ by

$$\mathcal{J}_h(e_K) := \begin{cases} -e_{K\setminus\{h\}} & \text{if } h \in K \\ e_{K\cup\{h\}} & \text{if } h \notin K \end{cases}.$$

From the definition, it follows immediately that $\mathcal{J}_h^2 = -id_{\mathbb{R}_n}$. In other words, the endomorphisms $\mathcal{J}_h$ are complex structures on $\mathbb{R}_n$. One can easily verify that $\mathcal{J}_1$ is the left multiplication by e_1, $\mathcal{J}_n$ is the right multiplication by e_n and, for every $h = 1,\ldots,n$, $\mathcal{J}_h$ coincides with the left multiplication by e_h on the complex plane $\mathbb{C}_{e_h} = \langle 1, e_h\rangle$ of $\mathbb{R}_n$.

Proposition 4. The complex structures $\mathcal{J}$ are pairwise commuting and therefore they define commuting *Cauchy-Riemann operators* w.r.t. $\mathcal{J}_h$:

$$\partial_h F = \frac{1}{2}\left(\frac{\partial F}{\partial\alpha_h} - \mathcal{J}_h\left(\frac{\partial F}{\partial\beta_h}\right)\right), \quad \bar\partial_h F = \frac{1}{2}\left(\frac{\partial F}{\partial\alpha_h} + \mathcal{J}_h\left(\frac{\partial F}{\partial\beta_h}\right)\right).$$

4.3 Slice regularity: several variables

Extend the complex structures $\mathcal{J}_h$ to $A\otimes\mathbb{R}_n$ by setting $\mathcal{J}_h(a\otimes x) = a\otimes\mathcal{J}_h(x)$ for every $a \in A, x \in \mathbb{R}_n$.

Definition 8. Let $F : D \to A \otimes \mathbb{R}_n$ be a C^1 stem function and let $f = \mathcal{I}(F) : \Omega_D \to A$. F is a *holomorphic stem function* if, for each $h = 1, \dots, n$ and each fixed $z^0 := (z_1^0, \dots, z_n^0) \in D$, the function $F_h^{z_0} : D_h \to (A \otimes \mathbb{R}_n, \mathcal{J}_h)$ defined by $z_h \mapsto F(z_1^0, \dots, z_{h-1}^0, z_h, z_{h+1}^0, \dots, z_n^0)$ is holomorphic on a domain $D_h \ni z_h^0$ of $\mathbb{C}$ or, equivalently, if $\bar{\partial}_h F = 0$ on D for every $h = 1, \dots, n$. If F is holomorphic, then we say that $f = \mathcal{I}(F)$ is a *slice regular function.*

Remark 1. For $A = \mathbb{H}$, several variables have been considered recently by Colombo, Sabadini and Struppa [1]. Slice functions defined via stem functions are the same, but regularity is different (they use the complex structure L_{e_h} in the place of $\mathcal{J}_h$).

Examples

1. For each $\ell \in \mathbb{N}^n$ and $a \in A$, the *monomial slice function* $x^\ell a : \mathcal{Q}_A^n \to A$ is regular. Therefore every *(ordered) polynomial function* $p(x) = \sum_{\ell \in L} x^\ell a_\ell$ with right coefficients in A is slice regular.
2. *Convergent power series* $\sum_{\ell \in \mathbb{N}^n} x^\ell a_\ell$ are slice regular functions on the intersection of $\mathcal{Q}_A^n$ with a product of balls centered in the origin.

Proposition 5. Let $F = \sum_{K \in P(n)} e_K F_K : D \to A \otimes \mathbb{R}_n$ be a stem function of class C^1. Let $f = \mathcal{I}(F) : \Omega_D \to A$. We denote by $f_I : \Omega_D \cap (\mathbb{C}_I)^n \to A$ the restriction of f on $\Omega_D \cap (\mathbb{C}_I)^n$. The following assertions are equivalent:

(1) f is slice regular.

(2) $\frac{\partial F_K}{\partial \alpha_h} = \frac{\partial F_{K \cup \{h\}}}{\partial \beta_h}$, $\frac{\partial F_K}{\partial \beta_h} = -\frac{\partial F_{K \cup \{h\}}}{\partial \alpha_h}$ for each K, h with $K \not\ni h$.

(3) There exists $I \in \mathbb{S}_A$ such that f_I is holomorphic w.r.t. the complex structures on $(\mathbb{C}_I)^n$ and on A defined by the left multiplication by I.

(3′) For every $I \in \mathbb{S}_A$, f_I is holomorphic w.r.t. the complex structures on $(\mathbb{C}_I)^n$ and on A defined by the left multiplication by I.

4.4 Products and derivatives

Proposition 6. Let $D = \prod_{h=1}^n D_h$. For each $h = 1, \dots, n$, let $F^h : D_h \to A \otimes \mathbb{C}_{e_h} \subseteq A \otimes \mathbb{R}_n$ be a (one variable) stem function of class C^1. Let $a \in A$ and $F : D \to A \otimes \mathbb{R}_n$ defined by $F(z_1, \dots, z_n) = \left(\overrightarrow{\prod}_{h \in \{1, \dots, n\}} F^h(z_h) \right) a$. Then F is a stem function, holomorphic if every F^h is holomorphic.

In general, the *pointwise* product of two slice functions is *not* a slice function. However, the pointwise product of stem functions (in the algebra $A \otimes \mathbb{R}_n$) is still a stem function.

Definition 9. Let $f = \mathcal{I}(F)$, $g = \mathcal{I}(G)$ slice functions. The *product* of f and g is the slice function $f \cdot g := \mathcal{I}(FG)$.

Proposition 7. If f, g are slice regular *and* $F = \sum_{K \in \mathcal{S}} e_K F_K$, $G = \sum_{H \in \mathcal{S}'} e_H G_H$, with $K \leqslant H$ for each $K \in \mathcal{S}$, $H \in \mathcal{S}'$, then $f \cdot g$ is slice regular.

Remark 2. The *ordering* of the variables is important for regularity: e.g. the function $x_2 \cdot x_1 = \mathcal{I}(\zeta_2 \zeta_1)$ is a slice function but it is not slice regular.

If $f = \mathcal{I}(F)$ is a slice function, of class C^1 on Ω_D, then the functions $\partial_h F$ and $\bar{\partial}_h F$ are stem functions on D.

Definition 10. We set

$$\frac{\partial f}{\partial x_h} := \mathcal{I}\left(\partial_h F\right), \quad \frac{\partial f}{\partial x_h^c} := \mathcal{I}\left(\bar{\partial}_h F\right), \quad h = 1, \ldots, n.$$

These functions are continuous slice functions on Ω_D.

The slice function f is slice regular if and only if $\frac{\partial f}{\partial x_h^c} = 0$ for every $h = 1, \ldots, n$. If f is slice regular, then also the derivatives $\frac{\partial f}{\partial x_h}$ are slice regular (it follows from the commutativity of the structures $\mathcal{J}_h$).

4.5 Cauchy integral formula

We now show that slice regular functions satisfy a Cauchy integral formula. As a consequence, we obtain that on a polydisc the class of slice regular functions coincides with that of convergent ordered power series.

Definition 11. Let $\Delta_y(x) = x^2 - t(y)x + n(y)$ and $\Gamma_A := \{(x, y) \in \mathcal{Q}_A \times \mathcal{Q}_A \mid \Delta_y(x) \neq 0\}$. We define the *Cauchy kernel of* A as the C^ω–function $C_A : \Gamma_A \to A$, slice regular in x, given by

$$C_A(x, y) := \left(\Delta_y(x)\right)^{-1}(y^c - x).$$

Fix $I \in \mathbb{S}_A$ and, for each $h = 1, \ldots, n$, a bounded open subset E_h of $\mathbb{C}$, whose boundary is piecewise of class C^1. Let $E_h(I) := \Omega_{E_h} \cap \mathbb{C}_I$ and let $\partial E_h(I)$ be the boundary of $E_h(I)$ in $\mathbb{C}_I$. Let $E := E_1 \times E_2 \times \ldots \times E_n \subset \mathbb{C}^n$. Denote by $\partial^* E(I)$ the *distinguished boundary* $\partial E_1(I) \times \partial E_2(I) \times \cdots \times \partial E_n(I)$ of $E(I) := E_1(I) \times E_2(I) \times \cdots \times E_n(I)$.

Theorem 1 (Cauchy integral formula). *If* $f \in \mathcal{SR}(\Omega_E, A) \cap C^0(\overline{\Omega}_E, A)$, *then*

$$f(x) = \frac{1}{(2\pi)^n} \int\limits_{\partial^* E(I)} C_A(x_1, \xi_1) \cdots C_A(x_n, \xi_n) d\xi_1 d\xi_2 \cdots d\xi_n I^{-n} f(\xi_1, \ldots, \xi_n)$$

for each $x = (x_1, \ldots, x_n) \in \Omega_E$ *if* A *is associative or for each* $x = (x_1, \ldots, x_n) \in E(I)$ *if* A *is not–associative. In particular,* f *is real analytic.*

Suppose that there exists a norm $\|\cdot\|_A$ on A which induces the euclidean topology on A and such that $\|x\|_A = \sqrt{\alpha^2 + \beta^2}$ for each $x = \alpha + J\beta \in \mathcal{Q}_A$. Let $r = (r_1, \ldots, r_n) \in (\mathbb{R}^+)^n$. Denote by B_r the polydisc $B(0, r_1) \times \cdots \times B(0, r_n)$ of $\mathbb{C}^n$ and by $B_A(r)$ the polydisc $\{(x_1, \ldots, x_n) \in A^n \mid \|x_1\|_A < r_1, \ldots, \|x_n\|_A < r_n\}$ of A^n. Note that $B_A(r)$ is an open neighborhood of 0 in A^n and $B_A(r) \cap \mathcal{Q}_A^n = \Omega_{B_r}$.

Corollary 1 (Ordered analyticity). Let $f \in \mathcal{SR}(\Omega_{B_r}, A) \cap C^0(\overline{\Omega}_{B_r}, A)$. Choose $I \in \mathbb{S}_A$ and, for each $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$, define $a_\ell \in A$ by setting

$$a_\ell := (2\pi I)^{-n} \int_{\partial^* B_r(I)} \xi_1^{-\ell_1 - 1} \cdots \xi_n^{-\ell_n - 1} d\xi_1 \cdots d\xi_n \, f(\xi_1, \ldots, \xi_n).$$

Then the ordered power series $\sum_{\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n} x_1^{\ell_1} \cdots x_n^{\ell_n} a_\ell$ converges uniformly on compact subsets of $B_A(r)$ and its sum is equal to $f(x)$ for each $x \in \Omega_{B_r}$.

Corollary 2. On Ω_{B_r}, the set of slice regular functions coincides with that of convergent ordered power series.

Corollary 3 (Cauchy's inequalities). Let $f \in \mathcal{SR}(\Omega_{B_r}, A) \cap C^0(\overline{\Omega}_{B_r}, A)$ and let $M > 0$ be a constant such that $\sup_{x \in \partial^* B_r(I)} \|f(x)\|_A \leqslant M$ for some $I \in \mathbb{S}_A$. Then there exists a constant N_A (depending only on $\| \cdot \|_A$) such that, for each $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$, it holds: $\|\partial_\ell f(0)\|_A \leqslant N_A \cdot M \cdot \ell! \cdot r_1^{-\ell_1} \cdots r_n^{-\ell_n}$, where $\partial_\ell := \partial^{\ell_1 + \ldots + \ell_n} / \partial \mathrm{Re}\,(x_1)^{\ell_1} \cdots \partial \mathrm{Re}\,(x_n)^{\ell_n}$.

4.6 Removability of singularities

Theorem 2 (Hartogs extension phenomenon). *Let* $D' \subset D \subset \mathbb{C}^n$ *open with compact closure* $K := \overline{D'} \subset D$ *such that* $D \setminus K$ *is connected. If* f *is a slice regular function on* $\Omega_{D \setminus K} = \Omega_D \setminus \overline{\Omega}_{D'}$, *then it extends uniquely to a slice regular function on the whole set* Ω_D.

Theorem 3. *Let* Θ *be a circular open subset of* A^n, *let* $Z = \Omega_W$ *be a proper closed subset of* Θ *with* W *locally analytic in* $\mathbb{C}^n$ *and let* $f \in \mathcal{SR}(\Theta \setminus Z, A)$. *Suppose* *that at least one of the following two condition holds:* (1) f *is locally bounded in* Θ*,* (2) $\operatorname{codim}(W) \geqslant 2$*. Then* f *extends to a slice regular function on the whole* Θ*.*

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CONCEPTS OF TRACE, DETERMINANT AND INVERSE OF CLIFFORD ALGEBRA ELEMENTS

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Abstract. In our paper we consider the notion of determinant of Clifford algebra elements. We present some formulas for determinant and inverse of Clifford algebra elements. Also we consider the notion of trace of Clifford algebra elements that relates to the notion of matrix trace. We use the generalization of the Pauli's theorem for 2 sets of elements that satisfy the main anticommutation conditions of Clifford algebra.

1 Introduction

The notion of determinant of Clifford algebra elements was considered in [3]. In our work we present some formulas for determinant and inverse of Clifford algebra elements. Also we consider the notion of trace of Clifford algebra elements that relates to the notion of matrix trace. We use the generalization of the Pauli's theorem for 2 sets of elements that satisfy the main anticommutation conditions of Clifford algebra.

After writing this paper author found the article [4] on the subject that is close to the subject of this paper. In particular, the article [4] contains the formulas that are similar to the formulas for the determinant in this paper. However, note that for the first time most of these formulas ($n = 1, 2, 3$) were introduced in [3].

2 Complex Clifford algebras and conjugations

Let p and q be nonnegative integers such that $p + q = n \geqslant 1$. We consider complex Clifford algebra $\mathcal{C}\ell(p, q)$. The construction of Clifford algebra $\mathcal{C}\ell(p, q)$ is discussed in details in [2].

Generators $e^1, e^2, \ldots, e^n$ satisfy the following conditions $e^a e^b + e^b e^a = 2\eta^{ab} e$, where $\eta = ||\eta^{ab}||$ is the diagonal matrix whose diagonal contains p elements equal to $+1$ and q elements equal to -1.

The elements $e^{a_1}\dots e^{a_k} = e^{a_1\dots a_k}$, $\quad 1 \leqslant a_1 < \dots a_k \leqslant n$, $\quad k = 1,2,\dots n$ together with the identity element e form a basis of Clifford algebra $\mathcal{C}\ell(p,q)$. The number of basis elements equals to 2^n.

Any Clifford algebra element $U \in \mathcal{C}\ell(p,q)$ can be written in the following form

$$U = ue + u_a e^a + \sum_{a_1<a_2} u_{a_1a_2} e^{a_1a_2} + \dots + u_{1\dots n} e^{1\dots n}, \tag{1}$$

where $u, u_a, u_{a_1a_2}, \dots u_{1\dots n}$ are complex constants.

We denote the vector subspaces spanned by the elements $e^{a_1\dots a_k}$ enumerated by the ordered multi-indices of length k by $\mathcal{C}\ell_k(p,q)$. The elements of the subspace $\mathcal{C}\ell_k(p,q)$ are denoted by $\overset{k}{U}$ and called elements of rank k.

Clifford algebra $\mathcal{C}\ell(p,q)$ is a superalgebra, so we have even and odd subspaces $\mathcal{C}\ell_{\mathrm{Even}}(p,q) = \oplus_k \mathcal{C}\ell_{2k}(p,q)$, $\quad \mathcal{C}\ell_{\mathrm{Odd}}(p,q) = \oplus_k \mathcal{C}\ell_{2k+1}(p,q)$ such that $\mathcal{C}\ell(p,q) = \mathcal{C}\ell_{\mathrm{Even}}(p,q) \oplus \mathcal{C}\ell_{\mathrm{Odd}}(p,q)$.

Let denote complex conjugation of matrix by $\overleftarrow{A}$, transpose matrix by A^T, Hermitian conjugate matrix (composition of these 2 operations) by $A^\dagger$.

Now let define some operations on Clifford algebra elements.

Complex conjugation. Operation of complex conjugation $U \to \bar{U}$ acts in the following way:

$$\bar{U} = \overleftarrow{u}\, e + \overleftarrow{u_a} e^a + \sum_{a_1<a_2} \overleftarrow{u_{a_1a_2}} e^{a_1a_2} + \sum_{a_1<a_2<a_3} \overleftarrow{u_{a_1a_2a_3}} e^{a_1a_2a_3} + \dots$$

Reverse and grade involution. Let define operation reverse and grade involution for $U \in \mathcal{C}\ell(p,q)$ in the following way

$$U^\sim = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \overset{k}{U}, \qquad U^\curlywedge = \sum_{k=0}^{n} (-1)^k \overset{k}{U}.$$

Pseudo-Hermitian conjugation. Let define Pseudo-Hermitian conjugation as composition of reverse and complex conjugation: $U^\ddagger = \bar{U}^\sim$.

Hermitian conjugation. In [2] we consider operation of Hermitian conjugation. We have the following formulas for these operation:

$$U^\dagger = \begin{cases} (e^{1\dots p})^{-1} U^\ddagger e^{1\dots p}, & \text{if } p \text{ - odd;} \\ (e^{1\dots p})^{-1} U^{\ddagger\curlywedge} e^{1\dots p}, & \text{if } p \text{ - even;} \\ (e^{p+1\dots n})^{-1} U^\ddagger e^{p+1\dots n}, & \text{if } q \text{ - even;} \\ (e^{p+1\dots n})^{-1} U^{\ddagger\curlywedge} e^{p+1\dots n}, & \text{if } q \text{ - odd.} \end{cases}$$

3 Matrix representations of Clifford algebra elements, recurrent method.

Complex Clifford algebras $\mathcal{C}\ell(p,q)$ of dimension n and different signatures $(p,q), p+q=n$ are isomorphic. Clifford algebras $\mathcal{C}\ell(p,q)$ are isomorphic to the matrix algebras of complex matrices. In the case of even n these matrices are of order $2^{\frac{n}{2}}$. In the case of odd n these matrices are block diagonal of order $2^{\frac{n+1}{2}}$ with 2 blocks of order $2^{\frac{n-1}{2}}$.

Consider the following matrix representations of Clifford algebra elements.

Identity element e of Clifford algebra $\mathcal{C}\ell(p,q)$ maps to identity matrix of corresponding order: $e \to \mathbf{1}$.

For $\mathcal{C}\ell(1,0)$ element e^1 maps to the following matrix $e^1 \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For $\mathcal{C}\ell(2,0)$ we have $e^1 \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\quad e^2 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Further, suppose we have a matrix representation for $\mathcal{C}\ell(2k,0)$, $n=2k$: $e^1,\ldots,e^n \to \gamma^1,\ldots,\gamma^n$.

Then, for Clifford algebra $\mathcal{C}\ell(2k+1,0)$ we have

$$e^a \to \begin{pmatrix} \gamma^a & 0 \\ 0 & -\gamma^a \end{pmatrix}, \quad a=1,\ldots,n, \quad e^{n+1} \to \begin{pmatrix} i^k\gamma^1\ldots\gamma^n & 0 \\ 0 & -i^k\gamma^1\ldots\gamma^n \end{pmatrix}.$$

For Clifford algebra $\mathcal{C}\ell(2k+2,0)$ we have the same matrices for $e^a, a=1,\ldots,n+1$ as in the previous case and for e^{n+2} we have $e^{n+2} \to \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$.

So, we have matrix representation for all Clifford algebras $\mathcal{C}\ell(n,0)$. In the cases of other signatures elements $e^a, a>p$ maps to the same matrices as in signature $(n,0)$ but with multiplication by imaginary unit i.

4 Operation of trace of Clifford algebra elements

Consider complex Clifford algebra $\mathcal{C}\ell(p,q)$ and introduce the operation of *trace of Clifford algebra element* $U \in \mathcal{C}\ell(p,q)$ as the following operation of projection onto subspace $\mathcal{C}\ell_0(p,q)$:

$$\mathrm{Tr}(U) = \langle U\rangle_0|_{e\to 1}. \tag{2}$$

For arbitrary element $U \in \mathcal{C}\ell(p,q)$ in the form (1) we have $\mathrm{Tr}(ue+u_a e^a+\ldots)=u$.

Theorem 1. . *Operation trace (2) of Clifford algebra element* $U \in \mathcal{C}\!\ell(p,q)$ *has the following properties:*

1) $\mathrm{Tr}(U+V) = \mathrm{Tr}(U) + \mathrm{Tr}(V)$, $\mathrm{Tr}(\alpha U) = \alpha \mathrm{Tr}(U)$, $\forall U, V \in \mathcal{C}\!\ell(p,q)$, $\forall \alpha \in \mathbb{C}$,

2) $\mathrm{Tr}(UV) = \mathrm{Tr}(VU)$, $\mathrm{Tr}(UVW) = \mathrm{Tr}(VWU) = \mathrm{Tr}(WUV)$, *but, in general* $\mathrm{Tr}(UVW) \neq \mathrm{Tr}(UWV), \forall U, V, W \in \mathcal{C}\!\ell(p,q)$.

3) $\mathrm{Tr}(U^{-1}VU) = \mathrm{Tr}(V)$ *for all* $V \in \mathcal{C}\!\ell(p,q)$, $U \in \mathcal{C}\!\ell^{\times}(p,q)$, *where* $\mathcal{C}\!\ell^{\times}(p,q)$ *is the set of all invertible Clifford algebra elements.*

4) $\mathrm{Tr}(U) = \mathrm{Tr}(U^{\curlywedge}) = \mathrm{Tr}(U^{\sim}) = \overleftarrow{\mathrm{Tr}(\overline{U})} = \overleftarrow{\mathrm{Tr}(U^{\ddagger})} = \overleftarrow{\mathrm{Tr}(U^{\dagger})}$.

There is a relation between operation trace Tr of Clifford algebra element $U \in \mathcal{C}\!\ell(p,q)$ and operation trace tr of quadratic matrix. To obtain this relation, at first, we will prove the following statement.

Lemma 1. *Consider recurrent matrix representation of Clifford algebra* $\mathcal{C}\!\ell(p,q)$ *(see above). For this representation* $U \to \underline{U}$ *we have*

$$\mathrm{tr}(\underline{U}) = 2^{[\frac{n+1}{2}]}\mathrm{Tr}(U), \qquad \mathrm{tr}(\underline{U}^{\curlywedge}) = \mathrm{tr}(\underline{U}).$$

Proof. Coefficient $2^{[\frac{n+1}{2}]}$ equals to the order of corresponding matrices. It is not difficult to see that trace of almost all matrices that correspond to basis elements equals to zero: $\mathrm{tr}(\underline{e}^A) = 0$, where A - any multi-index except empty. The only exception is identity element e, which corresponds to the identity matrix. In this case we have $\mathrm{tr}(\underline{e}) = 2^{[\frac{n+1}{2}]}$. Further we use linearity of trace and obtain $\mathrm{tr}(\underline{U}) = 2^{[\frac{n+1}{2}]}u = 2^{[\frac{n+1}{2}]}\mathrm{Tr}(U)$. The second property is a simple consequence of the first property, because $\mathrm{tr}(\underline{U}^{\curlywedge}) = 2^{[\frac{n+1}{2}]}\mathrm{Tr}(U^{\curlywedge}) = 2^{[\frac{n+1}{2}]}\mathrm{Tr}(U) = \mathrm{tr}(\underline{U})$. □

Theorem 2. *Consider complex Clifford algebra* $\mathcal{C}\!\ell(p,q)$. *Then*

$$\mathrm{Tr}(U) = \frac{1}{2^{[\frac{n+1}{2}]}}\mathrm{tr}(\gamma(U)), \tag{3}$$

where $\gamma(U)$ *- any matrix representation of Clifford algebra* $\mathcal{C}\!\ell(p,q)$ *of minimal dimension. Moreover, this definition of trace (3) is equivalent to the definition (2). New definition is well-defined because it doesn't depend on the choice of matrix representation.*

Proof. This property proved in the previous statement for the recurrent matrix representation. Let we have besides recurrent matrix representation $\underline{U} = U|_{e^a \to \gamma^a}$ another matrix representation $\underline{\underline{U}} = U|_{e^a \to \beta^a}$. Then, by Pauli's theorem in Clifford algebra of even dimension n there exists matrix T such that $\beta^a = T^{-1}\gamma^a T, \quad a =$

$1,\ldots,n$. Then, we have $\underline{\underline{U}} = T^{-1}\underline{U}T$ and $\mathrm{tr}(\underline{\underline{U}}) = \mathrm{tr}(T^{-1}\underline{U}T) = \mathrm{tr}(\underline{U})$. In the case of odd n we can have also another case (by Pauli's theorem), when two sets of matrices relate in the following way $\beta^a = -T^{-1}\gamma^a T, \quad a = 1,\ldots,n$. In this case we have $\underline{\underline{U}} = T^{-1}\underline{U^\curlywedge}T$. From $\mathrm{tr}(\underline{U^\curlywedge}) = \mathrm{tr}(\underline{U})$ (see Lemma 1) we obtain $\mathrm{tr}(\underline{\underline{U}}) = \mathrm{tr}(T^{-1}\underline{U^\curlywedge}T) = \mathrm{tr}(\underline{U^\curlywedge}) = \mathrm{tr}(\underline{U})$. □

5 Determinant of Clifford algebra elements

Determinant of Clifford algebra element $U \in \mathcal{C}\!\ell(p,q)$ is a complex number

$$\mathrm{Det}U = \det(\underline{U}), \tag{4}$$

which is a determinant of any matrix representation $\underline{U}$ of minimal dimension.

Now we want to show that this definition is well-defined. Let prove the following Lemma.

Lemma 2. *Consider the recurrent matrix representation (see above) of Clifford algebra* $\mathcal{C}\!\ell(p,q)$*. For this representation* $U \to \underline{U}$ *we have*

$$\det(\underline{U^\curlywedge}) = \det(\underline{U}).$$

Proof. In the case of Clifford algebra of even dimension n we have $U^\curlywedge = (e^{1\ldots n})^{-1}Ue^{1\ldots n}$. So, we obtain $\det(\underline{U^\curlywedge}) = \det(\underline{(e^{1\ldots n})^{-1}Ue^{1\ldots n}}) = \det(\underline{(e^{1\ldots n})^{-1}})\det(\underline{U})\det(\underline{e^{1\ldots n}}) = \det(\underline{U})$.

In the case of Clifford algebra of odd dimension generators maps to the block diagonal matrices and blocks are identical up to the sign: $e^a \to \begin{pmatrix} \gamma^a & 0 \\ 0 & -\gamma^a \end{pmatrix}$. It is not difficult to see that even part U_{Even} of arbitrary element $U = U_{\mathrm{Even}} + U_{\mathrm{Odd}}$ maps to the matrix with identical blocks, and odd part U_{Odd} of the element U maps to the matrix with the blocks differing in sign: $U_{\mathrm{Even}} \to \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, $U_{\mathrm{Odd}} \to \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}$. Then we have $U \to \begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix}$, $U^\curlywedge \to \begin{pmatrix} A-B & 0 \\ 0 & A+B \end{pmatrix}$ and $\det(U) = (A-B)(A+B) = \det(U^\curlywedge)$. □

Theorem 3. *Definition (4) is well-defined, i.e. it doesn't depend on the matrix representation.*

Proof. Consider the recurrent matrix representation $\underline{U} = U|_{e^a \to \gamma^a}$. The statement for this representation proved in the previous lemma. Let we have another matrix representation $\underline{\underline{U}} = U|_{e^a \to \beta^a}$. Then, by Pauli's theorem in Clifford algebra of even dimension n there exists a matrix T such that $\beta^a = T^{-1}\gamma^a T, \quad a = 1, \ldots, n$. Then we have $\underline{\underline{U}} = T^{-1}\underline{U}T$ and obtain $\det(\underline{\underline{U}}) = \det(T^{-1}\underline{U}T) = \det(T^{-1})\det(\underline{U})\det(T) = \det(\underline{U})$.

In the case of odd n, by Pauli's theorem we also have another case, where 2 sets of matrices relate in the following way $\beta^a = -T^{-1}\gamma^a T, \quad a = 1, \ldots, n$. In this case we have $\underline{\underline{U}} = T^{-1}\underline{U^\curlywedge}T$. From $\det(\underline{U^\curlywedge}) = \det(\underline{U})$ (see Lemma 2) we obtain $\det(\underline{\underline{U}}) = \det(T^{-1}\underline{U^\curlywedge}T) = \det(\underline{U^\curlywedge}) = \det(\underline{U})$. $\square$

Let formulate some properties of operation determinant of Clifford algebra element.

Theorem 4. *Operation determinant (4) of Clifford algebra element $U \in \mathcal{C}\!\ell(p,q)$ has the following properties*

1) $\mathrm{Det}(UV) = \mathrm{Det}(U)\mathrm{Det}(V), \quad \mathrm{Det}(\alpha U) = \alpha^{2^{[\frac{n+1}{2}]}}\mathrm{Det}(U), \quad \forall U, V \in \mathcal{C}\!\ell(p,q), \quad \forall \alpha \in \mathbb{C}$.

2) Arbitrary element $U \in \mathcal{C}\!\ell(p,q)$ is invertible if and only if $\mathrm{Det}U \neq 0$.

3) For any invertible element $U \in \mathcal{C}\!\ell(p,q)$ $\mathrm{Det}(U^{-1}) = (\mathrm{Det}U)^{-1}$.

4) $\mathrm{Det}(U^{-1}VU) = \mathrm{Det}(V) \quad \forall V \in \mathcal{C}\!\ell(p,q),\ U \in \mathcal{C}\!\ell^\times(p,q)$, *where $\mathcal{C}\!\ell^\times(p,q)$ is set of all invertible Clifford algebra elements.*

5) $\mathrm{Det}(U) = \mathrm{Det}(U^\curlywedge) = \mathrm{Det}(U^\sim) = \overleftarrow{\mathrm{Det}(\overline{U})} = \overleftarrow{\mathrm{Det}(U^\ddagger)} = \overleftarrow{\mathrm{Det}(U^\dagger)}$.

Definition (4) of determinant of Clifford algebra element $U \in \mathcal{C}\!\ell(p,q)$ is connected with its matrix representation. We have shown that this definition doesn't depend on matrix representation. So, determinant is a function of complex coefficients $u_{a_1 \ldots a_k}$ located before basis elements $e^{a_1 \cdots a_k}$ in (1). In the cases of small dimensions $n \leqslant 5$ we give expressions for determinant of Clifford algebra elements that doesn't relate to the matrix representation.

Now we need also 2 another operations of conjugations $\bigtriangledown$, $\triangle$:

$$(\overset{0}{U} + \overset{1}{U} + \overset{2}{U} + \overset{3}{U} + \overset{4}{U} + \overset{5}{U})^\bigtriangledown = \overset{0}{U} + \overset{1}{U} + \overset{2}{U} + \overset{3}{U} - \overset{4}{U} - \overset{5}{U}, \qquad n = 4, 5,$$

$$(\overset{0}{U} + \overset{1}{U} + \overset{2}{U} + \overset{3}{U} + \overset{4}{U} + \overset{5}{U})^\triangle = \overset{0}{U} + \overset{1}{U} + \overset{2}{U} + \overset{3}{U} + \overset{4}{U} - \overset{5}{U}, \qquad n = 5.$$

Theorem 5. *We have the following formulas for the determinant of Clifford algebra element $U \in \mathcal{C}\ell(p,q)$:*

$$\operatorname{Det} U = \begin{cases} U, & n=0; \\ UU^{\curlywedge}, & n=1; \\ UU^{\sim\curlywedge}, & n=2; \\ UU^{\sim}U^{\curlywedge}U^{\sim\curlywedge} = UU^{\sim\curlywedge}U^{\curlywedge}U^{\sim}, & n=3; \\ UU^{\sim}(U^{\curlywedge}U^{\sim\curlywedge})^{\triangledown} = UU^{\sim\curlywedge}(U^{\curlywedge}U^{\sim})^{\triangledown}, & n=4; \\ UU^{\sim}(U^{\curlywedge}U^{\sim\curlywedge})^{\triangledown}(UU^{\sim}(U^{\curlywedge}U^{\sim\curlywedge})^{\triangledown})^{\vartriangle}, & n=5. \end{cases} \tag{5}$$

Note, that these expressions are Clifford algebra elements of the rank 0. In this case we identify them with the constants: $ue \equiv u$.

Proof. The proof is by direct calculation. □

Note, that properties from Theorem 4 for small dimensions also can be proved with the formulas from Theorem 5. For example, in the case $n=3$ we have $\operatorname{Det}(UV) = UV(UV)^{\sim\curlywedge}(UV)^{\curlywedge}(UV)^{\sim} = UVV^{\sim\curlywedge}U^{\sim\curlywedge}U^{\curlywedge}V^{\curlywedge}V^{\sim}U^{\sim} = UU^{\sim\curlywedge}U^{\curlywedge}U^{\sim}VV^{\sim\curlywedge}V^{\curlywedge}V^{\sim} = \operatorname{Det}(U)\operatorname{Det}(V)$. We used the fact that $VV^{\sim\curlywedge}$ and $V^{\curlywedge}V^{\sim} = (VV^{\sim\curlywedge})^{\sim}$ are in Clifford algebra center $\mathcal{C}\ell_0(p,q) \oplus \mathcal{C}\ell_3(p,q)$ and commute with all elements.

Theorem 5 give us explicit formulas for inverse in $\mathcal{C}\ell(p,q)$. We have the following theorem.

Theorem 6. *Let U be invertible element of Clifford algebra $\mathcal{C}\ell(p,q)$. Then we have the following expressions for U^{-1}:*

$$(U)^{-1} = \begin{cases} \frac{e}{U}, & n=0; \\ \frac{U^{\curlywedge}}{UU^{\curlywedge}}, & n=1; \\ \frac{U^{\sim\curlywedge}}{UU^{\sim\curlywedge}}, & n=2; \\ \frac{U^{\sim}U^{\curlywedge}U^{\sim\curlywedge}}{UU^{\sim}U^{\curlywedge}U^{\sim\curlywedge}} = \frac{U^{\sim\curlywedge}U^{\curlywedge}U^{\sim}}{UU^{\sim\curlywedge}U^{\curlywedge}U^{\sim}}, & n=3; \\ \frac{U^{\sim}(U^{\curlywedge}U^{\sim\curlywedge})^{\triangledown}}{UU^{\sim}(U^{\curlywedge}U^{\sim\curlywedge})^{\triangledown}} = \frac{U^{\sim\curlywedge}(U^{\curlywedge}U^{\sim})^{\triangledown}}{UU^{\sim\curlywedge}(U^{\curlywedge}U^{\sim})^{\triangledown}}, & n=4; \\ \frac{U^{\sim}(U^{\curlywedge}U^{\sim\curlywedge})^{\triangledown}(UU^{\sim}(U^{\curlywedge}U^{\sim\curlywedge})^{\triangledown})^{\vartriangle}}{UU^{\sim}(U^{\curlywedge}U^{\sim\curlywedge})^{\triangledown}(UU^{\sim}(U^{\curlywedge}U^{\sim\curlywedge})^{\triangledown})^{\vartriangle}}, & n=5. \end{cases} \tag{6}$$

Note, that in denominators we have Clifford algebra elements of the rank 0. We identify them with the constants: $ue \equiv u$.

Proof. Statement follows from Theorem 5. □

Note, that formulas for determinant in Theorem 5 are not unique. For example, in the case of $n = 4$ we can use the following formulas, but only for even and odd Clifford algebra elements $U \in \mathcal{C}\!\ell_{\mathrm{Even}}(p,q) \cup \mathcal{C}\!\ell_{\mathrm{Odd}}(p,q)$.

Consider operation $+$, that acts on the even elements such that it changes the sign before the basis elements that anticommutes with e^1. For example, elements $e, e^{23}, e^{24}, e^{34}$ maps under $+$ into themselves, and elements $e^{12}, e^{13}, e^{14}, e^{1234}$ change the sign.

Theorem 7. *Consider $\mathcal{C}\!\ell(p,q)$, $n = p+q = 4$. Then for $U \in \mathcal{C}\!\ell_{\mathrm{Even}}(p,q)$ we have* $\mathrm{Det}U = UU^{\sim}U^{\sim+}U^{+}$. *For $U \in \mathcal{C}\!\ell_{\mathrm{Odd}}(p,q)$ we have* $\mathrm{Det}U = UU^{\sim}U^{\sim}U$.

Proof. The proof is by direct calculation. □

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II. Real Analysis, Functional Analysis and Operator Theory

II.2. Integral Transforms and Reproducing Kernels

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GABOR FRAMES, DISPLACED STATES AND THE LANDAU LEVELS: A TOUR IN POLYANALYTIC FOCK SPACES

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Abstract. We will observe that several results concerning Gabor frames, euclidean Landau levels and displaced Fock states, can be translated to the setting of polyanalytic Fock spaces. This will allow several observations. For instance, in the higher Landau level eigenspaces and in displaced Fock states there is an analogue of Perelomov results on completeness of systems of coherent states, which is equivalent to the completeness of Gabor systems with Hermite functions. Another observation is that the parameter m in the weight function $e^{-m|z|^2}$ of polyanalytic Fock spaces corresponds to the strengh of the magnetic field of the Schrödinger operator which leads to the Landau Laplacian. We will motivate our presentation with an analogy to a very simple multiplexing problem from signal analysis.

1 Introduction

The solutions of the Cauchy-Riemann equation

$$\frac{d}{d\overline{z}}F(z) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial \xi}\right)F(x+i\xi) = 0,$$

known as *analytic functions*, provide one of the most well studied and useful objects in Mathematics. The analiticity restriction is so important that non-analytic functions are often seen as "bad objects"and therefore not worthy of further study. However, there are intermediate classes of non-analytic functions which posses significant structure. They are called *polyanalytic functions.* A function $F(z)$, defined on a subset of $\mathbb{C}$, is said to be *polyanalytic of order* $n-1$ if it satisfies the generalized Cauchy-Riemann equations

$$\left(\frac{d}{d\overline{z}}\right)^n F(z) = \frac{1}{2^n}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial \xi}\right)^n F(x+i\xi) = 0. \tag{1}$$

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This is equivalent to saying that $F(z)$ is a polynomial of order $n-1$ in $\overline{z}$ with analytic functions $\{\varphi_k(z)\}_{k=0}^{n-1}$ as coefficients:

$$F(z) = \sum_{k=0}^{n-1} \overline{z}^k \varphi_k(z) \tag{2}$$

To gain insight from a simple example, we consider $F(z) = 1 - |z|^2 = 1 - z\overline{z}$ and observe that $\frac{d}{d\overline{z}}F(z) = -z$, while $\left(\frac{d}{d\overline{z}}\right)^2 F(z) = 0$. As a result, $F(z)$ is not analytic in z, but is polyanalytic with $n = 2$. This simple example highlights one of the reasons why the properties of polyanalytic functions can be rather intricated when compared with those of analytic functions: they can vanish on closed curves without vanishing identically, while analytic functions can not even vanish on a accumulation set of the complex plane! Still, many properties of analytic functions have found an extension to polyanalytic functions, often in a nontrivial form. The function theoretical aspects of polyanalytic functions have been investigated thoroughly, notably by the Russian school led by Balk [4]. More recently the subject gained a renewed interest within operator theory [5, 17]. Our investigations in the topic were originally motivated by applications in signal analysis, in particular by the results of Gröchenig and Lyubarskii on Gabor frames with Hermite functions [10, 11] but soon it was clear that Hilbert spaces of polyanalytic functions lie at the heart of several interesting mathematical topics and that they provide an explicit representation of the eigenspaces of the Landau levels.

We have organized the paper as follows. We start with a section on the Hilbert space theory of polyanalytic Fock spaces. The third section explains the connections to the theory of Gabor frames with Hermite functions. We quote some applications in Quantum Physics in section 4, namely the interpretation of the so-called true polyanalytic Fock spaces as the eigenspaces of the Euclidean Landau Hamiltonian with a constant magnetic field. In section 5 we take a close look to the reproducing kernels and some asymptotic results recently obtained in the study of random matrices.

2 Fock spaces of polyanalytic functions

2.1 The orthogonal decomposition

Write $\mathcal{L}_2(\mathbb{C})$ to denote the Hilbert space equipped with the norm

$$\|F\|^2_{\mathcal{L}_2(\mathbb{C})} = \int_{\mathbb{C}} |F(z)|^2 \, e^{-\pi|z|^2} \, d\mu(z),$$

where $d\mu(z)$ stands for area measure on $\mathbb{C}$. If we require the elements of the space to be analytic, we are lead to the Fock space $\mathcal{F}_2(\mathbb{C})$. Polyanalytic Fock spaces $\mathbf{F}_2^n(\mathbb{C})$ arise in an analogous manner, by requiring its elements to be polyanalytic. They seem to have been first considered by Balk [4, pag. 170] and, more recently, by Vasilevski [17], who obtained the following decompositions in terms of spaces $\mathcal{F}_2^n(\mathbb{C})$ which he called *true poly-Fock spaces*:

$$\mathbf{F}_2^n(\mathbb{C}) = \mathcal{F}_2^1(\mathbb{C}) \oplus ... \oplus \mathcal{F}_2^n(\mathbb{C}), \quad \mathcal{L}_2(\mathbb{C}) = \bigoplus_{n=1}^{\infty} \mathcal{F}_2^n(\mathbb{C}). \tag{3}$$

We will use the following definition of $\mathcal{F}_2^n(\mathbb{C})$ which is equivalent to the one given by Vasilevski: a function F belongs to the *true polyanalytic Fock space* $\mathcal{F}_2^{n+1}(\mathbb{C})$ if $\|F\|_{\mathcal{L}_2(\mathbb{C})} < \infty$ and there exists an entire function H such that

$$F(z) = \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} e^{\pi|z|^2} \left(\frac{d}{dz}\right)^n \left[e^{-\pi|z|^2} H(z)\right]. \tag{4}$$

It is easy to verify that the spaces $\mathcal{F}_2^n(\mathbb{C})$ are orthogonal using Green´s formula.

2.2 The multiplexing problem

Multiplexing is the transmission of several signals over a single channel, while allowing the receiver to recover the original signals. The orthogonal decomposition (3) can be used for this purpose, once we construct a map $\mathcal{B}^n$ sending an arbitrary $f \in L^2(\mathbb{R})$ to the space $\mathcal{F}_2^n(\mathbb{C})$. Then we can proceed as follows.

1. Given n signals $f_1, \ldots, f_n$, with finite energy ($f_k \in L^2(\mathbb{R})$ for every k), process each individual signal by evaluating $\mathcal{B}^k f_k$. This encodes each signal into one of the n orthogonal spaces $\mathcal{F}^1(\mathbb{C}), \ldots, \mathcal{F}^n(\mathbb{C})$.
2. Construct a new signal $F = \mathbf{Bf} = \mathcal{B}^1 f_1 + ... + \mathcal{B}^n f_n$ as a superposition of the n processed signals.
3. Sample, transmit, or process F.
4. Let P^k denote the orthogonal projection from $\mathbf{F}_2^n(\mathbb{C})$ onto $\mathcal{F}^k(\mathbb{C})$, then $P^k(F) = \mathcal{B}^k f_k$ by virtue of (3).
5. Finally, after inverting each of the transforms $\mathcal{B}^k$, we recover each component f_k in its original form.

The combination of n independent signals into a single signal $\mathbf{B}^n\mathbf{f}$ and the subsequent processing provides our multiplexing device. With two signals this can be outlined in the following scheme.

$$\begin{array}{ccccccccc}
f_1 & \to & \mathcal{B}f_1 & & & & \mathcal{B}f_1 & \to & f_1\\
 & & & \searrow & & P^1 \nearrow & & & \\
 & & & & \mathcal{B}f_1+\mathcal{B}^2 f_2 = \mathbf{Bf} & & & & \\
 & & & \nearrow & & P^2 \searrow & & & \\
f_2 & \to & \mathcal{B}^2 f_2 & & & & \mathcal{B}^2 f_2 & \to & f_2
\end{array}$$

We will use the above scheme as a source of ideas for our results. With some poetic license, we may consider that we apply signal analysis to mathematics.

2.3 The polyanalytic Bargmann transform

The construction of the map $\mathcal{B}^k$ above can be done as follows. To map the first signal $f_1 \in L^2(\mathbb{R})$ to the space $\mathcal{F}_2^1(\mathbb{C}) = \mathcal{F}_2(\mathbb{C})$ we can of course use the good old Bargmann transform $\mathcal{B}f(z) = \int_{\mathbb{R}} f(t)e^{2\pi tz-\pi z^2-\frac{\pi}{2}t^2}dt$. The remaining signals are mapped using

$$\mathcal{B}^{k+1}f(z) = \left(\frac{\pi^k}{k!}\right)^{\frac{1}{2}} e^{\pi|z|^2} \left(\frac{d}{dz}\right)^k \left[e^{-\pi|z|^2}\mathcal{B}f(z)\right]$$

It can be proved that $\mathcal{B}^k : L^2(\mathbb{R}) \to \mathcal{F}_2^k(\mathbb{C})$ is a Hilbert space isomorphism, by observing that the Hermite functions are mapped to the orthogonal basis $\{e_{k,m} : m \geqslant 0\}$ of $\mathcal{F}_2^k(\mathbb{C})$, where $e_{k,m}(z) = \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} e^{\pi|z|^2} \left(\frac{d}{dz}\right)^k \left[e^{-\pi|z|^2} z^m\right]$.

2.4 A polyanalytic Weierstrass function

In order to transmit the signal, we use the following analogue of the Whittaker-Shannon-Kotel´nikov sampling theorem. Let σ be the Weierstrass sigma function corresponding to Λ defined by

$$\sigma_\Lambda(z) = z \prod_{\lambda\in\Lambda\setminus\{0\}} \left(1-\frac{z}{\lambda}\right) e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}},$$

To simplify our notations we will write the results in terms of the square lattice, $\Lambda = \alpha(\mathbb{Z}+i\mathbb{Z})$ consisting of the points $\lambda = \alpha l + i\alpha m$, $k, m \in \mathbb{Z}$, but most of what we will say is also true for general lattices. To write down our explicit sampling formulas, the following polyanalytic extension of the Weierstrass sigma function is required:

$$S_{\Lambda}^{n+1}(z) = \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} e^{\pi|z|^2} \left(\frac{d}{dz}\right)^n \left[e^{-\pi|z|^2} \frac{(\sigma_\Lambda(z))^{n+1}}{n!z}\right].$$

Clearly, $S_\Lambda^1(z) = \sigma_\Lambda(z)/z$. Let $\sigma_{\Lambda^\circ}(z)$ be the Weierstrass sigma function associated to the adjoint lattice $\Lambda^\circ = \alpha^{-1}(\mathbb{Z} + i\mathbb{Z})$ of Λ and consider the corresponding "polyanalytic Weierstrass function" $S_{\Lambda^0}^n(z)$. With this terminology we have:

Theorem 1. *If $\alpha^2 < \frac{1}{n+1}$, then every $F \in \mathbf{F}_2^{n+1}(\mathbb{C})$ can be written as:*

$$F(z) = \sum_{\lambda \in \alpha(\mathbb{Z}+i\mathbb{Z})} F(\lambda) e^{\pi\overline{\lambda}z - \pi|\lambda|^2} \mathbf{S}_{\Lambda^0}^{n+1}(z - \lambda),$$

where $\mathbf{S}_{\Lambda^0}^n(z) = \sum\limits_{k=1}^{n} S_{\Lambda^0}^k(z)$.

Combining this with the decomposition (3) gives:

Corollary 1. If $\alpha^2 < \frac{1}{n+1}$, every $F \in \mathcal{F}_2^{n+1}(\mathbb{C})$ can be written as:

$$F(z) = \sum_{\lambda \in \alpha(\mathbb{Z}+i\mathbb{Z})} F(\lambda) e^{\pi\overline{\lambda}z - \pi|\lambda|^2} S_{\Lambda^0}^{n+1}(z), \tag{5}$$

3 The Gabor connection

3.1 The Gabor transform

The study of polyanalytic Fock spaces can be significantly enriched via a connection to Gabor analysis. Recall that the Gabor transform of a function or distribution f with respect to a window function g is defined to be

$$V_g f(x, \xi) = \int_{\mathbb{R}} f(t)\overline{g(t - x)} e^{-2\pi i\xi t} dt. \tag{6}$$

Given a point $\lambda = (\lambda_1, \lambda_2)$ in phase-space $\mathbb{R}^2$, the corresponding time-frequency shift is $\pi_\lambda f(t) = e^{2\pi i\lambda_2 t} f(t - \lambda_1)$, $t \in \mathbb{R}$. If we choose the Gaussian function $h_0(t) = 2^{\frac{1}{4}} e^{-\pi t^2}$ as a window in (6), then a simple calculation shows that the Bargmann transform is related to these special Gabor transforms as follows:

$$\mathcal{B}f(z) = e^{-i\pi x\xi + \pi\frac{|z|^2}{2}} V_{h_0} f(x, -\xi). \tag{7}$$

With a bit more effort, we can choose the *nth Hermite function* $h_n(t) = c_n e^{\pi t^2} \left(\frac{d}{dt}\right)^n \left(e^{-2\pi t^2}\right)$ as a special window in (6), and find a similar relation between Gabor transforms with Hermite functions and true polyanalytic Bargmann transforms:n

$$\mathcal{B}^{n+1} f(z) = e^{-i\pi x\xi + \frac{\pi}{2}|z|^2} V_{h_n} f(x, -\xi). \tag{8}$$

This formula connects polyanalytic Fock spaces with Gabor analysis. Using this connection it was possible to prove results that seemed hopeless using only complex variables. For instance, it was possible to prove that the sampling and interpolation lattices of $\mathbf{F}_2^n(\mathbb{C})$ can be characterized by their density, something previously known only for analytic functions.

3.2 Gabor expansions with Hermite functions

Let us see what Theorem 1 tells about Gabor expansions. From Corollary 1, if $\alpha^2 < \frac{1}{n+1}$, then every $F \in \mathcal{F}_2(\mathbb{C})$ can be written in the form (5). Now, applying the inverse Bargmann transform and doing some calculations involving time-frequency shifts and Fock shifts (see [2] for the details), one can see that this expansion is exactly equivalent to the Gabor expansion of an $L^2(\mathbb{R})$ function. More precisely, if $\alpha^2 < \frac{1}{n+1}$, every $f \in L^2(\mathbb{R})$ admits the following representation as a Gabor series

$$f(t) = \sum_{l,k \in \mathbb{Z}} c_{k,l} e^{2\pi i \alpha l t} h_n(t - \alpha k). \tag{9}$$

This sort of expansions are have been used before for practical purposes, for instance, in image analysis [8]. Their mathematical study (see [1, 2, 10–12]) exposed a aesthetic blend of ideas from signal and complex analysis, leading to one of those scarse examples in mathematics.

Stable Gabor expansions of the form (9) can be obtained from frame theory. For a countable subset $\Lambda \in \mathbb{R}^2$, one says that the Gabor system $\mathcal{G}(h_n, \Lambda) = \{\pi_\lambda h_n : \lambda \in \Lambda\}$ is a *Gabor frame* or *Weyl-Heisenberg frame* in $L^2(\mathbb{R})$, whenever there exist constants $A, B > 0$ such that, for all $f \in L^2(\mathbb{R})$,

$$A \|f\|^2_{L^2(\mathbb{R})} \leqslant \sum_{\lambda \in \Lambda} \left| \langle f, \pi_\lambda h_n \rangle_{L^2(\mathbb{R})} \right|^2 \leqslant B \|f\|^2_{L^2(\mathbb{R})}. \tag{10}$$

The first proof of the sufficiency of the condition $\alpha^2 < \frac{1}{n+1}$ for the expansion (9) is due to Gröchenig and Lyubarskii [10]. In the same paper, the authors provide some evidence to support the conjecture that the condition may even be sharp (it is known from a general result of Ramanathan and Steger [14] that $\alpha^2 < 1$ is necessary), a statement which would be surprising, since $\alpha^2 < \frac{1}{n+1}$ is exactly the sampling rate necessary and sufficient for the expansion of n functions using the superframe (the superframe [11] is a vectorial version of frame which has be seen to be equivalent to sampling in the polyanalytic space [1]). The following problem seems to be quite hard.

Problem [10]. Find the exact range of α such that $\mathcal{G}(h_n, \alpha(\mathbb{Z}+i\mathbb{Z}))$ is a frame.

Recently, Lyubarskii and Nes [12] found that $\alpha^2 = \frac{3}{5} > \frac{1}{2}$ is a sufficient condition for the case $n=1$. They also proved that, if $\alpha^2 = 1 - \frac{1}{j}$, no odd function in the Feichtinger algebra [7] generates a Gabor frame. In [12], supported by their results and by some numerical evidence, the authors formulated a conjecture.

Conjecture [12]. If $\alpha^2 < 1$ and $\alpha^2 = 1 - \frac{1}{j}$, then $\mathcal{G}(h_1, \alpha(\mathbb{Z}+i\mathbb{Z}))$ is a frame.

3.3 Sampling and Interpolation in $\mathbf{F}_2^n(\mathbb{C})$

We say that a set Λ is a set of sampling for $\mathbf{F}_2^n(\mathbb{C})$ if there exist constants $A, B > 0$ such that, for all $F \in \mathbf{F}_2^n(\mathbb{C})$,

$$A\,\|F\|^2_{\mathbf{F}_2^{n+1}(\mathbb{C})} \leqslant \sum_{\lambda\in\Lambda} |F(\lambda)|^2 e^{-\pi|\lambda|^2} \leqslant B\,\|F\|^2_{\mathbf{F}_2^{n+1}(\mathbb{C})}\,.$$

A set Λ is a set of interpolation for $\mathbf{F}_2^n(\mathbb{C})$ if for every sequence $\{a_{i(\lambda)}\} \in l^2$, we can find a function $F \in \mathbf{F}_2^n(\mathbb{C})$ such that $e^{i\pi\lambda_1\lambda_2 - \frac{\pi}{2}|\lambda|^2} F(\lambda) = a_{i(\lambda)}$, for every $\lambda \in \Lambda$. The sampling and interpolation lattices of $\mathbf{F}_2^n(\mathbb{C})$ can be characterized by their density. For the square lattice the results are as follows.

Theorem 2. *The lattice $\alpha(\mathbb{Z}+i\mathbb{Z})$ is a set of sampling for $\mathbf{F}_2^n(\mathbb{C})$ if and only if $\alpha^2 < \frac{1}{n+1}$. and it is a set of interpolation for $\mathbf{F}_2^n(\mathbb{C})$ if and only if $\alpha^2 > \frac{1}{n+1}$.*

So far, nobody has been able to find a proof of these results using only complex variables. The proof in [1] is based on the observation that, using the polyanalytic Bargmann transform, the sampling problem can be transformed in a problem about Gabor superframes with Hermite functions. Then, a remarkable structure result of Gabor analysis, the so called *Ron-Shen duality* [15] transforms the problem in a problem about Riesz sequences, which can be further transformed in a problem about multiple interpolation in the Fock space which has been solved in [6]. The dual of this argument proves the second theorem. The characterization of the lattices yielding Gabor superframes with Hermite functions had been previously obtained by Gröchenig and Lyubarskii in [11], using the Wexler-Rax orthogonality relations.

4 The Quantum connection

4.1 The Landau levels

The motion of a charged particle in a constant uniform magnetic field in $\mathbb{R}^2$ is described by the Schrödinger operator

$$H_B = -\frac{1}{4}\left((\partial_x + iBy)^2 + (\partial_y - iBx)^2\right) - \frac{1}{2}$$

acting on $L^2\left(\mathbb{R}^2\right)$. Here $B > 0$ is the strength of the magnetic field. Writing

$$\widetilde{\Delta_z} = e^{\frac{B}{2}|z|^2} H_B e^{-\frac{B}{2}|z|^2}$$

we obtain the following Laplacian on $\mathbb{C}$

$$\widetilde{\Delta_z} = -\frac{d}{dz}\frac{d}{d\overline{z}} + B\overline{z}\frac{d}{d\overline{z}}. \tag{11}$$

This Laplacian is positive and selfadjoint operator in the Hilbert space $\mathcal{L}_2(\mathbb{C})$ and the set $\{n, n \in \mathbb{Z}^+\}$ can be shown to be the pure point spectrum of $\widetilde{\Delta_z}$ in $\mathcal{L}_2(\mathbb{C})$. The eigenspaces of $\widetilde{\Delta_z}$, are known as the *Landau levels.* In [3] the authors consider

$$A^2_{n,B}(\mathbb{C}) = \{F \in \mathcal{L}_2(\mathbb{C}) : \widetilde{\Delta_{z,B}} F = nf\},$$

and obtain an orthogonal basis for the spaces $A^2_{n,B}$. When $B = \pi$ we can use the results in [3] (comparing either the orthogonal basis or the reproducing kernels of both spaces) to see that

$$A^2_{m,\pi}(\mathbb{C}) = \mathcal{F}^n_2(\mathbb{C}). \tag{12}$$

Now, using the true polyanalytic transform, the results about Gabor frames with Hermite function translate to sampling in true polyanalytic Fock: *The lattice $\alpha(Z + iZ)$ is a set of sampling for $F^n_2(\mathbb{C})$ if and only $G\left(h_n, \alpha(\mathbb{Z} + i\mathbb{Z})\right)$ is a Gabor frame.* Thus, we conclude that, in particular, if $\alpha^2 < \frac{1}{n+1}$, the subsystems of states constituted by the lattice $\alpha(\mathbb{Z} + i\mathbb{Z})$ are complete in the Landau levels. Now, take $B = 1$ and observe that

$$\widetilde{\Delta_z} = \left(-\frac{d}{dz} + \overline{z}\right)\left(\frac{d}{d\overline{z}}\right).$$

This suggests us to consider the operators

$$\mathfrak{a}^+ = -\frac{d}{dz} + \overline{z}, \quad \mathfrak{a}^- = \frac{d}{d\overline{z}},$$

which are formally adjoint to each other and satisfy the commutation relations for the quantum mechanic creation and annihilation operators. Vasilevski [17, Theorem 2.9] proved that the operators

$$\sqrt{\frac{(k-1)!}{(l-1)!}}\left(\mathfrak{a}^{+}\right)^{l-k}|_{\mathcal{F}_2^k(\mathbb{C})} \quad : \quad \mathcal{F}_2^k(\mathbb{C}) \to \mathcal{F}_2^l(\mathbb{C})$$
$$\sqrt{\frac{(k-1)!}{(l-1)!}}\mathfrak{a}^{l-k}|_{\mathcal{F}_2^k(\mathbb{C})} \quad : \quad \mathcal{F}_2^l(\mathbb{C}) \to \mathcal{F}_2^k(\mathbb{C})$$

are Hilbert spaces isomorphisms (and one is the inverse of the other). Given our identification (12) we conclude that *the operators a^+ and a^- are, respectively, the raising and lowering operators between two different Landau levels.*

4.2 Displaced Fock states

In [16], Wünsche derives a representation for the displaced Fock states $|z,n>$:

$$|z,n>=\frac{(-1)^n}{\sqrt{n!}}\left(-\frac{d}{dz}+\overline{z}\right)^n|z> . \tag{13}$$

Observing that $e^{|z|^2}\frac{d}{dz}\left[e^{-|z|^2}F(z)\right] = \frac{d}{dz}F(z)-\overline{z}F(z)$, one realizes that (13) is essentially the map $T:\mathcal{F}_2(\mathbb{C})\to\mathcal{F}_2^{n+1}(\mathbb{C})$ such that

$$T:F(z)\to e^{\pi|z|^2}\left(\frac{d}{dz}\right)^n\left[e^{-\pi|z|^2}F(z)\right].$$

Thus, the displaced Fock states are also true polyanalytic Fock spaces. We can now use Gröchenig and Lyubarskii result to show that if $\alpha^2<\frac{1}{n+1}$ then the subsystem of these coherent states constituted by the square lattice on the plane is overcomplete. From Ramathan and Steeger general result [14], we know that if $\alpha^2>1$ they are not. This can be seen as analogues of Perelomov completeness result [13] in the setting of displaced Fock states.

5 Reproducing kernels

The reproducing kernels of the polyanalytic Fock spaces have been computed using several different methods: invariance properties of the Landau laplacian $\widetilde{\Delta_z}$ [3], composition of unitary operators [17], Gabor transforms with Hermite functions [2], and the expansion in the kernel basis functions [8]. Nice formulas are obtained using the Laguerre polynomials

$$L_k^\alpha(x) = \sum_{i=0}^{k} (-1)^i \binom{k+\alpha}{k-i} \frac{x^i}{i!}.$$

The reproducing kernel of the space $\mathcal{F}_2^n(\mathbb{C})$, $\mathcal{K}^n(z,w)$, can be written as $\mathcal{K}^n(z,w) = \pi L_{n-1}^0(\pi |z-w|^2) e^{\pi z\overline{w}}$. This gives the explicit formula for the orthogonal projection P^n required at the step 5. in our theoretical multiplexing device of section 2:

$$(P^n F)(w) = \int_{\mathbb{C}} F(z) \pi L_{n-1}^0(\pi |z-w|^2) e^{\pi z(\overline{w}-\overline{z})} d\mu(z). \tag{14}$$

The reproducing kernel of the space $\mathbf{F}_2^n(\mathbb{C})$ is denoted by $\mathbf{K}^n(z,w)$. Using the formula $\sum_{k=0}^{n-1} L_k^\alpha = L_{n-1}^{\alpha+1}$, (3) gives $\mathbf{K}^n(z,w) = \pi L_{n-1}^1(\pi |z-w|^2) e^{\pi z\overline{w}}$. In [8], a variant of this setting is used in the investigation of the *polyanalytic Ginibre ensemble*. The authors consider the space with reproducing kernel

$$\mathbf{K}_m^n(z,w) = m L_{n-1}^1(m |z-w|^2) e^{mz\overline{w}}$$

and $Pol_{m,n,k} = span\{z^j \overline{z}^l : 0 \leqslant j \leqslant k-1, 0 \leqslant l \leqslant n-1\}$.

Several interesting asymptotic results are obtained. For instance, denoting the reproducing kernel of $Pol_{m,n,k}$ by $\mathbf{K}_{m,k}^n(z,w)$, it is proved that, if $z, w \in \mathbb{D}$, when $m, k \to \infty$ with $|m-k|$ bounded and $1 - |zw| \geqslant \tau > 0$, then

$$\mathbf{K}_{m,k}^n(z,w) = \mathbf{K}_m^n(z,w) + O(e^{-\frac{1}{2}m\tau^2} e^{m|zw|}).$$

Remark. Comparing this set up with section 4.1, one recognizes the parameter m as the strengh of the magnetic field B. Therefore, the physical interpretation of the above limit $m, k \to \infty$ consists of increasing the strength of the magnetic field and simultaneously the number of independent states in the system.

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ON DECOMPOSITION THEOREM IN NUMERICAL INVERSION PROBLEM OF LAPLACE INTEGRAL TRANSFORM

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Key words: Laplace transform, numerical inversion, decomposition theorem, Laguerre series

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Abstract. The different approaches for the decomposition of Laplace transforms based on the Laguerre series expansions of the functions-originals are introduced. The use of these decompositions in the theoretical sense is studied firstly for the extension of operational calculus rules into other classes of the functions-originals. Their use for the numerical inversion of Laplace integrals is studied secondly. The fundamental connection of Laguerre polynomials with Laplace integral transform was noticed in publications of outstanding mathematicians Widder D.V. [1], Tricomi F. [2], Bateman H. [3], Hille E. [4], Shohat J. [5], Pollard H. [6] and other beginning from thirties years of last century. These works have attracted attention of next generations of operational calculus investigators for a long time. The extensive bibliography of studies in this direction exist diffused in different monographs and papers at present. The decomposition of Laplace transforms is considered in two aspects: in theoretical sense – construction of generalized operational calculus on the basis of Euler principle of divergent series summing applied to Laguerre series; in applied sense – analysis of some technical methods of functions – originals decompositions into the Laguerre series.

1 Euler principle of originals space completion of operational calculus

Leonard Euler wrote in the book [7] "... Let's consider the result of summing of all terms as the sum of the series as usually. It's undobtedly, that it's possible to obtain sums only for these infinite series, which are convergent and give results the more closer to any definite value, the more biggest number of series are added... We'll assign the word "sum" the finite value different from ordinary. More precisely we'll assert that the sum of some infinite series is the finite expression from which decomposition this series arises".

The last sentence is the statement of famous Euler principle on summation of divergent series.

The “sum of the series” is the prime notion by Euler as the “series” is the secondary, derived. The picture is opposite in ordinary sense: the “series” is the initial, and the “sum of the series” is secondary.

The similar scheme may be used under the construction of the operational calculus. The transform of the original $f(t)$ is the function $F(p)$ of complex variable p, obtained by the multiplication of the original $f(t)$ and e^{-pt} and integration by t on the interval from 0 to ∞ in the original sense of the operational calculus. Therefore the original is the initial, and the transform is the secondary: the original produces the transform. The analog of Euler principle in the operational calculus assumes such procedure of the transforms space construction, that the transform is the initial, and the original — secondary: transform produces the original.

Traditionally the function – transform $F(p)$ is generated by the function – original $f(t)$ accordingly the Laplace–Karson integral transform:

$$\varphi(p) = p \int_0^\infty e^{-pt} f(t)dt. \tag{1}$$

The question about the convergence of improper integral (1) arises in this case inevitably. It is known [8], that sufficient and necessary condition of existence of integral (1) in the domain $\mathrm{Re}\, p > \vartheta_0$ for the function $f(t)$ summable on every interval $[0, T](T > 0)$ is the fulfilment of the condition $e^{-\vartheta_0 t} \int_0^t f(t)dt \to 0$ for $t \to \infty$. It's convenient to use the following notations for future presentation: a) let's replace the symbol of free parameter p of integral (1) on z according the formula $z = 1/p$, the integral (1) takes respectively the form

$$F(z) = \frac{1}{z} \int_0^\infty e^{-\frac{t}{z}} f(t)dt, \tag{2}$$

where $F(z) = \Phi(1/z)$; b) let's denote the relation between functions $f(t)$ and $F(z)$, defined by integral (2) on “z” — language, by symbol

$$f(t) \div F(z) \tag{3}$$

and designate (3) as operational congruence.

Let's condider the basic operational congruences. It's not difficult to obtain the validity of following equality

$$\frac{1}{z}\int\limits_0^{\infty}\frac{t^k}{k!}e^{-\frac{t}{z}}dt = z^k \ (\operatorname{Re} z > 0),$$

by means of integration by parts. The last equality means that the following operational congruence

$$\frac{t^k}{k!} \div z^k, \quad k \in N. \tag{4}$$

is valid.

In particular for the unitary Hevisaid function $H(t)$ the equality $H(t) \div 1$ is correct.

Further on the basis of (4) the operation congruence is valid

$$L_n(t) \div (z-1)^n, \quad \forall n \in N, \tag{5}$$

where $L_n(t)$ – Laguerre polynomial $L_n(t) = \sum\limits_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{t^k}{k}$. So if $f(t)$ has a Laguerre expansion

$$f(t) = \sum_{n=0}^{\infty} a_n L_n(t), \tag{6}$$

convergent uniformly on every finite interval [0,T], then z-transform Laplace–Karson appropriate it takes the form

$$F(z) = \sum_{n=0}^{\infty} a_n (z-1)^n, \ (z \in C)$$

that means $F(z)$ is an analytical in any vicinity of the point $z = 1$. In this connection if $F(z)$ admits an analytical continuation into the point $z = a \in C$, then in the vicinity of this point $F(z)$ has the expansion

$$F(z) = \sum_{n=0}^{\infty} c_n (z-a)^n. \tag{7}$$

It's easy to see that the operational correspondence is valid

$$a^n L_n(t/a) \div (z-a)^n. \tag{8}$$

The formal Laguerre series corresponds the transform $F(z)$ according to (7) and (8) in general

$$f(t) \sim \sum_{n=0}^{\infty} \alpha^n c_n L_n(t/\alpha). \tag{9}$$

Its coefficients satisfy the Cauchy-Hadamard condition because the convergence's radius of the series (7) differs from 0

$$\overline{\lim} \sqrt{|c_n|} \neq \infty. \tag{10}$$

Let's name the Laguerre series (9) with coefficients c_n satisfying the Cauchy-Hadamard condition, as generalized Laguerre series of the function $f(t)$, and denote the population of all generalized Laguerre series of the function $f(t)$ as the symbol $f(t)$.

Let's consider now the space L of all possible generalized Laguerre series of the type (9) (for different α). The space L^* of all possible power series corresponds to it. It's obviously that the correlation (8) set up the one-to-one correspondence between L and L^*. It's possible to describe the Euler principle of the generalization of transform's and original's notion by the following way now.

Let's introduce the equivalence relation (w) in the space L^* by the following rule: two power series $F_\alpha(z)$ and $F_\beta(z)$ in the vicinities of the points α and β correspondingly will be considered in one equivalence class if they are the elements of one analytic function $F(z)$, i.e. the simple contour exists connecting points $z = \alpha$ and $z = \beta$, along which the element $F_\alpha(z)$ may be continued analytically until $F_\beta(z)$.

The factor-space L^*/w on indicated equivalence relation w became equivalent to the space of complete analytic functions of complex variable z with which we identify it as a result.

It's clear that the equivalence relation in the space L^* performs the the equivalence relation λ in the space L by the following rule: two generalized Laguerre series $f_\alpha(t)$ and $f_\beta(t)$ relate to one equivalency class if the power series $F_\alpha(z)$ and $F_\beta(z)$ corresponding to them are equivalent in the meaning mentioned above. Let's denote the relevant factor-space of the generalized Laguerre series as symbol L/λ.

One-to-one correspondence takes the place according to the construction

$$L/\lambda \leftrightarrow L^*/w. \tag{11}$$

Let's name the equivalence classes on L, i.e. elements of the space L/λ as generalized originals and denote by symbol $\{f(t)\}$ (briefly written as g.o.). Let's name the space L/λ as the space of generalized originals, and the space L^*/w - as the space of transforms.

According (11) every g.o. $\{f(t)\}$ has one-to-one correspondence with the corresponding transform $F(t)$ written in the form $\{f(t)\} \div F(z)$. Therefore every operator A^*, acting in the transform space L^* corresponds to unique operator A, acting in the space g.o. L.

This scheme allows firstly to be free from the limitations of Laplace transforms convergence since not the original generates the transform, but the transform (arbitrary analytical function) generates the original which may be not the function in the general case. Secondly it allows to extend the operational rules $A^* F(z) \div A\{f(t)\}$ onto the operational correspondences $F(z) \div \{f(t)\}$.

So one-to-one correspondence between L^* and L predetermines the space of originals and various operational rules, the operation A^* over the transform $F(z)$ forms the operation A over the corresponding original in this case.

But it's necessary to take into the account the following. Let's remind that domail G is called as domain of the definition, and its boundary Γ — as natural boundary of analytical function $F(z)$, if it's impossible to extend $F(z)$ analytically over the boundary of the domain G.

It's known that $F_1(z) = \sum\limits_{k=0}^{\infty} z^{k!}$ and $F_2(z) = \sum\limits_{k=0}^{\infty} (z-2)^{2^k}$ have definition domains correspondingly $G_1(|z| < 1)$ and $G_2(|z-2| < 1)$. Every from these transforms generates g.o.

$$f_1(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}$$

and

$$f_2(t) = \sum_{k=0}^{\infty} 2^{2^k} L_{2^k}(t/2^{2^k}).$$

As $G_1(|z|) \frown G_2(|z-2| < 1) = 0$, then the operations of the type $F_1(z) \div F_2(z)$ and $F_1(z)F_2(z)$ lose the sense. Therefore, operations of summing and convolution g.o. may be defined not for all g.o. as the space $W = \frac{L^*}{w}$ and as the space $\Lambda = L/\lambda$ are not the rings. Let's assume in the connection with this fact further that the domains of definition of the considered transforms have nonempty domain of the intersection. In particular the analytical functions with all complex $z-$ plane with the exception of some special points of this plane (singular points of the transform) as a domain definition may be considred as the space of transforms.

The property of the linearity is valid under these assumptions also: if $\{f(t)\} \div F(z)$ and $\{g(t)\} \div G(z)$, then $\alpha\{f(t)\} + \beta\{g(t)\} \div \alpha F(z) + \beta G(z)$ for all $\alpha, \beta \in P$.

Let's denote g.o. corresponding the transform $\alpha F(z) + \beta G(z)$ by symbol $\{\alpha f(t) + \beta g(t)\}$. Then the following rules of the operations with braces in the space of g.o. are valid:

$$\{\alpha f(t)\} = \alpha\{f(t)\}$$

and

$$\{\alpha f(t) + \beta g(t)\} = \alpha\{f(t)\} + \beta\{g(t)\}.$$

Let's consider g.o. $\{f(t)\}$. This function may be considered as a representative of the equivalence classes $\{f(t)\}$ if at least one Laguerre series exists in this equivalence class with one-to-one function $f(t)$ associated by any regular kind (it's possible for example when Laguerre series converges to function $f(t)$ in one or other case) and it's possible to reject the brace in this case.

Example. Let's $F(z) = lnz$ (principal branch of logarithm, chosen by the condition $-\pi \leqslant \arg z < \pi$). The expansion is valid $\ln z = \sum\limits_{k=0}^{\infty}(-1)^{k-1}\frac{(z-1)^k}{k}$, which corresponds the expansion $f(t) = \sum\limits_{k=0}^{\infty}(-1)^{k-1}\frac{1}{k}L_k(t)$ in the space of g.o.

It's known that last Laguerre series converges for every $t > 0$ to the sum equal $\ln t + C$ (C – Euler constant). Therefore, $\{f(t)\} = \ln t + C$ and $\ln t + C \div \ln z$.

Remark 1. The uniqueness of the generalized original responding to the transform $F(z)$ is regarded some widely in the scope of the considered theory then it took place in the classical operational calculus. So, g.o. $\{f(t)\}$ responding to the multifunction $F(z)$ may consider different components corresponding different branches of the function $F(z)$. For example the original $\ln t + C$ corresponds to the principal branch of the logarithm $\mathrm{Ln}\, z$ and therefore the g.o. consisting from the components of the type $\ln t + C + 2k\pi$ corresponds to the function $\mathrm{Ln}\, z$.

So the sampling from the equivalence class $\{f(t)\}$ may be dependent from the choice of transform branch $F(t)$. This fact is important for the case of multifunction transform.

It's easy to show that the space of the functions-originals of the classical operational calculus is the part of the space of generalized originals. The action of all operational rules of the classical operational calculus is extended into the space of generalized originals on this basis.

2 Operational rules of the generalized operational calculus

Euler principle of the construction of the generalized operational calculus is formulated shortly as "no from the original to transform (by means of Laplace–Karson integral transform), but as from the transform to the original (by means of the comparison of the transform $F(z)$ – arbitrary complete analytical function, generalized original $\{f(t)\}$ – equivalence class of the generalized Laguerre series)".

Correspondingly it's possible to prove the rules of the operational calculus according the following scheme: if the operational relation is given $F(z) \div \{f(t|\}$, then every operator A^*, acting on the transform $F(z)$ generates in the space of g.o. the operator A, acting on g.o. $f(t)$.

Let's mention the following rules without the proof. If $F(z) \div f(t)$, then the following correct:

$$(i_1)\ zF(z) \div \int\limits_0^t \{f(t)\}d\tau \text{ (integration g.o.)}$$

$$(i_2)\ F(z)G(z) \div \frac{d}{dt}\int\limits_0^t \{f(t-\tau)\}\{g(\tau)\}d\tau \text{ (convolution g.o.)}$$

$$(i_3)\frac{d}{dz}F(z) \div \frac{d}{dt}t\frac{d}{dt}\{f(t)\}.$$

Operator $B := \frac{d}{dt}t\frac{d}{dt}f(t)$ is connected closely with the Bessel equation and called as Bessel operator.

Example. $B \ln t = \delta$.

Really, $\ln z - C \div \ln t$. Therefore $B \ln t \div \frac{1}{z}$, but $\frac{1}{z} + \delta(t)$.

$$(i_4)\ z\frac{d}{z}zF(z) \div t\{f(t)\} \text{ (multiplication g.o. on t).}$$

$$\left(z\frac{d}{z}z\right)^n F(z) \div t^n\{f(t)\}.$$

$$(i_5)\frac{1}{1-\lambda z}F\left(\frac{z}{1-\lambda z}\right)\div e^{\lambda t}\{f(t)\} \text{ (decay theorem - multiplication g.o.on exponent).}$$

If the integral exists then

$$(i_6)\frac{1}{z}z\int\limits_{(0)}^z \frac{1}{\xi}F(\xi)d\xi + \frac{\{f(t)\}}{t} \text{ (division g.o. on t).}$$

$$\frac{1}{(n-1)!z^n}\int\limits_{(0)}^{t}(z-\xi)^{n-1}F(\xi)d\xi+\frac{\{f(t)\}}{t^n},\quad n\in N.$$

$$(i_7)\int\limits_{(0)}^{t}F(\xi)d\xi\div\int\limits_{0}^{t}\frac{d\tau}{\tau}\int\limits_{0}^{\tau}\{f(\xi)\}d\xi(B-\text{ integral g.o.})$$

(i_8) Derivatives g.o. and their transforms.

If the conclusion of the preceding rules has regular character (i.e. directly rely on the definition of generalized Laguerre series), then the notion of g.j.derivative demands preliminary definition.

It's necessary to distinguish two types of derivatives as in the case of generalized functions: g.o.derivative $\{f(t)\}$ – written $\{f'\}^{(t)}$ and generalized derivative g.o. $\{f(t)\}$ – written $D_t\{f(t)\}$. It's necessary to define these notions by means of action on the transform $F(z)$ g.o. $\{f(t)\}$.

Let's turn our attention to the definition of the derivative. It's given $\{f(t)\}\div F(z)$. Let's the limit $\lim\limits_{z\to 0}F(z)=F(0)$ exists. As according the rule of B-derivative computation of g.o.

$$\frac{d}{dt}t\frac{d}{dt}f(t)\div F'(z),$$

then according to the integration rule

$$\int\limits_{0}^{t}\left(\frac{d}{d\tau}\tau\frac{d}{d\tau}\right)\{f(\tau)\}d\tau\div zF'(z).$$

Further according to the division on t rule

$$\frac{1}{t}\int\limits_{0}^{t}\left(\frac{d}{d\tau}\tau\frac{d}{d\tau}\right)\{f(\tau)\}d\tau+\frac{1}{z}\int\limits_{0}^{z}\frac{\xi F'(z)}{\xi}d\xi.$$

But

$$\frac{1}{z}\int\limits_{0}^{z}\frac{\xi F'(z)}{\xi}d\xi=\frac{1}{z}\int\limits_{0}^{z}F'(z)d\xi=\frac{F(z)-F(0)}{z}.$$

Let's define the generalized original $\frac{1}{t}\int_0^t B\{f(\tau)\}d\tau$ as the derivative of the generalized original $f(t)$ and denote it by symbol $f'(t)$.

It's easy to show that for the cases with differentiable function $f(t)$ having finite derivative in the point $t = 0$

$$\frac{1}{t}\int\limits_0^t \frac{d}{d\tau}\tau\frac{d}{d\tau}f(\tau)d\tau = f'(t).$$

Let's define the generalized derivative g.o. $\{f(t)\}$, given by transform $F(z)$, as g.o. corresponding the transform $F(z)/z$, and denote by symbol

$$D_t\{f(t)\} \div \frac{1}{z}F(z).$$

So every g.o. is infinitely differentiable (in generalized sense).

The question on formulation of the derivatives of high order is connected with the notion of the values g.o. in the point $t = +0$. As it takes place in the theory of generalized functions it's impossible in the general case to introduce the notion of the arbitrary g.o. in the arbitrary point t.

But basing on the Tauberian type theorems [9], which under some assumptions on originals, connect the values of the transforms on the endpoints of the interval $[0, \infty)$ with the values of the original on the endpoints of the interval $[0, \infty)$, it's appropriate to introduce the following definition. Let's define the general value g.o. f in the points $t = +0$ and $t = +\infty$ correspondingly the values of the transform limits $\lim\limits_{z\to+\infty} F(z)$ and $\lim\limits_{z\to+\infty} F(z)$ (if they exist). These limits are considered as g.o. values in given points in the case of Euler concept. We obtain by induction method using the definition of g.o. differentiation under the assumption that $F(0), F'(0), \ldots, F^{(n-1)}(0)$ exist

$$\{f^n(t)\} \div \frac{1}{z^n}\left[F(z) - F(0) - \frac{z}{1!}F'(0) - \ldots - \frac{z^{n-1}}{(n-1)!}F^{(n-1)}(0)\right].$$

Hence it's easy to establish the connection of the generalized derivative of $n - th$ order g.o. $\{f(t)\}$ with the derivative of $n - th$ order g.o. $\{f(t)\}$

$$D^n\{f(t)\} = \{f^{(n)}(t)\} + F(0)\delta^{(n)}(t) + \frac{F'(0)}{1!}\delta^{(n-1)}(t) + \ldots + \frac{F^{(n-1)}(0)}{(n-1)!}\delta(t),$$

where $\delta^{(k)}(t)$ is $k - th$ generalized derivative of g.o. $\delta(t)$. Here $\delta(t)$ — generalized original, corresponding to the transform $\frac{1}{z}$, which is the generalized derivative of the unit Hevisaid function $H(t)$. it's easy to see the difference between the definitions of the derivative and generalized derivative of the generalized original on the following example.

As $H(t) \div \hat{H}(z) \equiv 1$ then $\frac{d}{dt}H(t) \div \frac{\hat{H}(z)-\hat{H}(0)}{z} = 0$, i.e. $H'(t) = 0$, $\forall t \in [0, \infty)$. But the generalized Hevisaid function is different from $0:$

$$D_t H(t) \div \frac{\hat{H}(z))}{z} = \frac{1}{z}, \ i.e. \ D_t H(t) = \delta(t).$$

Following four rules have symmetric form

$$(i_9)\ z^m F^{(m)}(z) \div t^m D^m\{f(t)\}.$$

$$(i_{10})\ [z^n F(z)]^{(n)} \div D^m(t^m\{f(t)\}).$$

$$(i_{11})\ (z\frac{d}{dz})^m F(z) \div (tD)^m\{f(t)\}.$$

$$(i_{12})\ \left(\frac{d}{dz}z\right)^m F(z) \div (Dt)^m\{f(t)\}.$$

$$(i_{13})\ F(z)e^{-\tau/z} \div \{f(t-\tau)H(t-\tau)\} \text{ (shift theorem).}$$

$$(i_{14}) forall \lambda \in R : F(\lambda z) \div \{f(\lambda t)\} \text{ (similarity theorem).}$$

Remark 2. It's possible to prove the relation (i_{14}) easily for $\lambda > o$ in the traditional operational calculus. For the general case the shift theorem is arised onto the level of generalized operational calculus for the arbitrary complex-valued parameter λ with the conservation of all rules of the differentiation and integration on parameter λ, including Zhelezny lemma [10].

3 Harmonic analysis of Laguerre spectrum - numerical inversion of Laplace transform

Let's Fourier series is given

$$g(t) = \sum_{k=0}^{\infty} a_k t^{jkt}. \tag{12}$$

Obviously,

$$\forall s \in Z \ g(2\pi s + \tau) = g(t) \text{ for } 0 \leqslant \tau < 2\pi. \tag{13}$$

Let's assume $\tau := \frac{2\pi}{N}m,\ m \in Z_N$. Z_N - ring of residues modulo N. Let's calculate the values of function $g(\tau)$ in points $\tau := \frac{2\pi}{N}m,\ m \in Z_N$. Obviously,

$$g\left(\frac{2\pi}{N}m\right) = \sum_{k=0}^{\infty} a_k e^{j\frac{2\pi}{N}|mk|_N}. \tag{14}$$

Here the residues of integer number r on modulus N is denoted by $|r|_N$. Let's represent the index of summing "k" of the series (14) in the form $k = t + rN$, where $r \in N, t = |k|_N \in Z_N$. Then (14) transforms into the form

$$g\left(\frac{2\pi}{N}m\right) = \sum_{k+rN} a_{k+rN} e^{j\frac{2\pi}{N}mk}$$

or

$$g\left(\frac{2\pi}{N}m\right) = \sum_{k \in Z_N} \left(\sum_{r=0}^{\infty} a_{k+rN}\right) e^{j\frac{2\pi}{N}mk},\ m \in Z_N.$$

So the expansion (14) obtains the form of DTF (discrete Fourier transform with the sampling equal N):

$$g\left(\frac{2\pi}{N}m\right) = \sum_{k \in Z_N} \tilde{a}_k e^{j\frac{2\pi}{N}mk},\ m \in Z_N, \tag{15}$$

where

$$\tilde{a}_k = \sum_{r=0}^{\infty} a_{k+rN},\ k \in Z_N. \tag{16}$$

Converting DTF (15), we obtain

$$\tilde{a}_k = \frac{1}{N} \sum_{m \in Z_N} g\left(\frac{2\pi}{N}m\right) e^{-j\frac{2\pi}{N}mk}, \quad z \in Z_N. \tag{17}$$

The last expression transforms into the following form taking into the account (16)

$$a_k = \frac{1}{N} \sum_{m \in Z_N} g\left(\frac{2\pi}{N}m\right) e^{-j\frac{2\pi}{N}mk} + \Delta_k, \tag{18}$$

where $\Delta_k = -\frac{1}{N} \sum\limits_{r=1}^{\infty} a_{k+rN},\ z \in Z_N$.

The formula (6) expresses the values of first N coefficients of Fourier series over the values of function $g(t)$ in the point $i = \frac{2\pi}{N}m$, $m \in Z_N$ with the error equal value Δ_k.

Remarks.

1. Conclusion of the formula (8) assumes the fulfilment of conditions ensuring the validity of all transforms with infinite series.
2. Let's mention that the formula (6) is the particular interpretation of known Poisson summing formula of the Fourier integral theory applicable to Fourier series.
3. The precision of formula (8) depends from the value of the sampling N and decay velocity to zero of coefficients $a_k(k \to \infty)$ of the series N.

4 Computation of Laguerre spectrum of the function-orginal given by Laplace transform

Let's the operational correspondence is given

$$f(t) \div F(p), \ (\mathrm{Re}\, p \geqslant \gamma > \gamma_0), \tag{19}$$

where $F(p)$ – Laplace transform of function $f(t)$, and γ_0 – abscissa of absolute convergence of Laplace integral. Let's the condition for function $f(t)$ is valid

$$e^{-\gamma ht} f(ht) \in L_2(0, \infty). \tag{20}$$

Then the function $e^{-2\gamma ht} f(2ht)$ decomposes into the orthogonal Laguerre series

$$e^{-2ht} f(2ht) = \sum_{k=0}^{\infty} a_k \varphi_k(2t), \tag{21}$$

where $\varphi_k(t)$ – Laguerre function $\varphi_k(t) = e^{-t/2} L_n(t)$ and $\varphi_k(t) \div \frac{\left(\frac{1}{2}-p\right)^n}{\left(\frac{1}{2}+p\right)^{n+1}}$. The series

$$\frac{1}{2h} F\left(\frac{1}{2y} p + \gamma\right) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} - p\right)^k}{\left(\frac{1}{2} + p\right)^{k+1}} \tag{22}$$

corresponds to the Laguerre series in the space of the transforms or realizing the conformal mapping of the half-plane $\mathrm{Re}\, p > 0$ into the circle $|z| < 1$ by means of the function $z = \frac{1-p}{1+p}$ we rewrite the series (22) in the form

$$g(z) = \sum_{k=0}^{\infty} a_k z^k, \tag{23}$$

where

$$g(z) = \frac{1}{h(1+z)} F\left(\gamma + \frac{1}{2h}\frac{1-z}{1+z}\right).$$

Therefore the problem of the computation of coefficients a_k of expansion (31) reduces to the computation of coefficients a_k of the power series (23) of function $g(z)$. This comparatively simple method of the determination of Laguerre spectrum a_k of function-original by means of the decomposition of function-original $g(z)$ into the power series (23) is used for the conclusion of big number of expansions of special functions on Laguerre polynomials. It may be used also for the numerical calculation of the Laguerre spectrum of the function-original by means of the calculation of the values of the transform $g(z)$ in the complex points z on the circle $|z| = 1$. The theorem [11] has an esssential significance here. Let's $g(z) \in H_2$ and $L_n(w, z)$ is a polynomial of $n - th$ degree from z, interpolating function $g(z)$ in the roots of $n + 1 - th$ degree from the unit. Then $\lim L_n(w, z) = g(z), |z| < 1$, and the tending to the limit is uniform on every closed set inside the contour $C(|z| = 1)$. The computation of the coefficients a_k of the series $g(e^{it}) = \sum_{k=0}^{\infty} a_k e^{ikt}$ as it was shown above may be carried on by means of computation of function $g(eit)$ values in the points $t = \frac{2\pi}{|Nm|}$ $(m \in Z_N)$.

These two facts open the ways for multiform application of harmonic analysis for the numerical inversion of the Laplace integral transform.

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ON THE APPLICATION OF THE WIENER-TYPE TAUBERIAN THEOREM TO THE HANKEL TRANSFORM

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Key words: Abelian and Tauberian theorems, integral transforms, Hankel transforms, generalized function of slow growth; existence of quasi-asymptotics

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Abstract. In this paper we consider the application of the Wiener-type Tauberian theorem for generalized functions of slow growth to the Hankel integral transform with the function $j_\nu(\xi) = 2^\nu(\xi)^{-\nu}\Gamma(\nu+1)J_\nu(\xi)$ in the kernel. Here $J_\nu(\xi)$ is the Bessel function of the first kind of order ν, $\nu > -1/2$. Since the Hankel transform is the Mellin convolution type transform then we can apply the general Wiener-type Tauberian theorem in the spaces $S^{a,M}_{b,N,\delta}$, where a and M characterize the behaviour and smoothness of the test functions at infinity while b and N are the corresponding characteristics in a neighbourhood of zero. These spaces was first introduced and studied by Yu.N. Drozhzhinov and B.I. Zav'yalov. We consider some subspaces of these spaces and obtain the quasi-asymptotic properties for the Hankel transform.

1 Introduction

In 1932 the well-known Tauberian theorem was proved by Norbert Wiener [1, 2]. This theorem in its multiplicative version can be stated as follows:

Let $f(t) \in L_\infty(0,+\infty)$ and assume that $\varphi_0(t) \in L_1(0,+\infty)$ has Mellin transform

$$\widehat{\varphi}_0(x) = \int_0^{+\infty} t^{-ix}\varphi_0(t)dt \neq 0 \quad \text{for all} \quad x \in (-\infty,+\infty).$$

If the limit of the multiplicative convolution of $f(t)$ with $\varphi_0(t)$ exists:

$$\lim_{k\to+\infty}\int_0^{\infty} f(kt)\varphi_0(t)dt = c,$$

then the corresponding limit exists for the convolution of $f(t)$ with any function $\varphi(t) \in L_1(0,+\infty)$.

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Wiener's tauberian theorem plays an important role in harmonic analysis and has numerous applications in several areas of mathematics (see, for example, [3–6]). From the point of view of applications in modern mathematical physics, the condition that the functions f and φ belong to L_∞ and L_1, respectively, is very restrictive.

In [7] Yu. N. Drozhzhinov and B. I. Zav'yalov proved a series of Wiener-type theorems for generalized multiplicative convolutions in the spaces $S^{a,M}_{b,N,\delta}$, where a and M characterize the behaviour and smoothness of the test functions at infinity while b and N are the corresponding characteristics in a neighbourhood of zero. These spaces was first introduced in [8] and was studied the basic properties of these spaces and the Mellin transforms of their elements. Since the Hankel transform is the Mellin convolution type transform then we can apply the results of these researches and consider the quasi-asymptotic properties of Hankel transforms in the subspaces of the spaces $S^{a,M}_{b,N,\delta}$.

2 Definitions and notations

Let a and b be non-integers real numbers, $a > b$. Let Π^a_b be the strip

$$\Pi^a_b = \{z = x + iy \in \mathbb{C} : b < y < a, x \in \mathbb{R}\}$$

and let $\overline{\Pi^a_b}$ be its closure.

For non-integer $x < -1$ we define the function $<x> = [-x - 1]$, where $[\xi]$ is the integer part of ξ.

If $\varphi(t)$ has m derivatives at zero, then its mth-order Taylor polynomial at zero is denoted by

$$T^m_{\varphi(t)} \equiv T^m_\varphi = \sum_{\ell=0}^{m} \frac{\varphi^{(\ell)}(0)}{\ell!} t^\ell, \quad \varphi^{(\ell)}(0) = \frac{d^\ell}{dt^\ell}\varphi(t)\bigg|_t = 0.$$

Let $\rho(k)$ be a function that is positive and continuous for sufficiently large k. Such a function is said to be regularly varying if

$$\lim_{k\to+\infty} \frac{\rho(kt)}{\rho(k)} = \psi(t),$$

where the convergence is uniform in t on any compact set in $(0, \infty)$. It is obvious that $\psi(t) = t^\alpha$ for some α. In this case $\rho(k)$ is called a regularly varying function of order α (see [9] for details). Regularly varying functions play the role of an asymptotic scale.

The standard Schwartz space of infinitely differentiable rapidly decreasing functions is denoted as usual by $\mathcal{S}$. The symbol $\mathcal{S}'$ denotes the standard space of temperate distributions. The space of temperate distributions whose supports are contained in $[0,+\infty)$ is denoted by $\mathcal{S}'_+$. This space is conjugate to the space $\mathcal{S}_+$ of functions φ that are infinitely differentiable on $[0,+\infty)$ and satisfy

$$\max_{|\ell|\leqslant m}\sup_{0\leqslant t<+\infty}(1+t^m)|\varphi^{(\ell)}(t)|<\infty, \quad m=0,1,\dots\ .$$

Here we will consider the space of even functions beloging to $\mathcal{S}_+$ which we denote ${}_{ev}\mathcal{S}_+$ and the corresponding space of distributions is ${}_{ev}\mathcal{S}'_+$.

Definition 1. An $f(t)\in {}_{ev}\mathcal{S}'_+$ is said to have quasi-asymptotics relative to $\rho(k)$ at $\varphi(t)$ if

$$\lim_{k\to+\infty}\frac{1}{\rho(k)}(f(kt), \quad \varphi(t))=\text{const}.$$

If the limit

$$\lim_{k\to\infty}\frac{1}{\rho(k)}(f(kt), \quad \varphi(t))=c_\varphi$$

exists for any function $\varphi(t)\in {}_{ev}\mathcal{S}_+$, then $f(t)$ is said to have quasi-asymptotics relative to $\rho(k)$.

If $c_\varphi=0$ for every $\varphi\in {}_{ev}\mathcal{S}_+$, then $f(t)$ has the trivial quasi-asymptotics.

If

$$\frac{1}{\rho(k)}(f(kt),\varphi(t))=O(1) \quad \text{as} \quad k\to+\infty \quad \text{for any} \quad \varphi(t)\in {}_{ev}\mathcal{S}_+,$$

then $f(t)$ is said to be quasi-asymptotically bounded relative to $\rho(k)$.

If $f(t)$ has quasi-asymptotics relative to $\rho(k)$, then $\rho(k)$ is a regularly varying function of some order α (see [7]).

3 The spaces of test function and distributions

Assume that M and N are non-negative integers, a and b are real non-integers as before, $b<a<-1$, and let $\delta>0$. Assume further that $\varphi(t)$ is a smooth function defined on $[0,+\infty)$, and

$$\varphi^{(2\ell-1)}(0)=0 \quad \text{for} \quad \ell=1,2,\dots, \quad 2\ell-1\leqslant N.$$

Under these conditions we can assume there is a $\beta\in\mathbb{Z}_+$ such that $<b>=2\beta$ or $[-b]=2\beta+1$ is odd $(b<-1)$.

Let

$$Q_{b,N,\delta}\left[\varphi\right]=\max_{\ell\leqslant N}\int\limits_{0}^{\delta}t^{b}\left|t^{\ell}\frac{d^{\ell}}{dt^{\ell}}\{\varphi(t)-T_{\varphi}^{<b>}(t)\}\right|d\,t,\tag{1}$$

Further let

$$V^{a,M,\delta}\left[\varphi\right]=\max_{\ell\leqslant M}\int\limits_{\delta}^{+\infty}t^{a}\left|t^{\ell}\frac{d^{\ell}}{dt^{\ell}}\varphi(t)\right|d\,t.\tag{2}$$

Consider the norm

$$\mathcal{P}^{a,M}_{b,N,\delta}[\varphi]=Q_{b,N,\delta}[\varphi]+V^{a,M,\delta}[\varphi]+\sum_{\ell=0}^{\beta}|\varphi^{(2\ell)}(0)|.\tag{3}$$

The completion of the space of even functions infinitely differentiable on $[0,+\infty)$ and rapidly decreasing together with all their derivatives (in this norm) is denoted by ${}_{ev}\mathcal{S}^{a,M}_{b,N,\delta}$. This space is a subspace of $\mathcal{S}^{a,M}_{b,N,\delta}$ introduced and studied by Yu. N. Drozhzhinov and B. I. Zav'yalov in [8] (see also [7]).

Throughout the remainder of this paper we assume that $M\leqslant N$.

The functions $\varphi\in{}_{ev}\mathcal{S}^{a,M}_{b,N,\delta}$ can be described as follows:

$$\varphi(t)=c_0+\frac{c_2}{2!}t^2+\cdots+\frac{c_{2\beta}}{(2\beta)!}t^{2\beta}+\psi(t),$$

where

$$t^b|t^\ell\psi^{(\ell)}(t)|\in L_1(0,\delta),\quad 0\leqslant\ell\leqslant N,$$
$$t^a|t^\ell\varphi^{(\ell)}(t)|\in L_1(\delta-\varepsilon,+\infty),\quad 0\leqslant\ell\leqslant M.$$

The $c_{2\ell}$, $\ell=0,1,\dots,\beta$ are defined unambiguously. They are natural extensions of the even derivatives at zero of functions that belong to ${}_{ev}\mathcal{S}^{a,M}_{b,N,\delta}$. Throughout the rest of this paper we denote them by $\varphi^{(2\ell)}(0)$, that is $c_{2\ell}=\varphi^{(2\ell)}(0)$, $\ell=0,1,\dots,\beta$.

The projective limit of ${}_{ev}\mathcal{S}^{a,M}_{b,N,\delta}$ with respect to M and N is denoted by ${}_{ev}\mathcal{S}^{a}_{b}$. The projective limit of these spaces with respect to a and b is the space ${}_{ev}\mathcal{S}_{+}$, whence

$${}_{ev}\mathcal{S}_{+}=\bigcap_{a,b\in\mathbb{R}}{}_{ev}\mathcal{S}^{a}_{b}.$$

On an analogue of the paper [7] we can consider different subspaces of ${}_{ev}\mathcal{S}^{a,M}_{b,N,\delta}$, for example, ${}_{ev}\overset{0}{\mathcal{S}}{}^{a,M}_{b,N,\delta}$ and ${}_{ev}\underset{0}{\mathcal{S}}{}^{a,M}_{b,N,\delta}$. But we do not consider these results here as we do not use them for the formulation of the main theorems.

The corresponding spaces of distributions are defined to be the cojugate spaces (spaces of continuous linear functionals). Conjugation is denoted by prime. For example, $f \in \left({}_{ev}\mathcal{S}^{a,M}_{b,N,\delta}\right)'$ means that f is a continuous linear functional on ${}_{ev}\mathcal{S}^{a,M}_{b,N,\delta}$.

We will write $f \in \left({}_{ev}\mathcal{S}^{a,M}\right)'$ if for some $\eta > 0$ the functional f belongs to $\left({}_{ev}\mathcal{S}^{a,M}_{b,N,\eta}\right)'$ for any b and N. In a similar war, $f \in \left({}_{ev}\mathcal{S}_{b,N}\right)'$ if for some $\eta > 0$ the functional f belongs to $\left({}_{ev}\mathcal{S}^{a,M}_{b,N,\eta}\right)'$ for any a and M.

For $a > b$ the Mellin transform of a test function $\varphi(t) \in {}_{ev}\mathcal{S}^{a,M}_{b,N,\delta}$ is defined by

$$\mathcal{M}[\varphi] \equiv \widehat{\varphi}(z) = \int\limits_0^{+\infty} t^{-iz}\left[\varphi(t) - T^{<y>}_{\varphi}(t)\right]dt, \quad z = x + iy \in \overline{\Pi}^a_b. \tag{4}$$

This integral is well defined for $b \leqslant y \leqslant a$, with the possible exception of negative integers y, that lie between a and b, and defines an analytic function in Π^a_b that can have simple poles at the points $z = -ik, \quad k = 1, 2, \ldots,$ cjntained in this strip. A way from the poles (for example, in $\overline{\Pi}^a_b \cap \{|x| > 1\}$) the function $|\widehat{\varphi}(z)|$ is bounded. It is easy to verify that $\left(t\frac{d}{dt}\right)^p \varphi \in \mathcal{S}^{a,M-p}_{b,N-p,\delta}$ and

$$\mathcal{M}\left[\left(t\frac{d}{dt}\right)^p \varphi(t)\right] = (iz-1)^p\widehat{\varphi}(z)$$

for any integer $p \leqslant \min\{N, M\}$. Hence, $|\widehat{\varphi}(z)|$ decreases in $\overline{\Pi}^a_b$ as $|z| \to \infty$ at least as fast as $|z|^{-p}$.

Since any function $\varphi(t) \in {}_{ev}\mathcal{S}^{a,M}_{b,N,\delta}$ belongs to $\mathcal{S}^{a,M}_{b,N,\delta}$ then all corresponding results from the papers [7, 8] are valid in our case.

4 The main theorems

Let $f \in \left({}_{ev}\mathcal{S}^{a,M}_{b,N,\eta}\right)'$ and $\varphi \in {}_{ev}\mathcal{S}^{a,M}_{b,N,\delta}$. For large k the expression $(f(kt), \varphi(t)) = (1/k)(f(t), \varphi(t/k))$ is well defined. Indeed, under the condition $M \leqslant N$ for $k\delta > \eta$ we have $\varphi(t/k) \in {}_{ev}\mathcal{S}^{a,M}_{b,N,k\delta} \subset {}_{ev}\mathcal{S}^{a,M}_{b,N,\delta}$.

Lemma. *Let* $f \in \left({}_{ev}\mathcal{S}^{a,M}_{b,N,\eta}\right)'$ *and* $\varphi \in {}_{ev}\mathcal{S}^{a,M}_{b,N,\delta}$*, where* $M \leqslant N$*,* $b < a < -1$*. Then*

$$(f(kt), \varphi(t)) = \sum_{\ell=0}^{a} \frac{1}{k^{\ell+1}} \frac{\varphi^{(\ell)}(0)}{\ell!} \left(f(t), t^{\ell}\right) + k^{a} O(1), \quad k \to +\infty. \tag{5}$$

The sum in this formula contains only even terms.

This lemma is proved similarly to Lemma 3 from [8].

Corollary. Assume that $\rho(k)$ is a regularly varying function of order $\alpha \leqslant a < -1$ and the hypotheses of Lemma are fulfilled. Assume that $f(t)$ to have quasi-asymptotics relative to $\rho(k)$ at $\varphi(t)$, and $\varphi^{(2\ell)}(0) \neq 0$ for all ℓ such that $2\ell \leqslant \langle a \rangle$. Then

$$(f(t), t^{2\ell}) = 0 \quad \forall \ell \in \mathbb{Z}_{+}, \quad 2\ell \leqslant \langle a \rangle.$$

This corollary shows that the case when $\alpha \leqslant a$ is the most interesting in the study of quasi-asymptotics.

Theorem 1. *Let* $\varphi_0(t) \in {}_{ev}\mathcal{S}^{a,M}_{b,N,\delta}$*,* $M \leqslant N$*,* $b < a < -1$*,* $\delta > 0$*. Assume that* $\varphi_0(t)$ *satisfies the following conditions. There is a non-integral number* α_0*,* $b \leqslant \alpha_0 \leqslant a$*,* $A > 0$*, and* $B \geqslant 0$ *such that*

$$|\widehat{\varphi}_0(z)| \geqslant \frac{A}{1 + |z|^{B}} \quad \text{for} \quad \alpha_0 \leqslant \operatorname{Im} z \leqslant a, \tag{6}$$

where $\widehat{\varphi}_0(z)$ *is the Mellin transform of* $\varphi_0(t)$ *(see (4)). Farther,*

$$\varphi_0^{(2\ell)}(0) \neq 0 \quad \forall \ell \in \mathbb{Z}_{+}, \quad 2\ell < \langle \alpha_0 \rangle.$$

If there is the limit

$$\lim_{k \to \infty} \frac{1}{\rho(k)} (f(kt), \quad \varphi_0(t)) = \text{const} \tag{7}$$

for a regularly varying function $\rho(k)$ *of order* $\alpha > \alpha_0$ *and for any* $f(t) \in \left({}_{ev}\mathcal{S}^{a_1,M}_{b,N,\eta}\right)'$*, where* $a > a_1 > \alpha_0$*,* $\eta > 0$*, then* $f(t)$ *has quasi-asymptotics relative to* $\rho(k)$*, that is,*

$$\lim_{k \to +\infty} \frac{1}{\rho(k)} (f(kt), \varphi(t)) = c_{\varphi}, \quad \forall \varphi \in {}_{ev}S_{+}. \tag{8}$$

Now we apply the above results to Hankel integral transform and formulate the corresponding Tauberian theorem.

The distributional study of Hankel transform was started by Zemanyan [10] and continued by Altenburg [11]. These authors defined the Hankel transform of distributions of slow growth. The modern researches of Hankel integral transform and its convolution can be found in papers by Ja. J. Betancor and co-authors (see, for example, [12, 13]).

The Hankel integral transform is defined by [14]

$$h_\nu(f)(s) = \left(t^{2\nu+1} f(t), \quad j_\nu(st)\right) = \left(\tilde{f}(t), j_\nu(st)\right), \quad \nu > -1/2, \tag{9}$$

where the function $j_\nu(\xi)$ is associated with the Bessel function $J_\nu(\xi)$ of the first kind of order ν

$$j_\nu(\xi) = \frac{2^\nu \Gamma(\nu+1)}{(\xi)^\nu} J_\nu(\xi) \tag{10}$$

and it is the solution of the equation

$$\frac{d^2y}{dt^2} + \frac{2\nu+1}{t}\frac{dy}{dt} + y = 0$$

under conditions $y(0) = 1$ and $y'(0) = 0$.

Using the following asymptotic estimate for $j_\nu(\xi)$ at $\xi \to +\infty$

$$j_\nu(\xi) = \mathrm{const}\, \xi^{-\nu-1/2} \cos(\xi - \frac{\nu\pi}{2} - \frac{\pi}{4}) + O(\xi^{-\nu-3/2}) \tag{11}$$

we contain if $\nu > -1/2$ then $j_\nu(\xi) \to 0$ at $\xi \to +\infty$. Hence, for all $\xi \in \mathbb{R}_+$ the function (10) is bounded and does not belong to ${}_{ev}\mathcal{S}_+$. But $\forall b$, M, N such that $M < N$ and for $a < \nu - M - 1/2$ the function $j_\nu(\xi) \in {}_{ev}\mathcal{S}^{a,M}_{b,N,\delta}$.

Hence for the Mellin transform of the function $j_\nu(\xi)$ the condition (6) holds so the Theorem 1 is valid for the Hankel integral transform (9).

Theorem 2. *Let $M < N$, $b < a < \nu - M - 1/2 < -1$, $\delta > 0$. Assume that $\tilde{f}(t) \in \left({}_{ev}\mathcal{S}^{a_1,M}_{b,N,\eta}\right)'$, where $a > a_1 > \alpha_0$, $\eta > 0$ If there is the limit*

$$\lim_{k\to\infty} \frac{1}{\rho(k)} (\tilde{f}(kt), j_\nu(st)) = const, \quad s \in \mathbb{R}_+ \tag{12}$$

for a regularly varying function $\rho(k)$ of order $\alpha > \alpha_0$, then $\tilde{f}(t)$ has quasi-asymptotics relative to $\rho(k)$.

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APPLICATIONS OF REPRODUCING KERNELS TO FRACTIONAL FUNCTIONS AND CONVOLUTION INEQUALITIES

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Abstract. We shall introduce applications of reproducing kernels to fractional functions and convolution inequalities mostly based on our recent accepted two papers in refs. [2, 3].

1 Fractional function interpretation within reproducing kernel Hilbert spaces

Taking profit of reproducing kernel Hilbert spaces techniques [8, 9], in the present work – which may be viewed as a survey paper – we are going to present a global interpretation to the fractional function

$$\frac{g}{f} \tag{1}$$

concept for some very general functions g and f on a set E. In view of this, associated with (1), we shall consider the related equation

$$f_1(p)f(p) = g(p) \quad \text{on} \quad E \tag{2}$$

for some functions f_1 and g on the set E. If the solution f_1 of (5) on the set E exists, then the solution f_1 will give the meaning of the fractional function (1). So, the problem may be transformed to the very general and popular equation (5). At first, we observe that for an arbitrary function $f(p)$, there exist many reproducing kernel Hilbert spaces containing the function $f(p)$; the simplest reproducing kernel

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is given by $f(p)\overline{f(q)}$ on $E \times E$. In general, a reproducing kernel Hilbert space $H_K(E)$ on E admitting a reproducing kernel $K(p,q)$ on $E \times E$ is characterized by the very natural property that any point evaluation $f(p)$ is a bounded linear operator on $H_K(E)$ for any point $p \in E$. So, we shall consider such a reproducing kernel Hilbert space $H_{K_1}(E)$ admitting a reproducing kernel $K_1(p,q)$ containing the functions $f_1(p)$. Then, we note the very interesting fact that the products $f_1(p)f(p)$ determine a natural reproducing kernel Hilbert space that is induced by the reproducing kernel Hilbert space $H_{K_1}(E)$ and by a second reproducing kernel Hilbert space, say $H_K(E)$, containing the function $f(p)$. In fact, the space in question is a reproducing kernel Hilbert space $H_{K_1K}(E)$ that is determined by the product $K_1(p,q)K(p,q)$ and, furthermore, we obtain the inequality

$$\|f_1 f\|_{H_{K_1K}(E)} \leqslant \|f_1\|_{H_{K_1}(E)} \|f\|_{H_K(E)}. \tag{3}$$

This important inequality means that for the linear operator $\varphi_f(f_1)$ on $H_{K_1}(E)$ (for a fixed function f), defined by

$$\varphi_f(f_1) = f_1(p)f(p), \tag{4}$$

we obtain the inequality

$$\|\varphi_f(f_1)\|_{H_{K_1K}(E)} \leqslant \|f_1\|_{H_{K_1}(E)} \|f\|_{H_K(E)}.$$

This means that the mapping φ_f is a bounded operator from $H_{K_1}(E)$ into $H_{K_1K}(E)$.

2 Reproducing kernel Hilbert spaces machinery and Tikhonov regularization

Following refs. [8,9], we shall introduce a general theory for linear mappings in the framework of Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert (possibly finite-dimensional) space. Let E be an abstract set and $\mathbf{h}$ be a Hilbert $\mathcal{H}$-valued function on E. Then, we shall consider the linear transform

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H}, \tag{5}$$

from $\mathcal{H}$ into the linear space $\mathcal{F}(E)$ comprising all the complex valued functions on E. In order to investigate the linear mapping (6), we form a positive definite quadratic form function $K(p,q)$ on $E \times E$ defined by

$$K(p,q) = (\mathbf{h}(q), \mathbf{h}(p))_{\mathcal{H}} \quad \text{on} \quad E \times E. \tag{6}$$

Then, we obtain the following:

(I) *The range of the linear mapping (6) by $\mathcal{H}$ is characterized as the reproducing kernel Hilbert space $H_K(E)$ admitting the reproducing kernel $K(p,q)$ whose characterization is given by the following two properties: $K(\cdot,q) \in H_K(E)$ for any $q \in E$ and, for any $f \in H_K(E)$ and any $p \in E$, $(f(\cdot), K(\cdot . p))_{H_K(E)} = f(p)$.*

(II) *In general, we have the inequality $\|f\|_{H_K(E)} \leqslant \|\mathbf{f}\|_{\mathcal{H}}$. Here, for any member f of $H_K(E)$ there exists a uniquely determined $\mathbf{f}^* \in \mathcal{H}$ satisfying $f(p) = (\mathbf{f}^*, \mathbf{h}(p))_{\mathcal{H}}$ on E and*

$$\|f\|_{H_K(E)} = \|\mathbf{f}^*\|_{\mathcal{H}}. \tag{7}$$

For any two positive definite quadratic form functions $K_1(p,q)$ and $K_2(p,q)$ on $E \times E$, the usual product $K(p,q) = K_1(p,q)K_2(p,q)$ is again a positive definite quadratic form function on E. Then, the reproducing kernel Hilbert space H_K admitting the kernel $K(p,q)$ is the restriction of the tensor product $H_{K_1}(E) \otimes H_{K_2}(E)$ to the diagonal set:

Proposition 1. Let $\{f_j^{(1)}\}_j$ and $\{f_j^{(2)}\}_j$ be some complete orthonormal systems in $H_{K_1}(E)$ and $H_{K_2}(E)$, respectively. Then, the reproducing kernel Hilbert space H_K is comprised of all functions on E which are represented as

$$f(p) = \sum_{i,j} \alpha_{i,j} f_i^{(1)}(p) f_j^{(2)}(p) \quad \text{on} \quad E, \qquad \sum_{i,j} |\alpha_{i,j}|^2 < \infty, \tag{8}$$

in the sense of absolutely convergence on E, and its norm in H_K is given by

$$\|f\|_{H_K}^2 = \min \sum_{i,j} |\alpha_{i,j}|^2$$

where $\{\alpha_{i,j}\}$ are considered to satisfy (10).

Next, let L be any bounded linear operator from a reproducing kernel Hilbert space H_K into a Hilbert space $\mathcal{H}$. Then, for any member $\mathbf{d}$ of $\mathcal{H}$, the fundamental and classical problem of computing

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}} \tag{9}$$

is well-known as the best approximate mean square norm problem.

Proposition 2. For any member $\mathbf{d}$ of $\mathcal{H}$, there exists a function $\tilde{f}$ in H_K satisfying

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}} = \|L\tilde{f} - \mathbf{d}\|_{\mathcal{H}} \tag{10}$$

if and only if, for the reproducing kernel Hilbert space H_k admitting the kernel defined by $k(p,q) = (L^*LK(\cdot,q), L^*LK(\cdot,p))_{H_K}$,

$$L^*\mathbf{d} \in H_k. \tag{11}$$

Moreover, when there exists a function $\tilde{f}$ satisfying (12), there exists a uniquely determined function that minimizes the norms in H_K among the functions satisfying the equality, and such function $f_{\mathbf{d}}$ is represented as follows:

$$f_{\mathbf{d}}(p) = (L^*\mathbf{d}, L^*LK(\cdot,p))_{H_k} \quad \text{on} \quad E. \tag{12}$$

The extremal function $f_{\mathbf{d}}$ is the *Moore-Penrose generalized inverse* $L^{\dagger}\mathbf{d}$ of the equation $Lf = \mathbf{d}$. The criteria (13) is involved and so the Moore-Penrose generalized inverse $f_{\mathbf{d}}$ is not so good when the data contain error or noises in some practical cases. Therefore, to overcome this issue, we shall introduce the idea of the Tikhonov regularization within our framework.

We set, for a small $\lambda > 0$,

$$K_L(\cdot,p;\lambda) = \frac{1}{L^*L + \lambda I} K(\cdot,p),$$

where L^* denotes again the adjoint operator of L. Then, by introducing the inner product

$$(f,g)_{H_K(L;\lambda)} = \lambda(f,g)_{H_K} + (Lf, Lg)_{\mathcal{H}},$$

we construct the corresponding Hilbert space $H_K(L;\lambda)$ comprising all the functions of H_K. Furthermore, we directly obtain:

Proposition 3. The extremal function $f_{\mathbf{d},\lambda}(p)$ in the Tikhonov regularization

$$\inf_{f \in H_K} \{\lambda \|f\|^2_{H_K} + \|\mathbf{d} - Lf\|^2_{\mathcal{H}}\}$$

exists. Additionally, there is a unique element for which the corresponding minimum is attained and it is represented in terms of the kernel $K_L(p,q;\lambda)$ as follows:

$$f_{\mathbf{d},\lambda}(p) = (\mathbf{d}, LK_L(\cdot,p;\lambda))_{\mathcal{H}}\,. \tag{13}$$

Here, the kernel $K_L(p,q;\lambda)$ is the reproducing kernel for the Hilbert space $H_K(L;\lambda)$ and it is determined as the unique solution $\widetilde{K}(p,q;\lambda)$ of the equation

$$\widetilde{K}(p,q;\lambda) + \frac{1}{\lambda}(L\widetilde{K}_q, LK_p)_{\mathcal{H}} = \frac{1}{\lambda}K(p,q) \tag{14}$$

with $\widetilde{K}_q = \widetilde{K}(\cdot, q; \lambda) \in H_K$ for $q \in E$, $K_p = K(\cdot, p) \in H_K$ for $p \in E$.

The next proposition gives the inversion for errorness data.

Proposition 4. Suppose that $\lambda : (0,1) \to (0,\infty)$ is a function of δ such that

$$\lim_{\delta \downarrow 0} \left(\lambda(\delta) + \frac{\delta^2}{\lambda(\delta)} \right) = 0. \tag{15}$$

Let $D : (0,1) \to \mathcal{H}$ be a function such that $\|D(\delta) - \mathbf{d}\|_{\mathcal{H}} \leqslant \delta$ for all $\delta \in (0,1)$. If $\mathbf{d}$ is contained in the range set of the Moore–Penrose inverse, then we have

$$\lim_{\delta \downarrow 0} f_{D(\delta), \lambda(\delta)} = f_{\mathbf{d}}. \tag{16}$$

When $\mathbf{d}$ contains error or noises we need its error estimate. For this error estimate, we are able to invoke the next general result.

Proposition 5 (cf. ref. [1]). We have

$$|f_{\mathbf{d},\lambda}(p)| \leqslant \frac{1}{\sqrt{2\lambda}} \sqrt{K(p,p)}\, \|\mathbf{d}\|_{\mathcal{H}}.$$

3 General fractional functions

At first, we fix a reproducing kernel Hilbert space $H_K(E)$ containing the function f. Next, we shall consider a reproducing kernel Hilbert space $H_{K_1}(E)$ containing the *solutions* f_1. Then, the products $f_1 f$ belong to the natural reproducing kernel Hilbert space $H_{K_1 K}(E)$ admitting the reproducing kernel $K_1(p,q)K(p,q)$, and we obtain the following inequality by Proposition 1:

$$\|f_1 f\|_{H_{K_1 K}(E)} \leqslant \|f_1\|_{H_{K_1}(E)} \|f\|_{H_K(E)}. \tag{17}$$

That is, for fixed $f \in H_K(E)$, the linear operator $\varphi_f : H_{K_1}(E) \to H_{K_1 K}(E)$, given by $\varphi_f(f_1) = f_1(p)f(p)$, is bounded on $H_{K_1}(E)$. So, we can consider the Tikhonov functional, for any $g \in H_{K_1 K}(E)$:

$$\inf_{f_1 \in H_{K_1}(E)} \left\{ \lambda \|f_1\|^2_{H_{K_1}(E)} + \|g - \varphi_f(f_1)\|^2_{H_{K_1 K}(E)} \right\}.$$

The extremal function $f_{1,\lambda}$ exists, it is unique and we have, if (5) has the Moore-Penrose generalized inverse $f_1(p)$, $\lim_{\lambda \to 0} f_{1,\lambda}(p) = f_1(p)$ on E uniformly on where $K_1(p,p)$ is bounded. Furthermore, its convergence is also in the sense of the norm of $H_{K_1}(E)$. Sometimes, we can take $\lambda = 0$ and in this case we can represent the Moore-Penrose generalized solution in some direct form.

So, at first, we can introduce the *approximate fractional function* by the extremal function $f_{1,\lambda}$ above whose existence is always ensured in the above situation. In case that there exists the Moore–Penrose generalized inverse, we will call it the *generalized fractional function.* We can examine the above properties by the theory of reproducing kernels very well by the given propositions (cf. [2]).

We also would like to mention that even in the very difficult case of the numerical and real inversion formula of the Laplace transform, in some case of (14), Fujiwara [5,6] gave the solutions with $\lambda = 10^{-400}$ and **600 digits precision**. So, in this method, we will be able to give the approximate fractional functions by using Proposition 3, numerically for many cases containing the present situation.

4 Convolution norm inequalities

In order to state our recent results in ref. [3], we shall first introduce the relevant function spaces $\mathcal{F}(\rho)$ which are dependent on non-negative and integrable functions ρ on $\mathbb{R}$. We will say that $F \in \mathcal{F}(\rho)$ if and only if $\int |F(t)|^2/\rho(t)dt < \infty$ on the support of ρ, and $F = 0$ on the outside of the support of ρ.

We will consider the usual convolution and will additionally introduce the following three types in the just presented spaces:

$$
\begin{aligned}
(F_1 *_1 F_2)(t) &= \int_{\mathbb{R}} F_1(\xi)F_2(t-\xi)d\xi, (F_1 *_2 F_2)(t) = \int_{\mathbb{R}} F_1(\xi)\overline{F_2(\xi - t)}d\xi,\\
(F_1 *_3 F_2)(t) &= \int_{\mathbb{R}} \overline{F_1(\xi)}F_2(\xi + t)d\xi, (F_1 *_4 F_2)(t) = \int_{\mathbb{R}} \overline{F_1(\xi)F_2(-\xi - t)}d\xi.
\end{aligned}
$$

Then, from the Fourier integral transform and from the theory of reproducing kernels, we can obtain, naturally the following inequality.

Theorem 1. *The generalized convolution inequality*

$$
\int_{\mathbb{R}} \frac{|((F_1) *_1 (F_2) + (F_1) *_2 (F_2) + (F_1) *_3 (F_2) + (F_1) *_4 (F_2))(t)|^2}{(\rho_1 *_1 \rho_2)(t) + (\rho_1 *_2 \rho_2)(t) + (\rho_1 *_3 \rho_2)(t) + (\rho_1 *_4 \rho_2)(t)} dt \leqslant
$$

$$
\leqslant 4 \int_{\mathbb{R}} \frac{|F_1(t)|^2}{\rho_1(t)} dt \cdot \int_{\mathbb{R}} \frac{|F_2(t)|^2}{\rho_2(t)} dt
$$

holds true, for functions $F_j \in \mathcal{F}(\rho_j)$, $j = 1, 2$.

This result for the usual convolution was expanded in various directions with applications to inverse problems and partial differential equations through L_p ($p > 1$) versions and converse inequalities. See, for example, refs. [3, 7] and the references therein.

We derived many and entirely new inequalities by applying the theory of reproducing kernels. Some inequalities seem to be impossible to derive them if we do not apply the theory of reproducing kernels. See, for typical examples, ref. [9]. Furthermore, the equality problems in the inequalities that determine the cases holding the equalities are – in general – very difficult problems. See the deep theory of A. Yamada in ref. [10].

We can see from the informal communication and the manuscript ref. [7] that the authors were able to derive generalizations and many concrete applications to the boundedness of various integral transforms and the estimates of the solutions of integral equations that solved the equality problems, completely. However, it is worth mentioning that our results gave basic contributions to their paper already by creating entirely new type inequalities.

Theorem 1 gives a basic fundamental inequality for the induced convolution integral equations.

In order to state an example, we shall consider the function spaces $\mathcal{F}(\rho_j)$, for non-negative and integrable functions ρ_j on $\mathbb{R}$, $j = 1, 2, 3$ – defined at the beginning of the present section. For the space $\mathcal{F}(\rho_1)$, we will impose more additional assumptions due to the natural requests of our method. We assume that $\mathcal{F}(\rho_1)$ is the real-valued function space and the support of ρ_1 is $[a, b)$ $(-\infty < a < b \leqslant +\infty)$ and on this interval, ρ_1 is a positive continuous function.

We set $\Omega(t; \rho) = \rho_1 * (2\pi + \rho_2) + \int_{\mathbb{R}} \rho_1(\xi)\rho_3(\xi + t)d\xi$, for the usual convolution $*$.

For any fixed $F_j \in \mathcal{F}(\rho_j)$, $j = 2, 3$ (so that $F_2 \pm F_3$ are not zero identically), we consider the integral equation

$$2\pi\alpha F_1(t) + \int_{\mathbb{R}} F_1(\xi)F_2(t-\xi)d\xi + \int_{\mathbb{R}} F_1(\xi)F_3(t+\xi)d\xi = \widetilde{G}(t), \tag{18}$$

for any function $\widetilde{G}$ satisfying $\int_{\mathbb{R}} |\widetilde{G}(\tau)|^2 \Omega(\tau; \rho)^{-1} d\tau < \infty$. Then, by the convolution inequality, we can apply Proposition 3 to this integral equation (cf. [4]).

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OPERATOR-VALUED FEYNMAN INTEGRALS VIA CONDITIONAL EXPECTATIONS ON A BANACH SPACE

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Abstract. In this paper, we define an integral operator as an operator-valued Feynman integral over Wiener paths in abstract Wiener space. And then, we evaluate the operator-valued Feynman integrals for various types of functions which are of interest in quantum mechanics and Feynman integration theories, via the conditional Feynman integral over Wiener paths in abstract Wiener space as a kernel.

1 Introduction

Let $(\mathcal{H}, \mathbb{B}, m)$ be an abstract Wiener space [6]. In [5], the space $C(\mathbb{B})$ of all abstract Wiener space $\mathbb{B}$-valued continuous paths defined on $[0, t]$ was introduced and Ryu [7] developed several properties on $C_0(\mathbb{B})$, the space of all paths x in $C(\mathbb{B})$ with $x(0) = 0$. In [8], Yoo introduced the Banach algebra $\mathcal{S}''_{\mathbb{B}}$ which is a class of functions defined on $C_0(\mathbb{B})$ and corresponds to the Cameron and Storvick's Banach algebra $\mathcal{S}''$ in [1]. In that paper, he evaluated the analytic Feynman integrals of functions in $\mathcal{S}''_{\mathbb{B}}$. In [3], Cho et. al. introduced a concept of analytic conditional Feynman integral over Wiener paths in abstract Wiener space and derived a simple formula for conditional Wiener integral over Wiener paths in abstract Wiener space, which calculates directly conditional Wiener integrals in terms of ordinary Wiener integrals.

In this paper, we prove that the operator-valued Feynman integrals on $C_0(\mathbb{B})$ can be expressed in terms of analytic conditional Feynman integrals as kernels using the simple formula in [3]. In particular, for a function F defined on $C_0(\mathbb{B})$, the operator-valued Feynman integral $J_q^{an}(F) : L_1(\mathbb{B}) \to L_\infty(\mathbb{B})$ can be obtained

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by using the formula

$$(J_q^{an}(F)\psi)(\xi) = \int_{\mathbb{B}} E^{anf_q}[F|X](\xi)(\eta)\psi(\eta)dm_{t^{1/2}}(\eta), \tag{1}$$

where $E^{anf_q}[F|X]$ is the analytic conditional Feynman integral of F given X and $m_{t^{1/2}}$ is the probability distribution of X on $(\mathbb{B}, \mathcal{B}(\mathbb{B}))$ for suitable conditioning function X. Further, we show that for all F in $\mathcal{S}''_{\mathbb{B}}$, $J_q^{an}(F)$ is given by (1) and can be interpreted as a bounded linear operator from $L_1(\mathbb{B})$ to $L_\infty(\mathbb{B})$. And then, we prove that for appropriate ϑ, $J_q^{an}(F)$ is given by (1), where Fs are the functions of the forms

$$\exp\left\{\int_0^t \vartheta(s, x(s))ds\right\} \text{ and } \exp\left\{\int_0^t \vartheta(s, x(s))ds\right\}\varphi(x(t))$$

which are of interest in Feynman integration theories and quantum mechanics. Comparing with the definitions in [2], the definitions of the analytic conditional Feynman integral and the operator-valued Feynman integral are different from them in [2]. We also note that the concept of scale-invariant measurability is not assumed in this paper.

2 Definitions over paths in abstract Wiener space

Let $(\mathcal{H}, \mathbb{B}, m)$ be an abstract Wiener space [6]. Let $\{e_j : j \geqslant 1\}$ be a complete orthonormal set in the real separable Hilbert space $\mathcal{H}$ such that the e_js are in $\mathbb{B}^*$, the dual space of the real separable Banach space $\mathbb{B}$. For each $h \in \mathcal{H}$ and $y \in \mathbb{B}$, define the stochastic inner product $(h, y)^\sim$ by

$$(h, y)^\sim = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j\rangle (y, e_j), & \text{if the limit exists;} \\ 0, & \text{otherwise,} \end{cases}$$

where $(\cdot, \cdot)$ denotes the dual pairing between $\mathbb{B}$ and $\mathbb{B}^*$ [4].

Let $C(\mathbb{B})$ be the space of all $\mathbb{B}$-valued continuous paths defined on $[0, t]$ and let $C_0(\mathbb{B})$ be the space of all paths x in $C(\mathbb{B})$ with $x(0) = 0$. Then $C(\mathbb{B})$ is a real separable Banach space with the norm $\|x\|_{C(\mathbb{B})} \equiv \sup_{0\leqslant s\leqslant t} \|x(s)\|_{\mathbb{B}}$ and so is $C_0(\mathbb{B})$. The minimal σ-field making the mapping $x \to x(s)$ measurable is $\mathcal{B}(C_0(\mathbb{B}))$, the Borel σ-algebra on $C_0(\mathbb{B})$. Further, Brownian motion in $\mathbb{B}$ induces a probability measure $m_{\mathbb{B}}$ on $(C_0(\mathbb{B}), \mathcal{B}(C_0(\mathbb{B})))$ which is mean-zero Gaussian, where $\mathcal{B}(C_0(\mathbb{B}))$ is the Borel σ-field on $C_0(\mathbb{B})$ [7].

Let $F : C_0(\mathbb{B}) \to \mathbb{C}$ be integrable and let X be a random vector on $C_0(\mathbb{B})$ assuming that the value space of X is a normed space with the Borel σ-algebra. Then, we have the conditional expectation $E[F|X]$ of F given X from a well known probability theory. Further, there exists a P_X-integrable complex-valued function ψ on the value space of X such that $E[F|X](x) = (\psi \circ X)(x)$ for a.e. $x \in C_0(\mathbb{B})$, where P_X is the probability distribution of X. The function ψ is called the conditional Wiener integral of F given X and it is also denoted by $E[F|X]$.

Let $\tau : 0 = t_0 < t_1 < \ldots < t_k = t$ be a partition of $[0, t]$ and let x be in $C_0(\mathbb{B})$. Define the polygonal function $[x]$ of x on $[0, t]$ by

$$[x](s) = \sum_{j=1}^{k} \chi_{(t_{j-1}, t_j]}(s) \left[x(t_{j-1}) + \frac{s - t_{j-1}}{t_j - t_{j-1}} (x(t_j) - x(t_{j-1})) \right], \tag{2}$$

where $s \in [0, t]$. For each $\vec{\eta} = (\eta_1, \ldots, \eta_k) \in \mathbb{B}^k$, let $[\vec{\eta}]$ be the polygonal function of $\vec{\eta}$ on $[0, t]$ given by (2) with replacing $x(t_j)$ by $\eta_j (\eta_0 = 0)$.

The following lemma is useful for the definition of conditional Feynman integral over Wiener paths in abstract Wiener space. For the detailed proof, see [3].

Lemma 1. *Let F be defined and integrable on $C_0(\mathbb{B})$. Let $X_\tau : C_0(\mathbb{B}) \to \mathbb{B}^k$ be a random variable given by $X_\tau(x) = (x(t_1), \ldots, x(t_k))$. Then we have*

$$E[F|X_\tau](\vec{\eta}) = E[F(x - [x] + [\vec{\eta}])] \tag{3}$$

for P_{X_τ}-a.e. $\vec{\eta} \in \mathbb{B}^k$, where P_{X_τ} is the probability distribution of X_τ on $(\mathbb{B}^k, \mathcal{B}(\mathbb{B}^k))$.

Definition 1. $X : C(\mathbb{B}) \to \mathbb{B}$ be given by $X(x) = x(t)$ for $x \in C(\mathbb{B})$ and let $F : C(\mathbb{B}) \to \mathbb{C}$ be a function such that for $\lambda > 0$, $\int_{C_0(\mathbb{B})} |F(\lambda^{-\frac{1}{2}} x + \xi)| dm_{\mathbb{B}}(x) < \infty$ for $\xi \in \mathbb{B}$. Then for a.e. $\xi \in \mathbb{B}$, by (3), we have $E[F(\lambda^{-1/2} \cdot + \xi)|X(\lambda^{-1/2} \cdot)](\eta) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\eta] + \xi)]$ for a.e. $\eta \in \mathbb{B}$. If $E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\eta] + \xi)]$ has the analytic extension $J_\lambda(\xi, \eta)$ on $\mathbb{C}_+ \equiv \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$ as a function of λ, then we write $J_\lambda(\xi, \eta) = E^{anw_\lambda}[F|X](\xi)(\eta)$ for $\lambda \in \mathbb{C}_+$ and call it the conditional analytic Wiener integral of F given X with parameter λ. For non-zero real q, if the limit $\lim_{\lambda \to -iq} E^{anw_\lambda}[F|X](\xi)(\eta)$ exists where λ approaches to $-iq$ through $\mathbb{C}_+$, then we write $\lim_{\lambda \to -iq} E^{anw_\lambda}[F|X](\xi)(\eta) = E^{anf_q}[F|X](\xi)(\eta)$ and call it the analytic conditional Feynman integral of F given X with parameter q.

For a set E in $\mathcal{B}(\mathbb{B})$ let

$$m_{t^{1/2}}(E) = m(t^{-1/2}E) \tag{4}$$

where m is the abstract Wiener measure on $(\mathbb{B}, \mathcal{B}(\mathbb{B}))$. For $p = 1$ or $p = \infty$ we adopt the following notation for simplicity: $L_p(\mathbb{B}) = L_p(\mathbb{B}, \mathcal{B}(\mathbb{B}), m_{t^{1/2}})$. Now, we state the definition of the operator-valued Feynman integral as an element of $\mathcal{L}(L_1(\mathbb{B}), L_\infty(\mathbb{B}))$.

Definition 2. Let $F : C(\mathbb{B}) \to \mathbb{C}$ be a function. For any $\lambda > 0$, ψ in $L_1(\mathbb{B})$ and ξ in $\mathbb{B}$, let $(I_\lambda(F)\psi)(\xi) = \int\limits_{C_0(\mathbb{B})} F(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(t))dm_{\mathbb{B}}(x)$. If $I_\lambda(F)\psi$ is in $L_\infty(\mathbb{B})$ as a function of ξ and if the correspondence $\psi \to I_\lambda(F)\psi$ gives an element of $\mathcal{L} \equiv \mathcal{L}(L_1(\mathbb{B}), L_\infty(\mathbb{B}))$, the space of bounded linear operators from $L_1(\mathbb{B})$ to $L_\infty(\mathbb{B})$, we say that the operator-valued function space integral $I_\lambda(F)$ exists. Next suppose that there exists an $\mathcal{L}$-valued function which is analytic in $\mathbb{C}_+$ and agrees with $I_\lambda(F)$ on $(0, \infty)$. Then this $\mathcal{L}$-valued function is denoted by $I_\lambda^{an}(F)$ and is called the analytic operator-valued Wiener integral of F associated with λ. Finally, for any non-zero real q suppose that there exists an operator $J_q^{an}(F)$ in $\mathcal{L}$ such that for every ψ in $L_1(\mathbb{B})$, $\|I_\lambda^{an}(F)\psi - J_q^{an}(F)\psi\|_\infty \to 0$ as λ approaches to $-iq$ through $\mathbb{C}_+$. Then $J_q^{an}(F)$ is called the operator-valued Feynman integral of F with parameter q.

3 Operator-valued Feynman integral

Let $\mathcal{H}$ be an infinite dimensional separable real Hilbert space. Let $\Delta_n = \{(s_1, s_2, \ldots, s_n) \in [0, t]^n : 0 = s_0 < s_1 < s_2 < \ldots < s_n \leqslant t\}$ for any fixed $n \in \mathbb{N}$. Let $\mathcal{M}_n^{''} = \mathcal{M}_n^{''}(\Delta_n \times \mathcal{H}^n)$ be the class of all complex Borel measures on $\Delta_n \times \mathcal{H}^n$ and let $\|\mu\| = var\,\mu$, the total variation of μ in $\mathcal{M}_n^{''}$. Let $\mathcal{S}_{n,\mathbb{B}}'' = \mathcal{S}_{n,\mathbb{B}}''(\Delta_n \times \mathcal{H}^n)$ be the space of functions of the form

$$F_n(x) = \int\limits_{\Delta_n \times \mathcal{H}^n} \exp\left\{ i \sum_{k=1}^{n} (h_k, x(s_k))^\sim \right\} d\mu_{F_n}((s_1, \ldots, s_n), (h_1, \ldots, h_n)) \tag{5}$$

for a.e. $x \in C_0(\mathbb{B})$, where $\mu_{F_n} \in \mathcal{M}_n^{''}$. Here we take $\|F_n\|_n^{''} = \inf\{\|\mu_{F_n}\|\}$, where the infimum is taken over all μ_{F_n}s so that F_n and μ_{F_n} are related by (5). Let $\mathcal{M}'' = \mathcal{M}''(\sum \Delta_n \times \mathcal{H}^n)$ be the class of all sequences $\{\mu_n\}$ of measures such that each $\mu_n \in \mathcal{M}_n^{''}$ and $\sum\limits_{n=1}^{\infty} \|\mu_n\| < \infty$. Let $\mathcal{S}_{\mathbb{B}}'' = \mathcal{S}_{\mathbb{B}}''(\sum \Delta_n \times \mathcal{H}^n)$ be the space of functions of the form

$$F(x)=\sum_{n=1}^{\infty}F_n(x), \tag{6}$$

for a.e. $x\in C_0(\mathbb{B})$ where each $F_n\in\mathcal{S}''_{n,\mathbb{B}}$ and $\sum\limits_{n=1}^{\infty}\|F_n\|''_n<\infty$. The norm of F is defined by $\|F\|''=\inf\{\sum\limits_{n=1}^{\infty}\|F_n\|''_n\}$, where the infimum is taken over all representations of F given by (6).

Theorem 1. *Let $F_n\in\mathcal{S}''_{n,\mathbb{B}}$ be given by (5) and let $X:C(\mathbb{B})\to\mathbb{B}$ be given by $X(x)=x(t)$. Then for a.e. $(\xi,\eta)\in\mathbb{B}^2$, $E^{anf_q}[F_n|X](\xi)(\eta)$ exists and it is given by*

$$E^{anf_q}[F_n|X](\xi)(\eta)=$$
$$=\int\limits_{\Delta_n\times\mathcal{H}^n}\exp\Big\{i\sum_{k=1}^{n}(h_k,[\eta](s_k)+\xi)^{\sim}-\frac{i}{2q}\sum_{j=1}^{n+1}(s_j-s_{j-1})\Big|\sum_{k=j}^{n+1}h_k-\sum_{k=1}^{n+1}\frac{s_k}{t}h_k\Big|^2\Big\}$$
$$d\mu_{F_n}((s_1,\dots,s_n),(h_1,\dots,h_n))$$

for any nonzero real q, where $s_0=0$, $s_{n+1}=t$ and $h_{n+1}=0\in\mathcal{H}$.

Theorem 2. *Let $F\in\mathcal{S}''_{\mathbb{B}}$ be given by (6) and let X be as given in Theorem 1. Then a.e. $(\xi,\eta)\in\mathbb{B}^2$ and non-zero real q, $E^{anf_q}[F|X](\xi)(\eta)$ exists and is given by*

$$E^{anf_q}[F|X](\xi)(\eta)=\sum_{n=1}^{\infty}E^{anf_q}[F_n|X](\xi)(\eta),$$

where $E^{anf_q}[F_n|X](\xi)(\eta)$ is as given in Theorem 1.

Theorem 3. *Under the assumptions as given in Theorem 1, the operator-valued Feynman integral $J_q^{an}(F_n)$ exists as an element $\mathcal{L}$ and for each $\psi\in L_1(\mathbb{B})$ we have*

$$(J_q^{an}(F_n)\psi)(\xi)=\int\limits_{\mathbb{B}}E^{anf_q}[F_n|X](\xi)(\eta)\psi(\eta)dm_{t^{1/2}}(\eta),$$

for a.e. $\xi\in\mathbb{B}$, where $E^{anf_q}[F_n|X](\xi)(\eta)$ is as given in Theorem 1 and $m_{t^{1/2}}$ is given by (4).

Theorem 4. *Under the assumptions and notations as given in Theorems 1, 2 and 3 the operator-valued Feynman integral $J_q^{an}(F)$ exists as an element $\mathcal{L}$, and for* *each* $\psi \in L_1(\mathbb{B})$ *and a.e.* $\xi \in \mathbb{B}$ *we have*

$$(J_q^{an}(F)\psi)(\xi) = \int_{\mathbb{B}} E^{anf_q}[F|X](\xi)(\eta)\psi(\eta)dm_{t^{1/2}}(\eta) = \sum_{n=1}^{\infty}(J_q^{an}(F_n)\psi)(\xi).$$

4 Existence theorems

Let $\mathcal{M}(\mathcal{H})$ be the class of all complex Borel measures on $\mathcal{H}$ and let $\mathcal{G}$ be the set of all $\mathbb{C}$-valued functions ϑ on $[0,t] \times \mathbb{B}$ which have the following form

$$\vartheta(s,y) = \int_{\mathcal{H}} \exp\{i(h,y)^{\sim}\}d\sigma_s(h), \tag{7}$$

where $\{\sigma_s : s \in [0,t]\}$ is the family from $\mathcal{M}(\mathcal{H})$ satisfying the following conditions:

1. for each Borel subset E of $\mathcal{H}$, $\sigma_s(E)$ is a Borel measurable function of s on $[0,t]$,
2. $\|\sigma_s\| \in L_1([0,t])$.

Let $\vartheta \in \mathcal{G}$ be given by (7) and for a.e. x in $C_0(\mathbb{B})$ let

$$F_n(x) = \left[\int_0^t \vartheta(s,x(s))ds\right]^n \text{ and } F(x) = \exp\left\{\int_0^t \vartheta(s,x(s))ds\right\} \tag{8}$$

where n is any fixed natural number.

Theorem 5. *Let* X *be as given in Theorem 1 and let* F_n *be given by (8). For any Borel subset* E *of* $\Delta_n \times \mathcal{H}^n$ *let*

$$\mu_{F_n}(E) = \int_{\Delta_n}\int_{\mathcal{H}^n} n!\chi_E(\vec{s},\vec{h})d\left(\prod_{k=1}^{n}\sigma_{s_k}\right)(\vec{h})d\vec{s} \tag{9}$$

where $\vec{s} = (s_1,\ldots,s_n)$*. Then, for non-zero real* q, $E^{anf_q}[F_n|X](\xi)(\eta)$ *exists for a.e.* $(\xi,\eta) \in \mathbb{B}^2$*, and it is given by the expression in Theorem 1 with replacing* μ_{F_n} *by (9). Moreover, the operator-valued Feynman integral* $J_q^{an}(F_n)$ *exists as an element* $\mathcal{L}$ *and is given by the expression in Theorem 3.*

Theorem 6. *Let* X *be as given in Theorem 1, let* $m_{t^{1/2}}$ *be given by (4) and let* F *be given by (8). Then, for non-zero real* q, $E^{anf_q}[F|X](\xi)(\eta)$ *exists for a.e.*

$(\xi,\eta)\in\mathbb{B}^2$, and it is given by

$$E^{anf_q}[F|X](\xi)(\eta)=1+\sum_{n=1}^{\infty}\frac{1}{n!}E^{anf_q}[F_n|X](\xi)(\eta)$$

where $E^{anf_q}[F_n|X](\xi)(\eta)$ is as given in Theorem 5. Moreover, the operator-valued Feynman integral $J_q^{an}(F)$ exists as an element $\mathcal{L}$ and is given by, for ψ in $L_1(\mathbb{B})$ and a.e. ξ in $\mathbb{B}$,

$$(J_q^{an}(F)\psi)(\xi)=\int_{\mathbb{B}}E^{anf_q}[F|X](\xi)(\eta)\psi(\eta)dm_{t^{1/2}}(\eta)=$$

$$=\int_{\mathbb{B}}\psi(\eta)dm_{t^{1/2}}(\eta)+\sum_{n=1}^{\infty}\frac{1}{n!}(J_q^{an}(F_n)\psi)(\xi)$$

where $J_q^{an}(F_n)$ is as given in Theorem 5.

Let $\mathcal{F}(\mathbb{B})$ be the class of all functions of the form

$$\varphi(y)=\int_{\mathcal{H}}\exp\{i(h,y)^{\sim}\}d\nu(h) \tag{10}$$

for a.e. y in $\mathbb{B}$ where $\nu\in\mathcal{M}(\mathcal{H})$. For a.e. x in $C_0(\mathbb{B})$ let

$$K_n(x)=F_n(x)\varphi(x(t)) \text{ and } K(x)=F(x)\varphi(x(t)), \tag{11}$$

where F_n and F are given by (8).

Theorem 7. *Let X be as given in Theorem 1, let $m_{t^{1/2}}$ be given by (4) and let K_n be given by (11). Then, for for non-zero-real q, $E^{anf_q}[K_n|X](\xi)(\eta)$ exists for a.e. $(\xi,\eta)\in\mathbb{B}^2$, and it is given by*

$$E^{anf_q}[K_n|X](\xi)(\eta)=\varphi(\eta+\xi)E^{anf_q}[F_n|X](\xi)(\eta),$$

where $E^{anf_q}[F_n|X](\xi)(\eta)$ is as given in Theorem 5. Moreover, the operator-valued Feynman integral $J_q^{an}(K_n)$ exists as an element $\mathcal{L}$ and is given by, for $\psi\in L_1(\mathbb{B})$,

$$(J_q^{an}(K_n)\psi)(\xi)=\int_{\mathbb{B}}E^{anf_q}[K_n|X](\xi)(\eta)\psi(\eta)dm_{t^{1/2}}(\eta)$$

for a.e. $\xi\in\mathbb{B}$.

Theorem 8. *Let X be as given in Theorem 1, let $m_{t^{1/2}}$ be given by (4) and let K_n be given by (11). Then, for non-zero real q, $E^{anf_q}[K|X](\xi)(\eta)$ exists for a.e. $(\xi,\eta)\in\mathbb{B}^2$, and it is given by $E^{anf_q}[K|X](\xi)(\eta)=\varphi(\eta+\xi)E^{anf_q}[F|X](\xi)(\eta)$, where $E^{anf_q}[F|X](\xi)(\eta)$ is as given in Theorem 6. Moreover, the operator-valued Feynman integral $J_q^{an}(K)$ exists as an element $\mathcal{L}$ and is given by, for $\psi\in L_1(\mathbb{B})$,*

$$(J_q^{an}(K)\psi)(\xi)=\int\limits_{\mathbb{B}} E^{anf_q}[K|X](\xi)(\eta)\psi(\eta)dm_{t^{1/2}}(\eta)=$$

$$=\int\limits_{\mathbb{B}}\varphi(\eta+\xi)\psi(\eta)dm_{t^{1/2}}(\eta)+\sum_{n=1}^{\infty}\frac{1}{n!}(J_q^{an}(K_n)\psi)(\xi)$$

for a.e. $\xi\in\mathbb{B}$, where $J_q^{an}(K_n)$ is as given in Theorem 7.

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STATISTICAL INFERENCE WITH REPRODUCING KERNELS

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Abstract. Reproducing kernels has been recently applied to statistical inference problems by using the *kernel mean* expression of probability distributions. Given a random variable X taking values on a measurable space and a reproducing kernel Hilbert space $\mathcal{H}$ on that space with a positive definite kernel k, the *kernel mean* is defined by $E[k(\cdot, X)] \in \mathcal{H}$. This gives a mapping from the probabilities to $\mathcal{H}$. If this mapping is injective, the kernel mean uniquely identifies the probability. This class of kernel is useful for statistical inference and called *characteristic*. This paper gives a brief review on how characteristic kernels can be applied to derive practical methods for statistical inference problems, and discusses conditions that a positive definite kernel is characteristic.

1 Introduction

Statistical inference concerns problems of estimating or testing the relations and properties of distributions of random variables using finite number of data. There are various methods which solve specific tasks of statistical inference problems. Positive kernels or reproducing kernel and reproducing kernel Hilbert spaces have been proved to be useful for statistical data analysis since 1990's [12]. This paper explains a more recent methodology of statistical inference using kernel means.

Let X be a random variable taking values on a measurable space $(\mathcal{X}, \mathcal{B})$, where $\mathcal{B}$ is a σ-algebra on $\mathcal{X}$, and let $\mathcal{H}$ be a reproducing kernel Hilbert space (RKHS in short) on $\mathcal{X}$ defined by a bounded measurable positive definite kernel k. Define *kernel mean*, m_X^k, of X on $\mathcal{H}$ by

$$m_X^k = E[k(\cdot, X)] \quad \in \mathcal{H}. \tag{1}$$

Note that by reproducing property of k, m_X^k satisfies

$$\langle m_X^k, f \rangle_{\mathcal{H}} = E[f(X)] \tag{2}$$

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for any $f \in \mathcal{H}$. If there is no confusion, the kernel mean is simply denoted by m_X by omitting k. Since the kernel mean depends only on the probability distribution of X, it is also denoted by m_P^k or m_P if the distribution is P.

Let $\mathcal{P}$ be the set of probability measures on $(\mathcal{X}, \mathcal{B})$. A bounded measurable kernel k on $(\mathcal{X}, \mathcal{B})$ is called *characteristic* (with respect to $(\mathcal{X}, \mathcal{B})$) if the mapping

$$\mathcal{P} \to \mathcal{H}, \quad P \mapsto m_P^k$$

is injective. Because the kernel mean with a characteristic kernel uniquely identifies the probability, inference problems on the properties of probability distributions can be cast into the inference on the kernel means. For example, as we will see in Section 2, independence of two random variables X and Y can be tested by comparing the kernel mean of the joint variable (X, Y) and the product of kernel means of marginals.

In statistical inference, the kernel mean m_X should be estimated with finite number of data. Suppose $X_1, \ldots, X_n$ is an i.i.d. sample with the same distribution as X. The *empirical kernel mean* $\widehat{m}_X^{(n)}$ is defined by

$$\widehat{m}_X^{(n)} = \frac{1}{n} \sum_{i=1}^{n} k(\cdot, X_i).$$

It is known [2] that $\widehat{m}_X^{(n)}$ is a strongly consistent estimator, i.e., $\|\widehat{m}_X^{(n)} - m_X\|$ converges to zero in probability as $n \to \infty$. Moreover, $\sqrt{n}(\widehat{m}_X^{(n)} - m_X)$ converges to a Gaussian process. One of the advantages of using positive definite kernels and RKHS in statistical methods lies in the reproducing property: various useful quantities defined with the RKHS norm of the empirical kernel means can be exactly computed, while they are elements in infinite dimensional functional spaces. We will see some examples in Section 2.

In discussing the relation between two random variables, covariance is an essential notion. Let $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ and $(\mathcal{Y}, \mathcal{B}_\mathcal{Y})$ be measurable spaces, and (X, Y) be an random variable taking values in $\mathcal{X} \times \mathcal{Y}$. Suppose $\mathcal{H}_\mathcal{X}$ and $\mathcal{H}_\mathcal{Y}$ are RKHS's with bounded measurable positive definite kernels $k_\mathcal{X}$ on $\mathcal{X}$ and $k_\mathcal{Y}$ on $\mathcal{Y}$, respectively. The *cross-covariance operator* $\Sigma_{YX} : \mathcal{H}_\mathcal{X} \to \mathcal{H}_\mathcal{Y}$ is the operator that satisfies

$$\langle g, \Sigma_{YX} f \rangle_{\mathcal{H}_\mathcal{Y}} = E[g(Y)f(X)] - E[g(Y)]E[f(X)]$$

for any $f \in \mathcal{H}_\mathcal{X}$ and $g \in \mathcal{H}_\mathcal{Y}$. The cross-covariance operator can be also defined by $\Sigma_{YX} = E[m_{(YX)}^{k_\mathcal{Y} k_\mathcal{X}} - m_Y^{k_\mathcal{Y}} \otimes m_X^{k_\mathcal{X}}]$, where the product space $\mathcal{H}_\mathcal{Y} \otimes \mathcal{H}_\mathcal{X}$ associated with the product kernel $k_\mathcal{Y} k_\mathcal{X}$ is identified with the space of bounded linear operators from $\mathcal{H}_\mathcal{X}$ to $\mathcal{H}_\mathcal{Y}$ in a standard way. Obviously $\Sigma_{YX}^* = \Sigma_{XY}$, where A^* denotes the adjoint operator of A, and it is easy to see that Σ_{YX} is a Hilbert-Schmidt operator. When $Y = X$, the self-adjoint operator Σ_{XX} is called *covariance operator*. Σ_{XX} is a self-adjoint, trace class operator.

In a similar manner to the kernel mean, given $(X_1, Y_1), \dots, (X_n, Y_n)$ the *empirical cross-covariance operator* is defined by

$$\widehat{\Sigma}_{YX}^{(n)} = \frac{1}{n}\sum_{i=1}^n k_\mathcal{Y}(\cdot, Y_i) \otimes k_\mathcal{X}(\cdot, X_i) - \left(\frac{1}{n}\sum_{i=1}^n k_\mathcal{Y}(\cdot, Y_i)\right) \otimes \left(\frac{1}{n}\sum_{i=1}^n k_\mathcal{X}(\cdot, X_i)\right)$$

in the tensor form. $\widehat{\Sigma}_{YX}^{(n)}$ converges to Σ_{YX} in Hilbert–Schmidt norm at the rate of $n^{-1/2}$.

2 Statistical inference with kernel means

This section describes some examples of statistical inference with kernel means and cross-covariance operators.

2.1 Two sample test

Suppose we have two i.i.d. samples $X_1, \dots, X_\ell$ and $Y_1, \dots, Y_n$ with law P and Q, respectively. We wish to determine whether $P = Q$ or not. This problem is called two-sample homogeneity test, and has been long studied in statistical literature. In statistical terminology, the null hypothesis is $P = Q$, and the alternative hypothesis is $P \neq Q$. With a characteristic kernel, the problem of comparing two probabilities can be cast into the problem of comparing two kernel means. This motivates us to define the following test statistic:

$$T_{\ell,n} = \left\| \frac{1}{\ell}\sum_{i=1}^\ell k(\cdot, X_i) - \frac{1}{n}\sum_{j=1}^\ell k(\cdot, Y_j) \right\|^2$$

by introducing a positive definite kernel k. Note that this gives an empirical estimator for the squared distance measure of probabilities $\|m_P - m_Q\|^2$. By virtue of the reproducing property, we have

$$T_{\ell,n} = \frac{1}{\ell^2}\sum_{a,b=1}^\ell k(X_a, X_b) + \frac{1}{n^2}\sum_{c,d=1}^n k(Y_c, Y_d) - \frac{2}{\ell n}\sum_{a=1}^\ell \sum_{c=1}^n k(X_a, Y_c).$$

A small value of $T_{\ell,n}$ is expected under the null hypothesis $P = Q$, and the null hypothesis is rejected with the error probability α (significance level) if $T_{\ell,n}$ is larger than some threshold ϑ_α. The region of rejection is called critical region. For this test statistic, a better statistical property is obtained by debiasing it, which results in

$$U_{\ell,n} = \frac{1}{\ell(\ell-1)}\sum_{a=1}^{\ell}\sum_{b\neq a} k(X_a, X_b) + \frac{1}{n(n-1)}\sum_{c=1}^{n}\sum_{d\neq c} k(Y_c, Y_d) - \frac{2}{\ell n}\sum_{a=1}^{\ell}\sum_{c=1}^{n} k(X_a, Y_c).$$

It is known ([15], Chap. 12) that $U_{\ell,n}$ is in the class of U-statistics and the asymptotic distribution of $U_{\ell,n}$ under $\ell, n \to \infty$ with constraint $\ell/(\ell+n) \to \gamma \in (0,1)$ is a mixture of χ-square distributions. With this asymptotic distribution, we can determine ϑ_α for the test. Gretton et al. [8, 9] show some practical applications in comparing with other methods.

2.2 Independence test

Testing independence or dependence of two random variables is an important problem in many situations. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be an i.i.d. sample on $\mathcal{X} \times \mathcal{Y}$ with law P, and consider the statistical test for independence of X_i and Y_i: the null hypothesis is that they are independent, and the alternative hypothesis is that they are not. This problem can be regarded as a special case of two sample test, since we want to compare two probabilities P and $P_X \otimes P_Y$, where P_X and P_Y are marginal probabilities of X_i and Y_i, respectively. Prepare bounded measurable positive definite kernels $k_{\mathcal{X}}$ on $\mathcal{X}$ and $k_{\mathcal{Y}}$ for $\mathcal{Y}$ such that $k_{\mathcal{X}} k_{\mathcal{Y}}$ is a characteristic kernel on $\mathcal{X} \times \mathcal{Y}$. We can then use the statistic $\|\widehat{m}_{XY} - \widehat{m}_X \otimes \widehat{m}_Y\|^2$ for testing independence of X_i and Y_i. It is easy to see that this is equivalent to using the squared Hilbert-Schmidt norm of the empirical cross-covariance operator $\widehat{\Sigma}^{(n)}_{YX}$. The test statistics is thus given by

$$\|\widehat{\Sigma}^{(n)}_{YX}\|^2_{HS} = \frac{1}{n^2}\sum_{i,j=1}^{n} k_{\mathcal{X}}(X_i, X_j) k_{\mathcal{Y}}(Y_i, Y_j) -$$
$$- \frac{2}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{\mathcal{X}}(X_i, X_j)\sum_{\ell=1}^{n} k_{\mathcal{Y}}(Y_i, Y_\ell) + \frac{1}{n^4}\sum_{i,j=1}^{n} k_{\mathcal{X}}(X_i, X_j)\sum_{\ell,r=1}^{n} k_{\mathcal{Y}}(Y_\ell, Y_r).$$

In matrix notation,

$$\left\|\widehat{\Sigma}^{(n)}_{YX}\right\|^2_{HS} = \frac{1}{n^2}\mathrm{Tr}\left[K_X Q_n K_Y Q_n\right],$$

where K_X and K_Y are Gram matrices given by $(k_{\mathcal{X}}(X_i, X_j))_{ij}$ and $(k_{\mathcal{Y}}(Y_i, Y_j))_{ij}$, respectively, and $Q_n = I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T$ with $\mathbf{1}_n = (1, \ldots, 1)^T$.

In a similar manner to the two sample test, the asymptotic distribution of the test statistic for $n \to \infty$ is known and can be used for determining the critical region of the independence test at a significance level. Alternatively, we can also use permutation test, which simulates the distribution under the independence assumption by random permutation of either of (X_i) or (Y_i). For the details of this test statistic and numerical examples, see [8, 10].

Another important statistical notion is conditional independence, which is widely used in statistical inference for graphical modeling, causal inference, and Bayesian methods. This paper describes only a sketch of the kernel method for conditional independence, and leaves the details to the original papers [4, 6]. Suppose we have random variables (X, Y, Z) on $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$, and bounded measurable positive definite kernels $k_{\mathcal{X}}, k_{\mathcal{Y}}, k_{\mathcal{Z}}$ on $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, respectively. The respective RKHS are denoted by $\mathcal{H}_{\mathcal{X}}, \mathcal{H}_{\mathcal{Y}}, \mathcal{H}_{\mathcal{Z}}$. The *conditional cross-covariance operator* from X to Y given Z is the operator from $\mathcal{H}_{\mathcal{X}}$ to $\mathcal{H}_{\mathcal{Y}}$ defined by

$$\Sigma_{YX|Z} = \Sigma_{YX} - \Sigma_{YZ}\Sigma_{ZZ}^{-1}\Sigma_{ZX}. \tag{3}$$

Here the operator $\Sigma_{YZ}\Sigma_{ZZ}^{-1}\Sigma_{ZX}$ should be rigorously interpreted as $\Sigma_{YY}^{1/2}V_{YZ}V_{ZX}\Sigma_{XX}^{1/2}$, where V_{YZ} is given by the unique operator in the decomposition [1] $\Sigma_{YZ} = \Sigma_{YY}^{1/2}V_{YZ}\Sigma_{ZZ}^{1/2}$ with $|V_{YZ}| \leqslant 1$, $\mathcal{R}(V_{YZ}) \subset \overline{\mathcal{R}(\Sigma_{YY})}$ and $\mathcal{N}(V_{YZ})^{\perp} \subset \overline{\mathcal{R}(\Sigma_{ZZ})}$. V_{ZX} is given similarly.

The conditional cross-covariance operator is related to the conditional covariance as follows.

Proposition 1 (see [3]). Assume $k_{\mathcal{Z}}$ is characteristic. Then, for any $f \in \mathcal{H}_{\mathcal{X}}$ and $g \in \mathcal{H}_{\mathcal{Y}}$

$$\langle g, \Sigma_{YX|Z} f\rangle_{\mathcal{H}_{\mathcal{Y}}} = E[\mathrm{Cov}[f(X), g(Y)|Z]].$$

The above definition is a straightforward extension of the conditional covariance of the Gaussian random variables: for a Gaussian random vector (X, Y, Z) the conditional covariance between X and Y given Z is given by

$$C_{YX|Z} = C_{YX} - C_{YZ}C_{ZZ}^{-1}C_{ZX},$$

where the existence of C_{ZZ}^{-1} is assumed. It is well known that for Gaussian variables X and Y are conditionally independent given Z if and only if $C_{YX|Z} = 0$. As an extension of this fact, we have the following theorem.

Theorem 1 (see [3]). *Define $W = (X, Z)$ and use the product kernel $k_{\mathcal{W}} = k_{\mathcal{X}} k_{\mathcal{Z}}$ for W. Assume that $k_{\mathcal{Z}}$ and $k_{\mathcal{Y}} k_{\mathcal{W}}$ are characteristic kernels on $\mathcal{Z}$ and $\mathcal{Y} \times (\mathcal{X} \times \mathcal{Z})$, respectively. Then, X and Y are conditional independent given Z if and only if $\Sigma_{YW|Z} = O$.*

Note that in the above theorem the joint variable $W = (X, Z)$ is used to check the conditional independence. As is shown in Proposition 2, the conditional cross-covariance operator can handle the conditional covariance between $f(X)$ and $g(Y)$ given Z only on average over Z, though conditional independence requires $\mathrm{Cov}[f(X), g(Y)|Z] = 0$ for almost every Z. Intuitively, the joint variable $W = (X, Z)$ in $\Sigma_{WY|Z}$ makes it possible to handle each value of Z.

As $\Sigma_{YW|Z}$ is a Hilbert–Schmidt operator, the squared Hilbert-Schmidt norm $\|\Sigma_{YW|Z}\|_{HS}^2$ can be used for discussing conditional independence or dependence. Further discussions on this statistic can be found in [4], and an application to causal inference is proposed in [14].

3 Characteristic kernels on LCA group

As we have seen in the previous section, the characteristic property of a kernel is important in its statistical applications. This section discusses some conditions of this property. We first start with a general condition.

Proposition 2. Let $(\mathcal{X}, \mathcal{B})$ be a measurable space, and k be a measurable bounded positive definite kernel on $\mathcal{X}$ with RKHS $\mathcal{H}_k$. Then, k is characteristic if and only if $\mathcal{H}_k + \mathbb{R}$ is dense in $L^2(P)$ for any probability measure P on $(\mathcal{X}, \mathcal{B})$, where $\mathcal{H}_k + \mathbb{R} = \{f + c \mid f \in \mathcal{H}_k, c \in \mathbb{R}\}$.

Proof. Suppose $m_P = m_Q$ for different probabilities P and Q while $\mathcal{H}_k + \mathbb{R}$ is dense in $L^2(|P - Q|)$, where $|P - Q|$ is the total variation of $P - Q$. Then, for any $E \in \mathcal{B}$ and $\varepsilon > 0$ there is $f \in \mathcal{H}_k$ and $c \in \mathbb{R}$ such that $\int |f + c - \chi_E| d|P - Q| < \varepsilon$. Here χ_E is the indicator function of E. This means $|(\int f dP - P(E)) - (\int f dQ - Q(E))| < \varepsilon$. It follows from the assumption $m_P = m_Q$ that $\int f dP = \int f dQ$, which implies $|P(E) - Q(E)| < \varepsilon$. As $\varepsilon > 0$ is arbitrary, $P(E) = Q(E)$, which causes contradiction.

Next, suppose $\mathcal{H}_k + \mathbb{R}$ is not dense in $L^2(P)$ for some probability P. Then, there is nonzero $f \in L^2(P)$ such that $\int f g dP = 0$ for any $g \in \mathcal{H}_k$ and $\int f dP = 0$. Define two different probabilities Q_1 and Q_2 by $Q_1(E) = \int_E |f| dP / \|f\|_{L^1(P)}$ and $Q_2(E) = \int_E (|f| - f) dP / \|f\|_{L^1(P)}$. Then, for any $g \in \mathcal{H}_k$, $\int g dQ_1 - \int g dQ_2 = \int f g dP / \|f\|_{L^1(P)} = 0$. This implies $m_{Q_1} = m_{Q_2}$, hence k is not characteristic. $\square$

3.1 Harmonic Analysis on LCA group

This subsection gives a brief review of the harmonic analysis on locally compact group. For the details, see e.g. [11].

A complex-valued Radon measure μ on a locally compact space X is said to be *regular* if $|\mu|$ is outer regular, that is, $|\mu|(E) = \inf\{|\mu|(U) \mid U$ is an open set including $E\}$ holds for every Borel set E. The set of regular measures on X is denoted by $M(X)$. For a finite regular measure, there is the largest open set U with $|\mu|(U) = 0$. The complement of U is called the *support* of μ, and denoted by $\mathrm{supp}(\mu)$

Let G be a group. A function $\varphi : G \to \mathbb{C}$ is called *positive definite* if $k(x, y) = \varphi(y^{-1}x)$ is a positive definite kernel. This type of positive definite kernel is called *shift-invariant*, since $k(zx, zy) = k(x, y)$. There are many examples of shift-invariant positive definite kernels, which are used in practical applications in statistics: Gaussian RBF kernel $k(x, y) = \exp(-\|x - y\|^2/\sigma^2)$ and Laplacian kernel $k(x, y) = \exp(-\beta \sum_{i=1}^{n} |x_i - y_i|)$ are famous ones on the additive group $\mathbb{R}^n$.

A particulary interesting class of group in discussing positive definite kernels is locally compact Abelian groups (LCA group, in short), on which famous Bochner's theorem characterizes the continuous positive definite functions. For a LCA group the additive notation $x + y$ is employed for the group operation hereafter.

A function $\gamma : G \to \mathbb{C}$ is called a *character* of a LCA group G if $\gamma(x + y) = \gamma(x)\gamma(y)$ and $|\gamma(x)| = 1$ for all $x, y \in G$. The *dual group* $\widehat{G}$ of G is the set of all continuous characters of G. $\widehat{G}$ is an Abelian group with the value multiplication, which is conventionally denoted by addition, i.e., $(\gamma_1 + \gamma_2)(x) := \gamma_1(x)\gamma_2(x)$. For any $x \in G$, the function $\hat{x}$ on $\widehat{G}$ given by $\hat{x}(\gamma) = \gamma(x)$ $(\gamma \in \widehat{G})$ defines a character of $\widehat{G}$. It is known that $\widehat{G}$ is a LCA group if the weakest topology is introduced so that $\hat{x}$ is continuous for each $x \in G$. As is well known, the Pontryagin duality guarantees that the group homomorphism $G \to \widehat{\widehat{G}}$, $x \mapsto \hat{x}$ is isomorphism and homeomorphic, where $\widehat{\widehat{G}}$ is the dual group of $\widehat{G}$, and thus $\widehat{\widehat{G}}$ can be identified with G. In view of this duality, it is customary to write $(x, \gamma) := \gamma(x)$.

Let $f \in L^1(G)$ and $\mu \in M(G)$, the Fourier transform of f and μ are respectively defined by

$$\hat{f}(\gamma) = \int_G (-x, \gamma) f(x) dx, \quad \hat{\mu}(\gamma) = \int_G (-x, \gamma) d\mu(x), \quad (\gamma \in \widehat{G}), \tag{4}$$

where dx is the Haar measure of G. Note that $\hat{f}$ and $\hat{\mu}$ are continuous. For $f \in L^\infty(G)$, $g \in L^1(G)$, and $\mu \in M(G)$, the convolutions are defined respectively by

$$(g * f)(x) = \int_G f(x-y)g(y)dy, \quad (\mu * f)(x) = \int_G f(x-y)d\mu(y).$$

The convolution $g * f$ is uniformly continuous on G. For any $f, g \in L^1(G)$ and $\mu \in M(G)$, the following relations hold:

$$\widehat{f * g} = \hat{f}\hat{g}, \quad \widehat{\mu * f} = \widehat{\mu}\widehat{f}. \tag{5}$$

For a LCA group, the continuous positive definite functions are characterized in the following theorem.

Theorem 2 (Bochner's theorem). *A continuous function φ on G is positive definite if and only if there is a non-negative measure $\Lambda \in M(\widehat{G})$ such that*

$$\varphi(x) = \int_{\widehat{G}} (x, \gamma) d\Lambda(\gamma) \quad (x \in G). \tag{6}$$

Moreover, such Λ is unique for each φ.

Bochner's theorem implies that the continuous positive definite functions form a convex cone with the extreme points given by the dual group $\widehat{G}$.

3.2 Characteristic kernels on LCA group

Let G be a LCA group and k be a shift invariant positive definite kernel on G. We wish to give conditions that k is characteristic. Before going to the formal theorems, we show an intuitive explanation. First note that for a shift invariant kernel k, the kernel mean m_X for a random variable X with law P is given by

$$m_X(x) = \langle m_X, k(\cdot, x)\rangle = \int_G k(x-y)dP(y) = (\varphi * P)(x).$$

Thus k is characteristic if and only if $\varphi * (P - Q) \neq 0$ for any different probabilities P and Q. By Fourier transforms, this holds if $\widehat{\varphi}\widehat{\mu} \neq 0$ for any nontrivial finite signed measure μ. Based on Bochner's theorem, a sufficient condition is easily obtained.

Theorem 3 (see [5]). *Let φ be a continuous positive definite function on a LCA group G given by Eq. (6) with Λ. If $\operatorname{supp}(\Lambda) = \widehat{G}$, then the positive definite kernel $k(x, y) = \varphi(x - y)$ is characteristic.*

Proof. It suffices to prove that if $\mu \in M(G)$ satisfies $\mu * \varphi = 0$ then $\mu = 0$. By Fubini's theorem,

$$\int_G (\mu * \varphi)(x) d\mu(x) = \int_G \int_G \varphi(x-y) d\mu(y) d\mu(x) =$$

$$= \int_{\widehat{G}} \int_G (x,\gamma) d\mu(x) \int_G (-y,\gamma) d\mu(y) d\Lambda(\gamma) = \int_{\widehat{G}} |\widehat{\mu}(\gamma)|^2 d\Lambda(\gamma).$$

If $\mu * \varphi = 0$, it follows from the continuity of $\widehat{\mu}$ and $\mathrm{supp}(\Lambda) = \widehat{G}$ that $\widehat{\mu} = 0$, which means $\mu = 0$ by the duality. □

In real-valued cases, the condition $\mathrm{supp}(\Lambda) = \widehat{G}$ is almost necessary.

Theorem 4 (see [5]). *Let φ be a $\mathbb{R}$-valued continuous positive definite function on a LCA group G given by Eq. (6) with Λ. The positive definite kernel $k(x,y) = \varphi(x-y)$ is characteristic if and only if either of the following (i) or (ii) holds: (i) G is non-compact and* $\mathrm{supp}(\Lambda) = \widehat{G}$*, or (ii) G is compact and* $\mathrm{supp}(\Lambda) \supset \widehat{G} - \{0\}$.

Proof. It is obvious that k is characteristic if and only if so is $k(x,y)+1$. Since $\int_{\widehat{G}} (x,\gamma) d\delta_0$ is a positive constant for compact G, where δ_0 is the Dirac measure at $0 \in \widehat{G}$, we can assume w.l.o.g. that $0 \in \mathrm{supp}(\Lambda)$. The "if" part is thus given by Theorem 3.

For "only if" part, assuming $\mathrm{supp}(\Lambda) \neq \widehat{G}$, we will construct two different probabilities P_1 and P_2 such that $(P_1 - P_2) * \varphi = 0$. In the following, for a set A in G or $\widehat{G}$, the notations $-A = \{-x \mid x \in A\}$, $A - A = \{x - y \mid x, y \in A\}$, and $A + x = \{y + x \mid y \in A\}$ are used. Since φ is real-valued, $\Lambda(-E) = \Lambda(E)$ for every Borel set E. Thus $U := \widehat{G} \backslash \mathrm{supp}(\Lambda)$ is a non-empty open set with $-U = U$ and $0 \notin U$. Fix $\gamma_0 \in U$. By the continuity of $(\gamma_1, \gamma_2) \mapsto \gamma_1 - \gamma_2$, there exits an open neighborhood W of $0 \in \widehat{G}$ such that $\pm\gamma_0 \notin W - W$, $\mathrm{cl}(W - W) \pm \gamma_0 \subset U$, and $(W - W) + \gamma_0 \cap (W - W) - \gamma_0 = \emptyset$.

Let $g = \chi_W * \chi_{-W}$, where χ_E denotes the indicator function of E. It is easy to see that g is continuous and positive definite. By Bochner's theorem and Pontryagin duality, there is a nonzero, non-negative measure $\mu \in M(G)$ such that

$$g(\gamma) = \int_G (x,\gamma) d\mu(x) \quad (\gamma \in \widehat{G}).$$

Define a function h on $\widehat{G}$ by

$$h(\gamma) := g(\gamma - \gamma_0) + g(\gamma + \gamma_0) = \int_G (x, \gamma) d((\gamma_0 + \overline{\gamma_0})\mu)(x),$$

where $(\gamma_0 + \overline{\gamma_0})\mu$ is a signed measure defined by $((\gamma_0 + \overline{\gamma_0})\mu)(E) = \int_E (\gamma_0(x) + \overline{\gamma_0}(x)) d\mu(x)$. Note $\mathrm{supp}(g) \subset \mathrm{cl}(W - W)$. Since $\mathrm{cl}(W-W) + \gamma_0 \cap \mathrm{cl}(W-W) - \gamma_0 = \emptyset$ and g is nonzero, h is a nonzero function. Also $\mathrm{supp}(h) \subset \mathrm{cl}(W-W) + \gamma_0 \cup \mathrm{cl}(W - W) - \gamma_0 \subset U$, which does not contain 0. Thus, by setting $\gamma = 0$, we have

$$((\gamma_0 + \overline{\gamma_0})\mu)(G) = 0. \tag{7}$$

Let $m = |(\gamma_0 + \overline{\gamma_0})\mu|(G)$ $(\neq 0)$, and define two different probability measures by

$$P_1 = \frac{1}{m}|(\gamma_0 + \overline{\gamma_0})\mu|, \quad P_2 = \frac{1}{m}\{|(\gamma_0 + \overline{\gamma_0})\mu| - (\gamma_0 + \overline{\gamma_0})\mu\}.$$

From Fubini's theorem,

$$\begin{aligned} m \cdot ((P_1 - P_2) * \varphi)(x) &= \int_G \varphi(x - y)(\gamma_0(y) + \overline{\gamma_0}(y)) d\mu(y) = \\ &= \int_{\widehat{G}} (x, \gamma) \int_G \overline{\{(y, \gamma - \gamma_0) + (y, \gamma + \gamma_0)\}} d\mu(y) d\Lambda(\gamma) = \\ &= \int_{\widehat{G}} (x, \gamma)\{g(\gamma - \gamma_0) + g(\gamma + \gamma_0)\} d\Lambda(\gamma) = \int_{\widehat{G}} (x, \gamma) h(\gamma) d\Lambda(\gamma). \end{aligned}$$

Since $\mathrm{supp}(h) \subset U = \widehat{G} \backslash \mathrm{supp}(\Lambda)$, we have $(P_1 - P_2) * \varphi = 0$. □

Theorems 3 and 4 are generalization of the results in [13]. From Theorem 4, we can see that the characteristic property is stable under the product for real-valued shift-invariant continuous kernels.

Corollary 1 (see [5]). Let $\varphi_1(x - y)$ and $\varphi_2(x - y)$ be $\mathbb{R}$-valued continuous shift-invariant characteristic kernels on a LCA group G. If (i) G is non-compact, or (ii) G is compact and $2\gamma \neq 0$ for any nonzero $\gamma \in \widehat{G}$. Then $(\varphi_1\varphi_2)(x - y)$ is characteristic.

Proof. We show the proof only for (i). Let Λ_1, Λ_2 be the non-negative measures to give φ_1 and φ_2, respectively, in Eq. (6). By Theorem 4, $\mathrm{supp}(\Lambda_1) = \mathrm{supp}(\Lambda_2) =$

$\widehat{G}$. This means $\mathrm{supp}(\Lambda_1 * \Lambda_2) = \widehat{G}$. The proof is completed because $\Lambda_1 * \Lambda_2$ gives a positive definite function $\varphi_1\varphi_2$. $\square$

Example 1. $(\mathbb{R}^n, +)$: Gaussian RBF kernel $\exp(-\frac{1}{2\sigma^2}\|x-y\|^2)$ and Laplacian kernel $\exp(-\beta\sum_{i=1}^n |x_i - y_i|)$ are characteristic on $\mathbb{R}^n$, since the corresponding non-negative measures are $\exp(-\frac{\sigma^2}{2}\|\omega\|^2)$ and $\prod_{j=1}^n 1/(1+\omega_j^2)$, respectively, up to positive constant. An example of a positive definite kernel that is *not* characteristic on $\mathbb{R}^n$ is $\mathrm{sinc}(x-y) = \frac{\sin(x-y)}{x-y}$: the Fourier transform is the indicator function of a bounded interval.

Example 2. $([0, 2\pi), +)$: The addition is made modulo 2π. The dual group is $\{e^{\sqrt{-1}nx} \mid n \in \mathbb{Z}\}$, and expression of the Bochner's theorem is given by Fourier expansion,

$$\varphi(x) = \sum_{n=-\infty}^{\infty} a_n e^{\sqrt{-1}nx}, \quad a_n \geqslant 0, \quad \sum_{n=-\infty}^{\infty} a_n < \infty.$$

Among these positive definite functions, the characteristic kernels are given by the ones with coefficients $a_0 \geqslant 0$ and $a_n > 0$ $(n \neq 0)$. The examples of characteristic kernels are $k_1(x,y) = (\pi - (x-y)_{mod\, 2\pi})^2$ $(a_0 = \pi^2/3, a_n = 2/n^2$ $(n \neq 0))$, and $k_2(x,y) = 1/(1-2\alpha\cos(x-y)+\alpha^2)$ (Poisson kernel) given by $a_n = \alpha^{|n|}$ $(\alpha \in (0,1))$. Examples of *non*-characteristic kernels on $[0, 2\pi)$ include $\cos(x-y)$, Féjer, and Dirichlet kernel.

The above conditions of characteristic properties can be extended in part to the case of compact groups using the unitary representations [5].

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ON BLIND SOURCE SEPARATION PROBLEM IN TIME-FREQUENCY SPACE

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Key words: time-frequency analysis, blind source separation

AMS Mathematics Subject Classification: 42C40

Abstract. On the blind source separation problem, there is a method to use the quotient function of complex valued time-frequency informations of two observed signals. Under some assumptions, studying a commutative distribution function of the quotient function, we can estimate the number of sources. We will review a mathematical formulation of the method and give some remarks on the method in terms of application.

1 Introduction

To treat blind source separation problem, in many cases, either statistical independence or statistical orthogonality of the sources has been assumed. If we have as many observed signals as sources, we can separate sources from the observed signals under the above assumption. Jourjine et al. [7] considered the problem of separating by using signals observed at different positions and sources were assumed to be *W-disjoint orthogonal*, which means that the windowed Fourier transforms of the sources are mutually orthogonal. Then Balan-Rosca [3] relaxed the assumption to that of statistical independence with some ergodicity hypotheses. Napoletani et al. [8] considered the problem of detecting the number of sources by using two observed signals assuming the linear independence of the windowed Fourier transforms of the sources and the continuity of some density functions. The fundamental idea to detect the number of sources employed in [3, 7, 8] is to consider the ratio of the windowed Fourier transforms of two observed signals. In [1] and [4], we gave a mathematical formulation for the estimation problem of the number of sources. Later, Ashino et al. [2] treated problem considering time delay.

In this note, we will review our results in [1, 4–6], and give some remarks. Numerical experiments of our method were tested by Professor A. Morimoto of Osaka Kyoiku University. The author would like to express her sincere gratitude to him for helping her to complete this note.

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2 Underling theorems

2.1 Complex valued quotient function

Let d be an integer and $\mathbf{E} = \mathbf{R}^d, d \geqslant 2$ or $\mathbf{E} = \mathbf{C}^d$. For $M > 0$ put

$$\mathbf{E}_M = \{z = (z_1, \ldots, z_d) \in \mathbf{E}; |z_k| < M,\ k = 1, \ldots, d\}.$$

Let $n \geqslant 2$ be an unknown integer and $S_1, \ldots, S_n$ be linearly independent complex valued functions on X. Set

$$D = \bigcup_{j=1}^{n} \{z \in \mathbf{E};\ S_j(z) \neq 0\}, D_M = \bigcup_{j=1}^{n} \{z \in \mathbf{E}_M;\ S_j(z) \neq 0\},$$
$$E_j = \{z \in D;\ S_j(z) \neq 0,\ S_k(z) = 0\,(k \neq j)\}\,, E = \bigcup_j E_j,$$
$$E_j(M) = E_j \cap D_M,\ E_M = E \cap D_M,\ E^c(M) = D_M \setminus E_M.$$

For real numbers a_j, b_j $(j = 1, \ldots, n)$, put

$$X_1 = a_1 S_1 + \ldots + a_n S_n, X_2 = b_1 S_1 + \ldots + b_n S_n.$$

We consider a "quotient function"

$$Q(z) = \frac{X_1(z)}{X_2(z)} = \frac{a_1 S_1(z) + \ldots + a_n S_n(z)}{b_1 S_1(z) + \ldots + b_n S_n(z)}\ ,\ z \in X.$$

First we assume the condition:

Condition A. $a_j \neq 0,\ b_j \neq 0\ (j = 1, \ldots, n),\ a_j b_k - a_k b_j \neq 0 (j \neq k)$; that is, $q_j \equiv a_j/b_j$ are mutually distinct non zero numbers.

On E_j, $Q(z) = q_j \in \mathbf{R}$. Therefore, under Condition A, if $D = E$, then we can detect the number of sources by counting the number of elements of $Q(D) = Q(E) = \{q_j \in \mathbf{R}; j = 1, \ldots, n\}$. In general, such a set is too small. Therefore, for $\eta > 0$, denoting the imaginary part of Q by $\operatorname{Im} Q$, we consider the function

$$Q_\eta(z) = \begin{cases} Q(z) & (|\operatorname{Im} Q(z)| < \eta) \\ 0 & (|\operatorname{Im} Q(z)| \geqslant \eta) \end{cases}\ .$$

2.2 Commutative distribution function on R

We denote the Lebesgue measure of a measurable set $X \in \mathbf{E}$ by $\mu(X) = \int_X dz$, where dz is the Lebesgue measure on $\mathbf{E}$.

Put $\nu(X) = \mu(\{z \in X; \mathrm{Im}\, Q(z) = 0,\ Q(z) \neq 0\})$. For $\eta > 0, M > 0$ and $x \in \mathbf{R}$, define

$$\begin{aligned}(G_\eta(M))(x) &= \frac{\mu(\{z \in D_M; ReQ_\eta(z) < x,\ Q_\eta(z) \neq 0\})}{\mu(\{z \in D_M;\ Q_\eta(z) \neq 0\})}, \\ (G_0(M))(x) &= \frac{\mu(\{z \in D_M; ReQ(z) < x, ImQ(z) = 0,\ Q(z) \neq 0\})}{\nu(D_M)},\end{aligned}$$

where $\mathrm{Re}\, Q$ is the real part of Q. Note that $(G_\eta(M))(x)$ is a monotone increasing function in x. For $(G_\eta(M))(x)$ to be well-defined, we assume the condition.

Condition B. $\nu(D_M) > 0$.

Proofs for the following theorems see [1].

Theorem 1.

(i) $|(G_\eta(M))(x) - (G_0(M))(x)| \leqslant \dfrac{\mu\left(\{z \in D_M;\ 0 < |\,\mathrm{Im}\, Q(z)| < \eta\}\right)}{\mu(\{z \in D_M;\ Q_\eta(z) \neq 0\})}$.

(ii) $\lim_{\eta \to 0} (G_\eta(M))(x) = (G_0(M))(x)$.

2.3 A condition to detect the number of sources

Under some conditions, $(G_0(M))(x)$ will be a step function whose gaps are at $x = q_j, j = 1, \ldots, n$.

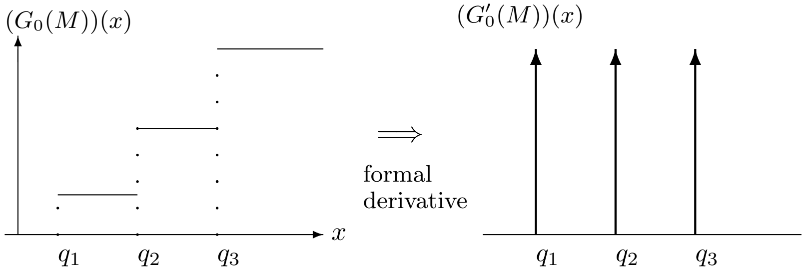
Theorem 2. *Assume $\nu(E_M) > 0$. Put $\alpha_M = \nu\left(E^c(M)\right)/\nu(D_M)$, and let $(H_0(M))(x)$ denote a step function defined by*

$$(H_0(M))(x) = \sum_{j=1}^{n} \frac{\nu(E_j(M))}{\nu(E_M)} Y\left(x - q_j\right),\ Y(x) = \begin{cases} 1 & (x \geqslant 0) \\ 0 & (x < 0) \end{cases}.$$

Then we have $|(G_0(M))(x) - (H_0(M))(x)| \leqslant \alpha_M$.

Corollary 1. Assume $\nu(E_M) > 0$. If $\nu\left(E^c(M)\right) = 0$, then

$$\lim_{\eta \to 0} (G_\eta(M))(x) = (G_0(M))(x) = (H_0(M))(x).$$



Note 1. For any $\varepsilon > 0$, take $\eta > 0$ such that

$$\frac{\mu\left(\{z \in D_M;\ 0 < |\operatorname{Im} Q(z)| < \eta\}\right)}{\mu(\{z \in D_M;\ Q_\eta(z) \neq 0\})} < \varepsilon.$$

Then we have $|(G_\eta(M))(x) - (H_0(M))(x)| < \epsilon + \alpha_M$. Especially for $\alpha_M = 0$, we have $|(G_\eta(M))(x) - (H_0(M))(x)| < \epsilon$.

Note 2. Beside the number of steps or peaks, the points q_j's are important to find S_j's. Once we detect the number of S_j's, say n, take another $X_j, j = 3, \ldots, n$. By using X_j and q_j, $j = 1, \ldots, n$, we find $m_j S_j, j = 1, \ldots, n$, where $m_j \in \mathbf{R}$ are flexible multipliers. For the detail see [4].

3 Generalization

The assumption $\nu(E_M) > 0$ seems quite restrictive. Let us consider an example.

For $s \in L^2(\mathbf{R})$ and $\omega_0 > 0$, the Gabor wavelet transform on $\mathbf{R}^2$ is defined by

$$S(t, \omega) = e^{i\omega t} \sqrt{|\omega|/\omega_0} \int s(x) e^{\left(-(x-t)^2\omega^2/2\omega_0^2\right)} e^{-i\omega x}\, dx. \tag{1}$$

Then for $s(x) = \sin \alpha x$,

$$S(t, \omega) = -i\sqrt{2\pi\omega_0/|\omega|}\left(e^{-(\omega_0 - \alpha\omega_0/\omega)^2/2} e^{it\alpha} - e^{-(\omega_0 + \alpha\omega_0/\omega)^2/2} e^{-it\alpha}\right).$$

Thus $S(t, \omega)$ for $s(x) = \sin \alpha x$ will not vanish and $E_j = \emptyset$. However by Figure 1, we find that the small value are ignored.

Numerical experiment. For three signals $s_1(t) = \sin(4\pi t)$, $s_2(t) = \sin(12\pi t)$ and $s_3(t) = \sin(18\pi t)$, take $a_1 = 0.7567$, $a_2 = 0.8795$, $a_3 = 0.8764$, $b_1 = 0.6205$, $b_2 = 0.9967$, $b_3 = 0.5550$ and set $x_1 = a_1 s_1 + a_2 s_2 + a_3 s_3, x_2 = b_1 s_1 + b_2 s_2 + b_3 s_3$. However $\nu(E) = 0$ for these sources, we will try to apply our method.

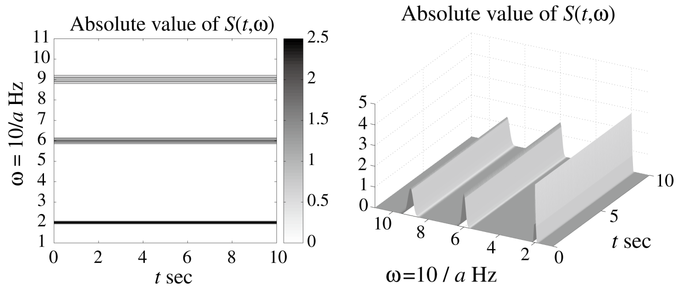


Figure 1. $|S(t,\omega)|$ for $s(x) = \sin(4\pi x) + \sin(12\pi x) + \sin(18\pi x)$ with $w_0 = 20\pi$ in (1)

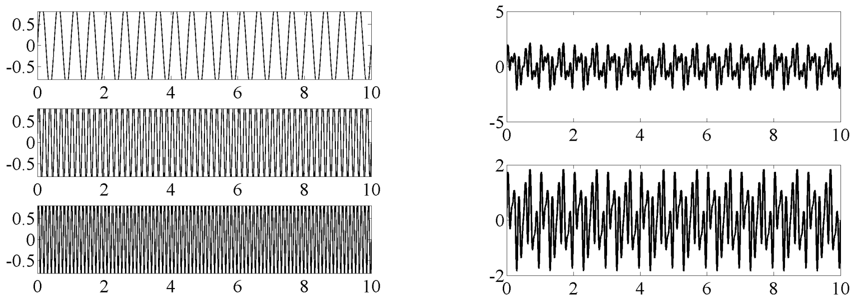


Figure 2. Sources s_1, s_2, s_2 (left) and Oveserved signals x_1, x_2 (right).

Let X_1 and X_2 are the Gabor wavelet transforms of x_1 and x_2, respectively. The steps of the graph of $Q_\eta(M)$ are expected to appear at

$$q_1 = a_1/b_1 = 1.2195,\ q_2 = a_2/b_2 = 0.88241,\ q_3 = a_2/b_2 = 1.5791.$$

The graphs in Figure 3 is a result of numerical experiment employing our method by using MATLAB. By Figure 3, we can detect the number of sources and q_j's.

We would like to weaken the assumption. Consider the case that one S_j is dominant over the other S_k's ($k \neq j$). For $\delta > 0$, put

$$E_j(\delta) = \{z \in D; |b_k S_k(z)| \leqslant \delta |b_j S_j(z)|\ (k \neq j),\ S_j \neq 0\}, E(\delta) = \cup_j E_j(\delta),$$

$$E_j(\delta; M) = E_j(\delta) \cap D_M,\ E(\delta; M) = E(\delta) \cap D_M,\ E^c(\delta; M) = (D \setminus E(\delta)) \cap D_M$$

Instead of assuming $\nu(E_M) > 0$, we discuss the problem in the situation where $\nu(E(\delta; M)) > 0$ holds for some $\delta > 0$.

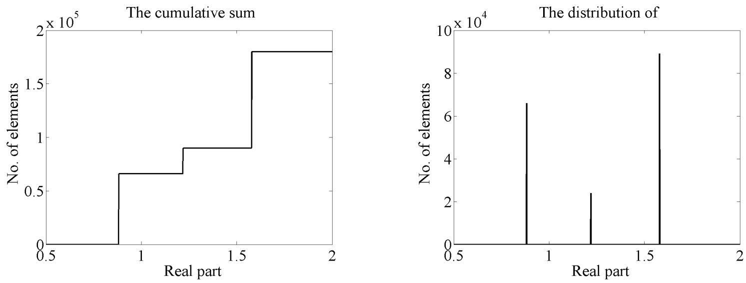


Figure 3

Theorem 3. *Let* $\eta > 0$ *and* $1/(n-1) > \delta > 0$. *Put* $\Delta = \max_{j,k} |q_j - q_k|$, $\gamma(\delta) = (n-1)\delta\Delta/(1-(n-1)\delta)$, $\alpha_M(\delta) = \nu(E^c(\delta; M))/(\nu(D_M))$ *and*

$$\rho_\eta(\delta; M) = \alpha_M(\delta) + \frac{\mu\left(\{z \in D_M;\ 0 < |\operatorname{Im} Q(z)| < \eta\}\right)}{\mu(\{z \in D_M;\ Q_\eta(z) \neq 0\})}.$$

Here we take δ so small as $\gamma(\delta) < \min_{k \neq j} |q_j - q_k|/2$. Assume that $\nu(E(\delta; M)) > 0$. Putting $\nu_j(E, \delta, M) = \nu(E_j(\delta; M))/\nu(E(\delta; M))$, define

$$(H_0(\delta; M))(x) = \sum_{j=1}^{n} \nu_j(E, \delta, M) Y\left(x - q_j\right),$$

$$(\overline{H}_\eta(\delta; M))(x) = \left(\sum_{j=1}^{n} \nu_j(E, \delta, M) + \rho_\eta(\delta; M)\right) Y(x - (q_j - \gamma(\delta))),$$

$$(\underline{H}_\eta(\delta; M))(x) = \left(\sum_{j=1}^{n} \nu_j(E, \delta, M) - \rho_\eta(\delta; M)\right) Y(x - (q_j + \gamma(\delta))).$$

Then the graph of $G_\eta(M)$ is contained in the closed domain surrounded by the graph $\mathcal{G}(\overline{H}_\eta(\delta; M))$ of $\overline{H}_\eta(\delta; M)$ and the graph $\mathcal{G}(\overline{H}_\eta(\delta; M))$ of $\underline{H}_\eta(\delta; M)$ (See Figure 4).

When we have to consider time delay, we have to consider the function such as

$$Q(z; 0) = Q(z),$$

$$Q(z; c) = \frac{X_1(z; c_1)}{X_2(z; c_2)} = \frac{a_1 S_1(z - c_{11}) + \ldots + a_n S_n(z - c_{n1})}{b_1 S_1(z - c_{12}) + \ldots + b_n S_n(z - c_{n2})}.$$

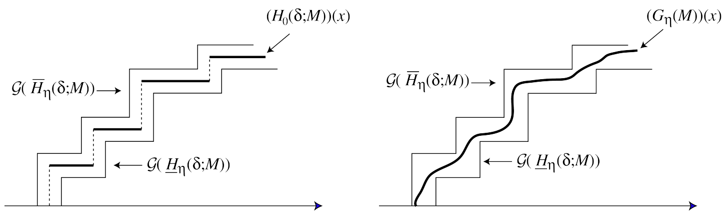


Figure 4

If $c \neq 0$, $Q(z;c)$ will take various values on $E_j = \{z \in D;\ S_j \neq 0,\ S_k = 0\,(k \neq j)\}$. Therefore we can not expect the graph of the commutative distribution function will be a step function even if $D = \cup_j E_j$.

4 Remarks

Remark 1. Let $s_j(x) = \sin(jx), j = 1, \dots, n$ and $x_1(x) = a_1 s_1(x) + \dots + a_n s_n(x)$, $x_2(x) = b_1 s_1(X) + \dots + b_n s_n(x)$ be observed signals. And let $X_1(t,\omega), X_2(t,\omega)$ be the Gabor wavelet transforms of x_1, x_2. Then $\mathrm{Supp} S_1 = \dots = \mathrm{Supp} S_n = \mathbf{R}^2$. Thus $E_j = \emptyset$, $(j = 1, \dots, n)$. But we can detect the number of sources because we can take δ such that $\nu\,(E^c(\delta;M))$ and $\gamma(\delta)$ are small enough. Thus there are some case that we can detect the number of sources even though $\mathrm{Supp} S_j \subset \mathrm{Supp} S_k, j \neq k$.

Remark 2 (On conditions in Theorem 3). (i) To detect the number of sources both δ and $\nu\,(E^c(\delta;M))$ are expected to be small. But if we take δ small enough, then, in general, $\nu(E^c(\delta;M))$ will be large. Thus we can not expect that both δ and $\nu\,(E^c(\delta;M))$ to be small.

(ii) To detected the number of sources, neighboring $q_j, j = 1, \dots, n$ are not so close.

Remark 3. If there are many sources, it will be difficult to detect the number of source. The reason is as follows: For fixed δ and Δ, $\gamma(\delta)$ is a increasing function in n. Thus we need to take δ or Δ so small if we have many sources. But by Remark 2 (ii), it is better that Δ is not small. Further by Remark 2 (i), in general, if δ become small, then $\nu\,(E^c(\delta;M))$ may become large.

Remark 4. We also need to choose η carefully because unnecessary element increase as η becomes large. On the other hand, if η is too small, it is difficult to detect the number of sources except the trivial case. Because $\nu(\{ Q_\eta(z) \neq 0\})$ will also become small.

Conclusion: It seems to be a few cases on which our method can apply.

Question: Is there some "good" transformation that transform original functions to new functions which satisfy the assumptions in Theorem 3?

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INTEGRAL GEOMETRY AND I.M. GELFAND'S QUESTION

M. Graev, Gr. Litvinov

Key words: Integral geometry, hypergroups, FOURIER transform, generalized RADON transform

AMS Mathematics Subject Classification: 42A38, 44A12, 51B25, 20N20

Abstract. The subject of integral geometry (in the sense of [1]) is formed by integral transforms mapping functions on a manifold (or space) X to their integrals over submanifolds in X forming a family $\hat{X}$ of submanifolds in X, so new functions defined on $\hat{X}$ appear. One of basic problems in integral geometry is a reconstruction of the initial function on X starting from its image on $\hat{X}$. This talk is an attempt to give an answer for the following old I. M. Gelfand's question: why some important problems of integral geometry (e.g., the Radon transform and others) are related to harmonic analysis on groups but for other quite similar problems such relations are not clear? In the talk we indicate standard problems of integral geometry generating harmonic analysis (the Plancherel theorem etc.) on pairs of commutative hypergroups in a duality of Pontryagin's type. As a result new meaningful examples of hypergroups are constructed.

1 Hypergroups and generalized Fourier transforms

Suppose that X and $\hat{X}$ have structures of smooth manifolds or diffeological spaces with fixed measures dx and dy respectively.

Let us examine transforms of the form

$$F : f(x) \mapsto \hat{f}(y) = \int f(x)e(x,y)\, dx, \tag{1}$$

where $e(x,y)$ is a generalized function on $X \times \hat{X}$. Suppose that this transform generates an isomorphism $L^2(X,dx) \to L^2(\hat{X},dy)$, such that the following generalized *Plancherel formula* is valid:

$$\int f(x)\overline{g(x)}dx = \int \hat{f}(y)\overline{\hat{g}(y)}dy. \tag{2}$$

In addition suppose that this isomorphism can be extended to δ-functions and set $g(x) = \delta_x(\cdot)$; then from (3) it follows that $\hat{\delta}_x = e(x,y)$ and the following inversion formula is valid:

$$f(x) = \int \hat{f}(y)\overline{e(x,y)}dy. \tag{3}$$

On the other hand the Plancherel formula (3) follows from (1) and (4). We suppose that in the space $L^2(X, dx)$ there is a dense locally convex linear subspace S (or lineal for the sake of brevity), such that S consists of continuous functions (but not all of them) and the lineal $\widehat{S} = F(S)$ possesses the same properties with respect to the space $L^2(\widehat{X}, dy)$. It is assumed that S and $\widehat{S}$ are algebras with respect to the usual multiplication of functions and the isomorphism (1) can be extended to δ-functions belonging to the spaces M and $\widehat{M}$, dual to S and $\widehat{S}$ respectively.

The described construction is a formalization and generalization of heuristic ideas of B. M. Levitan [3].

In the case at hand, when formulas (1) and (4) and hence (3) are valid, we shall say that the transform F is a *generalized Fourier transform* (or GFT for the sake of brevity); functions $e(x, y)$ and $\overline{e(x,y)}$ will be called *generalized exponential functions.*

Proposition 1. Under the specified assumptions, X has a structure of *commutative hypergroup* in the following sense: generalized translation operators act in S and for these operators the associativity axiom of J. Delsarte [2] is valid; generally speaking, it is not supposed that X has a neutral element. These generalized translation operators are defined by the formula

$$R^y f(x) = \int \hat{f}(\chi) \cdot \overline{e(y,\chi)} \cdot \overline{e(x,\chi)}d\chi, \tag{4}$$

where $\chi \in \widehat{X}$ defines the character $\chi(x) = e(x, \chi)$ on X. Similarly $x(\chi) = \overline{e(x,\chi)}$ is a character on $\widehat{X}$.

The hypergroup commutativity means that every two generalized translation operators commute. Then $\widehat{X}$ is a hypergroup dual to X; of course, the hypergroup X is dual to $\widehat{X}$ and $F^{-1}(\delta_y) = \overline{e(x,y)}$.

For example, if $X = \widehat{X} = \mathbb{R}^n$, dx and dy are normalized invariant measures, then F is the usual Fourier transform, $e(x, y)$ is the exponential function $e^{i\langle x,y\rangle}$, and for S it is possible to use the L. Schwartz space of functions which are rapidly decreasing with all the derivatives. In this case the space M consists of all tempered generalized functions (distributions). Below in Section 4 the space S is the space of all smooth functions on a compact hypergroup X; in other sections S is the L. Schwartz space on a hypergroup X.

Usually generalized translation operators act also in the spaces of smooth functions, summable functions, measures and generalized functions with compact supports etc. Different versions of the concept of hypergroup are discussed, e.g., in [4]. Of course, every group (or semigroup) G is an example of a hypergroup; in this case translations R^x are of the form $R^x : f(t) \mapsto f(tx)$, where tx is the product of elements of the group.

There are quite few examples of nontrivial meaningful hypergroups related to harmonic analysis. The most well known one is the Gelfand pair related to harmonic analysis of spherical functions, see, e.g., [4]. One of our aims in this talk is to extend the collection of examples of this type.

2 GFT associated with the generalized Radon transform on $\mathbb{C}^n$

The Radon transform on $\mathbb{C}^n$ maps functions f on $\mathbb{C}^n$ to their integrals over hyperplanes in $\mathbb{C}^n$, i.e. functions $\mathcal{R}f$ on the manifold of hyperplanes in $\mathbb{C}^n$. For an explicit description of this transform we shall define hyperplanes by the equations $x_n = \sum_{i=1}^{n-1} a_i x_i + a_n$ and we shall use $a = (a_1, \ldots, a_n)$ as local coordinates for the manifold of hyperplanes in $\mathbb{C}^n$. Using these notations and the delta-functions notation we can render the Radon transform in the following form:

$$(\mathcal{R}f)(a) = \int\limits_{\mathbb{C}^n} f(x)\delta(x_n - a_1x_1 - \ldots - a_{n-1}x_{n-1} - a_n)d\mu(x),$$

where $\delta(\cdot)$ is the delta-function on $\mathbb{C}$, and $d\mu(x)$ is the Lebesgue measure on $\mathbb{C}^n$. It is known that the following inversion formula is valid: if $\varphi = \mathcal{R}f$, then

$$f(x) = \int\limits_{\mathbb{C}^n} \varphi(a)\delta^{(n-1,n-1)}(x_n - a_1x_1 - \ldots a_{n-1}x_{n-1} - a_n)d\mu(a),$$

where $\delta^{(n-1,n-1)}(t) = \frac{\partial^{2n-2}}{\partial t^{n-1}\partial \bar{t}^{n-1}}\delta(t)$,

We shall say that a *generalized Radon transform* on $\mathbb{C}^n$ associated with an arbitrary generalized function $u(t)$ on $\mathbb{C}$ is the integral transform

$$(\mathcal{R}_u f)(a) = \int\limits_{L^n} f(x)u(x_n - a_1x_1 - \ldots - a_{n-1}x_{n-1} - a_n)d\mu(x).$$

Theorem 1. *If the Fourier transform $\widetilde{u}(c)$ of $u(t)$ is an usual function and this function is nonzero almost everywhere, then the generalized Radon transform $\mathcal{R}_u$ is* *invertible and the inversion formula is of the form* [1]*: if* $\varphi = J_u f$*, then*

$$f(x) = \int\limits_{\mathbb{C}^n} \varphi(a) U(x_n - a_1 x_1 - \dots a_{n-1} x_{n-1} - a_n) d\mu(a), \tag{5}$$

where $U(t) = \int_{\mathbb{C}} [\widetilde{u}(c)]^{-1} |c|^{2(n-1)} e^{i \operatorname{Re}(ct)} d\mu(c)$.

Corollary 1. If $|\widetilde{u}(c)| = |c|^{n-1}$, then the generalized Radon transform J_u is a GFT, i.e. $U(t) = \overline{u(t)}$.

Examples of GFT. Functions $\widetilde{u}(c) = c^k \bar{c}^{n-k-1}$, $k = 0, 1, \dots, n-1$ correspond to local GFT with kernels $u(t) = \delta^{(k,n-k-1)}(t)$. Functions $\widetilde{u}(c) = c^\lambda \bar{c}^\mu$, which differ from functions indicated above, where $\operatorname{Re}(\lambda + \mu) = n - 1$ and $\lambda - \mu$ is an integer number, correspond to GFT with nonlocal kernels $u(t) = t^{-\lambda-1} \bar{t}^{-\mu-1}$.

3 GFT associated with generalized Radon transform on $\mathbb{R}^n$

Definitions of the usual and generalized Radon transform on $\mathbb{R}^n$ are similar to the complex case. In the real case the generalized Radon transform $\mathcal{R}$ is a GFT if the Fourier transform $\widetilde{u}(c)$ of the kernel $u(t)$ satisfies the equation $|\widetilde{u}(c)| = |c|^{\frac{n-1}{2}}$. Hence we have one and only one local GFT for odd n, $n = 2k+1$ only. This kernel corresponds to the function $\widetilde{u}(c) = c^k$, and its kernel has the form $\delta^k(t)$. Examples of GFT with nonlocal kernels are GFT with kernels $u(t) = |t|^{-\frac{n+1}{2} - i\rho}$, $\rho \in \mathbb{R}$.

Remark. Similarly it is possible to construct GFT for generalized Radon transforms on L^n, where L is a continuos non-Archimedean locally compact field.

4 GFT associated with a transform of functions on the sphere $S^n \subset \mathbb{R}^{n+1}$

Let us examine an integral transform mapping *even* functions on the sphere S^n to their integrals on geodesic hypersurfaces (analogs of big circles on S^2). In the spherical coordinates ω these hypersurfaces are defined by the equations $\langle \xi, \omega \rangle = 0$, and the integral transform can be presented in the form

$$f(\omega) \to (Jf)(\xi) = \int\limits_{S^n} f(\omega) \delta(\langle \xi, \omega \rangle) d\omega, \tag{6}$$

[1]Here and in the next integral formulas we suggest that coefficients which usually are placed before the integral symbol are included in the corresponding measure normalization.

where $d\omega$ is an invariant measure on the sphere. There exists an inversion formula representing the initial function f via its image under the transform Jf.

For every $\lambda \in \mathbb{C}$ let us define the following generalized transform for even functions on S^n by the equation

$$(J_\lambda f)(\xi) = \int\limits_{S^n} f(\omega) \frac{|\langle \xi, \omega\rangle|^\lambda}{\Gamma(\frac{\lambda+1}{2})} d\omega. \tag{7}$$

The integral converges for $\operatorname{Re}\lambda > -1$ and it is defined for every $\lambda \in \mathbb{C}$ as an analytic continuation in λ. In particular, $J_{-1} = J$.

Theorem 2. *The following inversion formula holds:*

$$\textit{if} \quad \varphi = J_\lambda f, \quad \textit{then} \quad f = J_{-\lambda-n-1}\varphi. \tag{8}$$

Corollary 2. The transform J_λ is a GFT for $\lambda = -\frac{n+1}{2} + i\rho$, $\rho \in \mathbb{R}$.

In particular, for $n = 4k+1$ and $\lambda = -2k-1$ we have the GFT with the local kernel $\delta^{2k}(\cdot)$.

Similarly define the transform J_u for *odd* functions on the sphere replacing $|\langle \xi, \omega\rangle|^\lambda$ with $|\langle \xi, \omega\rangle|^\lambda \operatorname{sgn}(\langle \xi, \omega\rangle)$ in (7). For this definition, the same inversion formula (8) holds and the transform J_u is a GFT under the same conditions as in the case of even functions. The difference is that local GFT exist in the case of $n = 4k-1$, and in this case the kernel is the odd generalized function $\delta^{2k-1}(\cdot)$. Note that our hypergroup is compact and the dual hypergroupm is compact too.

5 GFT related to the complex of k-dimensional planes in $\mathbb{C}^n$

Define a family of n-dimensional submanifolds (complexes) in the manifold of all k-dimensional planes in the space $\mathbb{C}^n$, $0 < k < n-1$.

Let the space $\mathbb{C}^n$ be represented in the form of direct sum $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^l$, $k + l = n$ of the spaces with coordinates $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_l)$. Define k-dimensional planes in $\mathbb{C}^n$ by the following equations solved with respect to the coordinates $y_i : y_i = u_{i1}x_1 + \ldots + u_{ik}x_k + \alpha_i$, $i = 1, \dots, l$.

Let us fix an arbitrary (l,k)-matrix $u(t) = \|u_{ij}(t)\|$ whose elements are polynomials of $t = (t_1, \dots, t_k) \in \mathbb{C}^k$. For this matrix let us construct the submanifold of k-dimensional planes defined by the equations

$$y_i = u_{i1}(t)x_1 + \ldots + u_{ik}(t)x_k + \alpha_i, \quad i = 1, \dots, l, \tag{9}$$

or $y = u(t)x + \alpha$ for short.

The condition that the submanifold K just defined is a complex, i.e. $\dim K = n$, is equivalent to the condition of nonsingularity for the map $t \mapsto u(t)$ of the space $\mathbb{C}^k$ to the space of (l,k)-matrices. We shall use vectors $\alpha = (\alpha_1, \ldots, \alpha_l)$ and $t = (t_1, \ldots, t_k)$ as coordinates for K.

Complexes K have a simple geometric structure. Specifically, equations (9) for $\alpha = 0$ define a k-dimensional family of $(k-1)$-dimensional planes on a hyperpane in $\mathbb{C}^n$ at infinity. The complex K consists of all k-dimensional planes containing at least one of these $(k-1)$-dimensional planes. In particular, for $k = 1$ the complex K consists of straight lines in $\mathbb{C}^n$ that intersec a fixed curve in a hyperplane at infinity.

The complex K generates an integral transform J mapping functions $f(x,y)$ on $\mathbb{C}^n$ to their integrals over planes of the complex. Using the delta-functions notation, it is convenient to present this transform in the form

$$(Jf)(\alpha,t) = \int_{\mathbb{C}^n} f(x,y)\,\delta(y - u(t)x - \alpha)\,d\mu(x)\,d\mu(y), \tag{10}$$

where $\delta(\cdot)$ is the delta-function on $\mathbb{C}^k$, and $d\mu(x)$, $d\mu(y)$ are Lebesgue measures on $\mathbb{C}^k$ and $\mathbb{C}^l$.

We now examine general transforms J_a which are defined by replacing in definition (10) the delta-functions $\delta(s)$ with arbitrary generalized functions $a(s,t)$ and find inversion formulas for these transforms. By definition we have

$$(J_a f)(\alpha,t) = \int f(x,y)\,a(y - u(t)x - \alpha, t)\,d\mu(x)\,d\mu(y). \tag{11}$$

Note that $J_a f$ is a result of convolution with respect to α of the function $\varphi = Jf$ with the function $a(s,t)$:

$$(J_a f)(\alpha,t) = \int_{\mathbb{C}^l} \varphi(\alpha + s, t)\,a(s,t)\,d\mu(s). \tag{12}$$

There is a simple relation between the Fourier transform $\widetilde{f}(\eta,\xi)$ of the function $f(x,y)$ and the Fourier transform $\widetilde{\varphi}(\xi,t)$ with respect to α of the function $\varphi = J_a f$:

$$\widetilde{\varphi}(\xi,t) = \widetilde{f}(-\xi u(t),\, \xi)\widetilde{a}(-\xi,t), \tag{13}$$

where $\widetilde{a}(\xi,t)$ is the Fourier transform with respect to s of the function $a(s,t)$.

Let us check that *if* $\widetilde{a}(\xi,t)$ *is an usual function which is nonzero almost everywhere, then the function* f *can be reconstructed from the function* $\varphi = J_a f$ in a unique way.

Note that by virtue of (13) this condition implies that the function $F(\eta,\xi) = \widetilde{f}(-\xi u(t),\xi)$ can be reconstructed from the function φ uniquely. On the other hand from the nonsingularity of K and the analyticity condition it follows that for almost every pair (η,ξ) there exists $t\in\mathbb{C}^k$ such that $-\xi u(t)=\eta$. It then follows that the function $\widetilde{f}$, and hence the function f can be reconstructed from the function F. Let us describe an inversion formula explicitly.

Definition 1. We shall say that *the Crofton function* related to the complex K, is the function $\mathrm{Cr}_K(\eta,\xi)$ on $\mathbb{C}^k\oplus\mathbb{C}^l$, equal to the number of solutions t of the equation $\eta=-\xi u(t)$. If the Crofton function is constant almost everywhere, then this constant is called *the Crofton number.* Denote this number by Cr_K.

From the condition on the complex K it follows that its Crofton function is nonzero and constant almost everywhere.

Theorem 3. *If* $\widetilde{a}(\xi,t)$ *is a usual function which is nonzero almost everywhere, then there is the following inversion formula for the integral transform* $f\mapsto\varphi = J_a f$:

$$f(x,y)=\int\varphi(\alpha,t)\,A(y-u(t)x-\alpha,t)\,d\mu(\alpha)\,d\mu(t), \tag{14}$$

where

$$A(s,t)=\frac{1}{\mathrm{Cr}_K}\int[\widetilde{a}(\xi,t)]^{-1}\left|\frac{\partial(\xi u(t))}{\partial t}\right|^2 e^{i\,\mathrm{Re}\,\langle s,\xi\rangle}d\mu(\xi). \tag{15}$$

Corollary 3. Under the conditions of the theorem 3 the integral transform J_a is an GFT iff the Fourier transform $\widetilde{a}(\xi,t)$ with respect to s of the kernel $a(s,t)$ satisfies the equation

$$|\widetilde{a}(\xi,t)|=\mathrm{Cr}_K^{-1/2}\,|\omega(\xi,t)|,\quad \text{where}\quad \omega(\xi,t)=\frac{\partial(\xi u(t))}{\partial t}. \tag{16}$$

Examples of GFT are the integral transforms J_a with the kernels $a(s,t)$, for which the Fourier transforms with respect to s are functions $\widetilde{a}(\xi,t)$ of the form

$$\widetilde{a}(\xi,t)=\mathrm{Cr}_K^{-1/2}\,\omega(\xi,t)\prod_{p=1}^{l}\xi_p^{\lambda_p}\overline{\xi}_p^{-\lambda_p},\quad \lambda_p\in\mathbb{C}. \tag{17}$$

Let us describe the kernels $a(s,t)$ explicitly. Since $\omega(\xi,t)$ is a homogeneous polynom of $\xi_1,\dots,\xi_l$ of the degree k, then the function $\widetilde{a}(\xi,t)$ can be presented in the form

$$\widetilde{a}(\xi,t)=\sum_{m_1+\ldots+m_l=k}\left[u_{m_1,\ldots,m_l}(t)\prod_{p=1}^{l}\left(\xi_p^{m_p+\lambda_p}\overline{\xi}_p^{-\lambda_p}\right)\right].$$

Therefore

$$a(s,t)=\sum_{m_1+\ldots+m_l=k}\left[u_{m_1,\ldots,m_l}(t)\prod_{p=1}^{l}\mathcal{F}\left(\xi_p^{m_p+\lambda_p}\overline{\xi}_p^{-\lambda_p}\right)\right],$$

where $\mathcal{F}$ is the inverse Fourier transform. In particular, (see [1]), if $\lambda_p\neq 0$, $p=1,\ldots,l$, then $\mathcal{F}(\xi_p^{m_p+\lambda_p}\overline{\xi}_p^{-\lambda_p})$ is, up to a factor, the fuction $s_p^{-m_p-\lambda_p-1}\overline{s}_p^{\lambda_p-1}$.

In the special case $\lambda_1=\ldots=\lambda_l=0$ the GFT is local and its kernel $a(s,t)$ has the form:

$$a(s,t)=\mathrm{Cr}_K^{-1/2}\,\omega\left(\frac{\partial}{\partial s};t\right)\delta(s).$$

6 GFT related to the complex of k-dimensional planes in $\mathbb{R}^n$

There are similar results in the spirit of Section 5.

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A NOTE ON THE CONSTITUTIVE EQUATION IN A LINEAR FRACTIONAL VISCOELASTIC BODY MODEL

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Abstract. We give a review of recent results which propose a new model for a linear viscoelastic body. It is described by a stress-strain constitutive equation involving the Riemann-Liouville fractional derivatives. Such an equation generalizes several known models. We derive thermodynamical restrictions on coefficients and orders of fractional derivatives using the entropy inequality for isothermal processes.

1 Introduction

Fractional calculus is a powerful tool for modeling various problems arising in different branches of science such as mechanics, physics, engineering, economics, finance, medicine, biology, chemistry, etc. It allows differential and integral operators to be of arbitrary real (or even complex) order. Although the pioneering work in the field of derivatives and integrals of real order was made at the end of 17th century, and the solid foundation of the theory was given in the first half of the 19th century, the significant expansion of fractional calculus has started about four decades ago. Since then, many authors have contributed to this field, which has resulted in several extensive monographs (e.g. [5]) and a great number of scientific papers, covering different aspects of the theory of fractional calculus and its applications.

Theory of viscoelasticity was among the first areas where fractional calculus has been applied successfully. Indeed, it has turned out that real order derivatives and integrals are more appropriate for characterizing viscoelastic material properties. The method which is most often used for introducing such operators in the mathematical models describing different physical phenomena (e.g. waves) in a viscoelastic body is the so-called 'direct fractionalization'. More precisely, in the constitutive equation, which reflects specific properties of viscoelastic materials, the integer order derivatives are replaced by real-order fractional ones, while other equations in the model (e.g. equilibrium equation or the strain measure) remain unchanged. This approach is completely physically based, and therefore acceptable for further study. However, one has to be careful with determining restrictions on parameters appearing in that new fractional constitutive equation, in order to preserve the well-known physical laws (such as e.g. The Second Law of Thermodynamics).

In this note we are concerned with the generalized linear fractional constitutive equation of the form

$$\sum_{n=0}^{N} a_n \, {}_0D_t^{\alpha_n} \sigma = \sum_{m=0}^{M} b_m \, {}_0D_t^{\beta_m} \varepsilon, \qquad t > 0. \tag{1}$$

It has been obtained from a model of linear viscoelastic body with arbitrary finite number of springs and dashpots (cf. [4]), in which integer order derivatives are replaced by the Riemann-Liouville fractional real order ones. Our purpose is to examine validity of Eq. (1) by determining restrictions on both coefficients and orders of fractional derivatives appearing in the equation.

To the end of Introduction we briefly recall notions and facts which will be used in the sequel. In Section 2 we present a detailed analysis of the generalized linear fractional constitutive Eq. (1), extract necessary and sufficient conditions for its well-posedness, and discuss several special cases which appear in applications more frequently. Further details, as well as an application of these results, can be found in [2].

Let $\eta \in [0,1]$. The left η-th order Riemann-Liouville fractional derivative ${}_0D_t^\eta y$ of a function $y \in AC([0,T])$, $T > 0$, is defined as

$${}_0D_t^\eta y(t) := \frac{1}{\Gamma(1-\eta)} \frac{d}{dt} \int_0^t \frac{y(\tau)}{(t-\tau)^\eta} \, d\tau,$$

where Γ is the Euler gamma function and $AC([0,T])$, $T > 0$, denotes the space of absolutely continuous functions on $[0,T]$.

Also, recall that the Fourier transform of function f is defined as

$$\mathcal{F}[f(t)](\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt, \quad \omega \in \mathbb{R}.$$

It holds $\mathcal{F}[{}_0D_t^\eta f(t)](\omega) = (i\omega)^\eta \hat{f}(\omega)$.

We refer to [5] for a detailed account of fractional calculus.

2 Thermodynamical restrictions for the linear fractional constitutive equation

The goal of this section is to provide well-posedness of the generalized linear fractional constitutive Eq. (1), i.e.,

$$\sum_{n=0}^{N} a_n\, {}_0D_t^{\alpha_n}\sigma = \sum_{m=0}^{M} b_m\, {}_0D_t^{\beta_m}\varepsilon, \quad t>0,$$

by determining restrictions on parameters $\{\alpha_n\}_{n=0,\dots,N}$, $\{\beta_m\}_{m=0,\dots,M}$, $\{a_n\}_{n=0,\dots,N}$ and $\{b_m\}_{m=0,\dots,M}$, in such a way that the generalized linear fractional model of a viscoelastic body satisfies the requirements of the Second Law of Thermodynamics.

In Eq. (1) we used σ to denote the stress, ε for the strain, and ${}_0D_t^{\eta}$, $\eta\in[0,1]$, denotes the operator of the left Riemann-Liouville fractional derivation. The coefficients $\{a_n\}_{n=0,\dots,N}$ and $\{b_m\}_{m=0,\dots,M}$ are given real numbers, having the physical meaning of relaxation times. For technical purposes, the orders of the fractional derivatives in Eq. (1) are assumed to satisfy

$$0\leqslant\alpha_0<\alpha_1<\ldots<\alpha_N\leqslant 1, \quad 0\leqslant\beta_0<\beta_1<\ldots<\beta_M\leqslant 1. \tag{2}$$

Notice here that the general linear fractional constitutive Eq. (1) includes in itself many known constitutive equations of one-dimensional viscoelasticity as special cases:

1. The classical Zener model is obtained for $\alpha_0=\beta_0=0$, $\alpha_1=\beta_1=1$, $a_0=b_0=1$, $a_1=a$, $b_1=b$, and all other coefficients vanishing (cf. [3]).
2. The generalized Zener model is the result of the choice $a_0=b_0=1$, $\alpha_0=\beta_0=0$, $a_1=a$, $b_1=b$, $\alpha_1=\beta_1=\alpha$ ($\alpha\in[0,1]$), and the rest of coefficients being zero (cf. [1]).

In both cases the constants a and b have to satisfy the inequality $a\leqslant b$, which comes as a consequence of the Clausius-Duhem inequality. Therefore, a natural question arises: What is a generalization of such inequality for the model (1), or more precisely, which restrictions on the coefficients, as well as on the orders of derivatives in generalized linear fractional constitutive Eq. (1) should be assumed in order to obtain a model obeying The Second Law of Thermodynamics. This kind of problem has already been considered in [3, 5], resulting in properly formulated thermodynamical constraints (based on a nonnegative rate of energy dissipation and nonnegative internal work), which lead to a well-behaved mathematical description of the viscoelastic phenomenon. Hence, we shall follow the approach proposed in [3] and use the entropy inequality for isothermal processes, in order to determine thermodynamical restrictions on parameters in Eq. (1).

The procedure of finding restrictions on parameters in Eq. (1) consists of several steps, which can be briefly described as follows. First, one has to apply the Fourier transform on Eq. (1). From the transformed equation the complex modulus should be defined. Requesting the real and imaginary part of the complex modulus to be nonnegative one can then extract desired restrictions on parameters $\{\alpha_n\}_{n=0,\ldots,N}$, $\{\beta_m\}_{m=0,\ldots,M}$, $\{a_n\}_{n=0,\ldots,N}$ and $\{b_m\}_{m=0,\ldots,M}$.

We therefore first apply the Fourier transform to Eq. (1) and obtain

$$\hat{\sigma}(\omega)\sum_{n=0}^{N} a_n (i\omega)^{\alpha_n} = \hat{\varepsilon}(\omega)\sum_{m=0}^{M} b_m (i\omega)^{\beta_m}, \quad \omega \in \mathbb{R}. \tag{3}$$

Set

$$\hat{E}(\omega) := \frac{\sum\limits_{m=0}^{M} b_m (i\omega)^{\beta_m}}{\sum\limits_{n=0}^{N} a_n (i\omega)^{\alpha_n}}, \quad \omega \in \mathbb{R},$$

to be the complex modulus. Then Eq. (3) takes the form $\hat{\sigma}(\omega) = \hat{E}(\omega)\hat{\varepsilon}(\omega),\ \omega \in \mathbb{R}$. According to (cf. [3,5]), The Second Law of Thermodynamics in case of the isothermal process implies that

$$\operatorname{Re}\hat{E}(\omega) \geqslant 0, \qquad \forall\,\omega > 0, \tag{4}$$

$$\operatorname{Im}\hat{E}(\omega) \geqslant 0, \qquad \forall\,\omega > 0. \tag{5}$$

Closer analysis of conditions (4) and (5) should lead to determination of restrictions on parameters α_n, β_m, a_n and b_m in Eq. (1).

After a straightforward calculation one obtains:

$$\hat{E}(\omega) = \frac{\sum\limits_{m=0}^{M} b_m \omega^{\beta_m} \cos\frac{\beta_m \pi}{2} + i\sum\limits_{m=0}^{M} b_m \omega^{\beta_m} \sin\frac{\beta_m \pi}{2}}{\sum\limits_{n=0}^{N} a_n \omega^{\alpha_n} \cos\frac{\alpha_n \pi}{2} + i\sum\limits_{n=0}^{N} a_n \omega^{\alpha_n} \sin\frac{\alpha_n \pi}{2}}.$$

Set

$$\hat{E}'(\omega) := \hat{E}(\omega)\left|\sum_{n=0}^{N} a_n\,(i\omega)^{\alpha_n}\right|^2.$$

We stress here that the sign of real and imaginary parts of the complex modulus $\hat{E}(\omega)$ and the modified complex modulus $\hat{E}'(\omega)$ coincides, thus we shall examine the latter. We have

$$\operatorname{Re}\hat{E}'(\omega) = \left(\sum_{n=0}^{N} a_n\omega^{\alpha_n}\cos\frac{\alpha_n\pi}{2}\right)\left(\sum_{m=0}^{M} b_m\omega^{\beta_m}\cos\frac{\beta_m\pi}{2}\right) + \\ + \left(\sum_{n=0}^{N} a_n\omega^{\alpha_n}\sin\frac{\alpha_n\pi}{2}\right)\left(\sum_{m=0}^{M} b_m\omega^{\beta_m}\sin\frac{\beta_m\pi}{2}\right), \quad (6)$$

$$\operatorname{Im}\hat{E}'(\omega) = \left(\sum_{n=0}^{N} a_n\omega^{\alpha_n}\cos\frac{\alpha_n\pi}{2}\right)\left(\sum_{m=0}^{M} b_m\omega^{\beta_m}\sin\frac{\beta_m\pi}{2}\right) - \\ - \left(\sum_{n=0}^{N} a_n\omega^{\alpha_n}\sin\frac{\alpha_n\pi}{2}\right)\left(\sum_{m=0}^{M} b_m\omega^{\beta_m}\cos\frac{\beta_m\pi}{2}\right). \quad (7)$$

The real part of $\hat{E}'(\omega)$ is a sum whose terms contain sine and cosine of angles $\frac{\alpha_n\pi}{2}, \frac{\beta_m\pi}{2}$. By assumption $\alpha_n, \beta_m \in [0,1]$, $n = 0,1,\ldots,N$, $m = 0,1,\ldots,M$, thus $\frac{\alpha_n\pi}{2}, \frac{\beta_m\pi}{2} \in [0, \frac{\pi}{2}]$, and consequently, sine and cosine of those angles are positive. We may then conclude that conditions $a_n, b_m \geqslant 0$ imply $\operatorname{Re}\hat{E}'(\omega) \geqslant 0$, as well as (4). Therefore, we shall restrict our attention to the case $a_n, b_m \geqslant 0$.

Further calculation of (7) yields that (5) holds if and only if

$$\operatorname{Im}\hat{E}'(\omega) = -\sum_{\substack{n\in\{0,1,\ldots,N\}\\ m\in\{0,1,\ldots,M\}}} \omega^{\alpha_n+\beta_m}\sin\frac{(\alpha_n-\beta_m)\pi}{2}a_n b_m \geqslant 0, \qquad \forall\,\omega > 0. \quad (8)$$

The following claim gives a necessary condition for inequality (8).

Theorem 1. *Let*

1. $0 \leqslant \alpha_0 < \alpha_1 < \ldots < \alpha_N \leqslant 1$ *and* $0 \leqslant \beta_0 < \beta_1 < \ldots < \beta_M \leqslant 1$ *(i.e.,* (2) *holds);*
2. $a_n, b_m \geqslant 0$, $n = 0,1,\ldots,N$, $m = 0,1,\ldots,M$*;*
3. $\alpha_N \neq \beta_M$.

Then a necessary condition for (8) is that $\alpha_N < \beta_M$.

In other words, the highest order of fractional derivatives of stress in (1) *could not be greater than the highest order of fractional derivatives of strain.*

Proof. We observe that for large ω the sign of $\operatorname{Im}\hat{E}'(\omega)$ coincides with the sign of the term in the sum on the right-hand side of (8) with the largest power of ω. It follows from (2) that the latter is achieved for α_N and β_M. Therefore, in order that $\operatorname{Im}\hat{E}'(\omega) > 0$, α_N has to be less than β_M, as claimed. □

Remark 1. (i) In the same way it can be proved that in the case $\alpha_N = \beta_M$ a necessary condition from Th. 1 transforms to $\alpha_n < \beta_m$, for the largest α_n and β_m which do not coincide. The case when the orders of fractional derivatives of the stress and strain are the same is discussed below.

(ii) Th. 1 can also be used as a test for accepting or rejecting an equation which is a candidate for the constitutive equation in a mathematical model of viscoelastic body. For instance, equation $a_0\sigma + a_\alpha\, {}_0D_t^\alpha \sigma = b_0\varepsilon$, $0 < \alpha < 1$, can not be accepted as a constitutive equation, since it does not obey The Second Law of Thermodynamics. The latter can be seen from Eq. (8), which in this case reads $-\omega^\alpha \sin\frac{\alpha\pi}{2} a_\alpha b_0$, and which is strictly less than zero for all $\omega > 0$.

So far we have established a necessary criterion for well-posedness of the generalized linear fractional constitutive Eq. (1). On the other hand, the question of finding sufficient conditions from Eq. (4) and Eq. (5) is more delicate, and in turn, it is quite difficult to give any answer in general case. However, the solution to this problem can be sought in examination of certain special cases, which occur more often in applications. Hence, in the sequel we list four interesting classes of constitutive equations:

1. $\alpha_n \neq \beta_m$, for all n, m, i.e., there are $N+1$ and $M+1$ terms of different order in Eq. (1).
   Then the following two conditions
   (a) $a_n, b_m \geqslant 0$, $n = 0, 1, \ldots, N$, $m = 0, 1, \ldots, M$
   (b) $0 \leqslant \alpha_0 < \alpha_1 < \ldots < \alpha_N < \beta_0 < \beta_1 < \ldots < \beta_M \leqslant 1$
   guarantee validity of (5).
   We remark here that in [2] we study a linear viscoelastic body model with the constitutive equation which belongs to this class.
2. $M > N$ and $\alpha_i = \beta_i$, $i = 0, 1, \ldots, N$, i.e., there are $N+1$ first terms of the same order and $M - N$ terms left in Eq. (1).
   In this case the following set of conditions
   (a) $a_n, b_m \geqslant 0$, $n = 0, 1, \ldots, N$, $m = 0, 1, \ldots, M$
   (b) $\dfrac{a_0}{b_0} \geqslant \dfrac{a_1}{b_1} \geqslant \ldots \geqslant \dfrac{a_N}{b_N} \geqslant 0$
   (c) $0 \leqslant \alpha_0 < \alpha_1 < \ldots < \alpha_N < \beta_{N+1} < \ldots < \beta_M \leqslant 1$
   provides that (5) holds.

3. $N > M$ and $\alpha_{N-M+i} = \beta_i$, $i = 0, 1, \ldots, M$, i.e., there are $M + 1$ last terms of the same order and $N - M$ terms left in Eq. (1).
   Here, if one supposes
   (a) $a_n, b_m \geqslant 0$, $n = 0, 1, \ldots, N$, $m = 0, 1, \ldots, M$
   (b) $\dfrac{a_{N-M}}{b_0} \geqslant \dfrac{a_{N-M+1}}{b_1} \geqslant \ldots \geqslant \dfrac{a_N}{b_M} \geqslant 0$
   then (5) is satisfied.
4. $N = M$ and $\alpha_i = \beta_i$, $i = 0, 1, \ldots, N$, i.e., there are $N + 1$ terms of the same order on both sides of Eq. (1).
   In this case, (5) holds if
   (a) $a_n, b_m \geqslant 0$, $n = 0, 1, \ldots, N$, $m = 0, 1, \ldots, M$
   (b) $\dfrac{a_0}{b_0} \geqslant \dfrac{a_1}{b_1} \geqslant \ldots \geqslant \dfrac{a_N}{b_N} \geqslant 0.$

Remark 2. In all cases studied above we determined conditions which imply that all terms in the sum in (5) are positive. If one wants to consider other instances which are not listed above (e.g. when there are $K < \min\{N, M\}$ terms of the same order in Eq. (1)) it is not possible to make all terms in (8) nonnegative simultaneously, and thus it is more difficult to find some particular conditions on parameters a_n, b_m, α_n and β_m which implies (5).

Remark 3. We also stress here that there is a strong connection between thermodynamical restrictions on coefficients in Eq. (1) and conditions for the existence of the inverse Laplace transform of equation of motion (in a model describing waves in viscoelastic media). It has been proved that the thermodynamical restrictions guarantee the existence of solutions (cf. [6, 7]).

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INVERSION FORMULAS AND NUMERICAL EXPERIMENTS IN THE WAVE EQUATION BY REPRODUCING KERNELS AND TIKHONOV REGULARIZATION

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Key words: Wave equation, inverse problem, reproducing kernel, Tikhonov regularization

AMS Mathematics Subject Classification: 35L05, 44A35, 47A52, 31A30

Abstract. In this paper we shall give practical inversion formulas in the one dimensional wave equation using reproducing kernels and Tikhonov regularization. And we show their numerical experiments by using computers.

1 Introduction and main results

We shall consider the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad (c : constant > 0). \tag{1}$$

The solutions $u(x,t)$ satisfying the condition

$$u_t(x,t)\big|_{t=0} = F(x), \quad u(x,0) = 0 \quad \text{on} \quad \mathbf{R}, \tag{2}$$

is represented for some function F as follows:

$$u_F(x,t) = \frac{1}{2c} \int_{\mathbf{R}} F(\xi)\theta(ct - |x-\xi|)d\xi. \tag{3}$$

Here, $\theta(x)$ is a step function.

We shall use the first order Sobolev Hilbert space H_S comprising absolutely continuous functions F on $\mathbf{R}$ with the norm

$$\|F\|_{H_S}^2 = \int_{-\infty}^{\infty} (F(x)^2 + F'(x)^2)dx.$$

This Hilbert space admits the reproducing kernel

$$K(x,y) = \frac{1}{2}e^{-|x-y|} = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \frac{1}{1+\xi^2} e^{i(x-y)\xi} d\xi. \tag{4}$$

Similarly we shall use the first order Sobolev Hilbert space $H_{S,e}$ comprising absolutely continuous functions F on $\mathbf{R}$ which are even functions with respect to the origin with the norm

$$\|F\|^2_{H_{S,e}} = \frac{2}{\pi}\int\limits_0^{\infty} (F(x)^2 + F'(x)^2)dx.$$

This Hilbert space admits the reproducing kernel

$$K_e(x,y) = \frac{\pi}{4}\left(e^{-|x-y|} + e^{-x}e^{-y}\right) = \int\limits_0^{\infty} \frac{1}{1+\xi^2}\cos x\xi \cos y\xi d\xi. \tag{5}$$

For these formulas we see easily by using Fourier's transform (cf. [2], page 58).

In the typical situation (2) and (3) we shall give the results:

Theorem 1. *For any function $g \in L_2(\mathbf{R})$, for any $\lambda > 0$ and for any fixed $t > 0$, the best approximate function $F^*_{g,\lambda,t}$ in the sense*

$$\inf_{F\in H_S} \left\{\lambda\|F\|^2_{H_S} + \|g - \partial_x u_F(x,t)\|^2_{L_2(\mathbf{R})}\right\} = \\ = \lambda\|F^*_{g,\lambda,t}\|^2_{H_S} + \|g - \partial_x u_{F^*_{g,\lambda,t}}(x,t)\|^2_{L_2(\mathbf{R})} \tag{6}$$

exists uniquely and $F^*_{g,\lambda,t}$ is represented by

$$F^*_{g,\lambda,t}(x) = \int\limits_{\mathbf{R}} g(\xi)P_{\lambda,t}(\xi - x)d\xi \tag{7}$$

for

$$P_{\lambda,t}(\xi - x) = \frac{-ci}{2\pi}\int\limits_{\mathbf{R}} \frac{\sin(ct\eta)e^{-i\eta(\xi-x)}d\eta}{\lambda c^2(1+\eta^2) + \sin^2(ct\eta)}.$$

If, for $F \in H_S$ we consider the $u_F(x,t)$ and we take $\partial_\xi u_F(\xi,t)$ as $g(\xi)$, then we have the favourable result:

$$as \quad \lambda \to 0, \quad F^*_{g,\lambda,t} \to F, \tag{8}$$

uniformly.

Theorem 2. *For any function $g \in L_2(\mathbf{R}^+)$, for any $\lambda > 0$ and for fixed $x = 0$, the best approximate function $F^*_{g,\lambda,0}$ in the sense*

$$\inf_{F \in H_{S,e}} \left\{ \lambda \|F\|^2_{H_{S,e}} + \|g - \partial_t u_F(0,t)\|^2_{L_2(\mathbf{R}^+)} \right\} = \\ = \lambda \|F^*_{g,\lambda,0}\|^2_{H_{S,e}} + \|g - \partial_t u_{F^*_{g,\lambda,0}}(0,t)\|^2_{L_2(\mathbf{R}^+)} \tag{9}$$

exists uniquely and $F^*_{g,\lambda,0}$ is represented by

$$F^*_{g,\lambda,0}(\xi) = \int_0^\infty g(t) Q_{\lambda,0}(t,\xi)\, dt \tag{10}$$

for

$$Q_{\lambda,0}(t,\xi) = \int_0^\infty \frac{\cos(ct\eta)\cos(\xi\eta)\, d\eta}{\lambda(1+\eta^2) + \pi/(2c)}.$$

If, for $F \in H_{S,e}$ we consider the output $u_F(x,t)$ and we take $\partial_t u_F(0,t)$ as $g(t)$, then we have the favourable result:

$$as \quad \lambda \to 0, \quad F^*_{g,\lambda,0} \to F, \tag{11}$$

uniformly.

The motivations and results in Theorems 1 and 2 are clear; that is, we are establishing the inversion formulas that:

(a) from the observation $\partial_x u_F(x,t)$ for any fixed $t > 0$, we determine the initial velocity F, and

(b) from the observation $\partial_t u_F(0,t)$ for fixed $x = 0$, we determine the initial velocity F, indeed, we can only determine the even part of F with respect to the origin ([2], Pp. 146-157), respectively.

We shall show that our inversion formulas (7) and (10) are practical by showing numerical experiments. In particular, Theorem 2 shows a practical formula for some general "principle of telethoscope" in [4].

2 Background Theorems

We shall use the following two general theorems.

Theorem 3. *Let H_K be a Hilbert space admitting the reproducing kernel $K(p,q)$ on a set E. Let $L : H_K \to \mathcal{H}$ be a bounded linear operator on H_K into $\mathcal{H}$. For $\lambda > 0$ introduce the inner product in H_K and call it H_{K_λ} as*

$$\langle f_1, f_2 \rangle_{H_{K_\lambda}} = \lambda \langle f_1, f_2 \rangle_{H_K} + \langle Lf_1, Lf_2 \rangle_{\mathcal{H}} \tag{12}$$

then H_{K_λ} is the Hilbert space with the reproducing kernel $K_\lambda(p,q)$ on E satisfying the equation

$$K(\cdot, q) = (\lambda I + L^* L) K_\lambda(\cdot, q), \tag{13}$$

where L^* is the adjoint of $L : H_K \to \mathcal{H}$.

Theorem 4. *Let H_K, L, $\mathcal{H}$, E and K_λ be as in Theorem 3. Then, for any $\lambda > 0$ and for any $g \in \mathcal{H}$, the extremal function in*

$$\inf_{f \in H_K} \left(\lambda \|f\|^2_{H_K} + \|Lf - g\|^2_{\mathcal{H}} \right) \tag{14}$$

exists uniquely and the extremal function

$$f^*_{\lambda,g}(p) = \langle g, LK_\lambda(., p) \rangle_{\mathcal{H}} \tag{15}$$

is the member of H_K which attains the infimum in (14).

3 Proof of Theorem 1

As we see from the conservative law of energy, for any $t > 0$

$$\int\limits_{\mathbf{R}} F(x)^2 dx = \frac{1}{2} \int\limits_{\mathbf{R}} ((\partial_t u_F(x,t))^2 + c^2 (\partial_x u_F(x,t))^2) dx,$$

the mapping L_t in (3) $L_t : F \in H_S \longrightarrow \partial_x u_F(x,t)$ is a bounded linear operator from H_S into $L_2(\mathbf{R})$ for any fixed $t > 0$. Then we can see directly that

$$K_\lambda(x, y; L_t) = \frac{c^2}{2\pi} \int\limits_{\mathbf{R}} \frac{e^{i\eta(x-y)}}{\lambda c^2 (1 + \eta^2) + \sin^2(ct\eta)} d\eta \tag{16}$$

satisfies the functional equation (13) in our situation; that is, it is the reproducing kernel for the Hilbert space with the norm square $\lambda\|F\|_{H_S}^2 + \|\partial_x u_F(x,t)\|_{L_2(\mathbf{R})}^2$. In particular, we thus obtain (7) from Theorems 3 and 4.

In order to prove the result (8), as we see from (4) note that any member $F \in H_S$ is represented uniquely by a function $\mathbf{F}$ in the form

$$F(x) = \frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{ix\eta}}{1+\eta^2}\mathbf{F}(\eta)d\eta \tag{17}$$

satisfying

$$\frac{1}{2\pi}\int_{\mathbf{R}} \frac{1}{1+\eta^2}|\mathbf{F}(\eta)|^2 d\eta < \infty$$

and

$$\|F\|_{H_S}^2 = \frac{1}{2\pi}\int_{\mathbf{R}} \frac{1}{1+\eta^2}|\mathbf{F}(\eta)|^2 d\eta. \tag{18}$$

Then, we insert this F in (3) and we have $u_F(x,t)$. Then, we set $\partial_\xi u_F(\xi,t) = g(\xi)$ in (7) and we obtain, directly

$$F^*_{\partial_\xi u_F(\xi,t),\lambda,t}(x) = \frac{1}{2\pi}\int_{\mathbf{R}} \frac{\sin^2(ct\eta)e^{ix\eta}}{(\lambda c^2(1+\eta^2)+\sin^2(ct\eta))(1+\eta^2)}\mathbf{F}(\eta)d\eta. \tag{19}$$

From (17) and (19) we thus obtain the desired desired result (8).

4 Proof of Theorem 2

In (3), note that for any fixed x

$$\int_0^\infty (\partial_t u_F(x,t))^2 dt = \min \frac{1}{2c}\int_{\mathbf{R}} F(\xi)^2 d\xi,$$

where the minimum is taken over the functions F satisfying (3). Moreover, the minimum is attained by F^* if and only if F^* is the even parts of any F satisfying (3)([2], page 114.). Hence, the mapping L_x

$$L_x : F \in H_S \longrightarrow \partial_t u_F(x,t)$$

is a bounded linear operator from H_S into $L_2(\mathbf{R}^+)$ for any fixed x. Then we can see directly that

$$K_\lambda(x', x''; L_0) = \int_{\mathbf{R}^+} \frac{\cos(x'\eta)\cos(x''\eta)}{\lambda(1+\eta^2)+\pi/(2c)} d\eta \tag{20}$$

satisfies the functional equation (13) in our situation; that is, it is the reproducing kernel for the Hilbert space with the norm square

$$\lambda\|F\|^2_{H_{S,e}} + \|\partial_t u_F(0,t)\|^2_{L_2(\mathbf{R}^+)}.$$

In particular, we thus obtain (10) from Theorems 3 and 4.

In order to prove the result (11), as we see from (5) note that any member $F \in H_{S,e}$ is represented uniquely by a function $\mathbf{F}$ in the form

$$F(x) = \int_{\mathbf{R}^+} \frac{\cos(x\eta)}{1+\eta^2}\mathbf{F}(\eta) d\eta$$

satisfying

$$\int_{\mathbf{R}^+} \frac{1}{1+\eta^2}|\mathbf{F}(\eta)|^2 d\eta < \infty$$

and

$$\|F\|^2_{H_{S,e}} = \int_{\mathbf{R}^+} \frac{1}{1+\eta^2}|\mathbf{F}(\eta)|^2 d\eta.$$

We insert this F in (3) and we have $u_F(x,t)$. Then, we set $\partial_t u_F(0,t) = g(t)$ in (10) and we obtain, directly

$$F^*_{\partial_t u_F(0,t),\lambda,0}(\xi) = \frac{\pi}{2c}\int_{\mathbf{R}^+} \frac{\cos(\xi\eta)}{(\lambda(1+\eta^2)+\pi/(2c))(1+\eta^2)}\mathbf{F}(\eta) d\eta. \tag{21}$$

From (21) we thus obtain the desired result (11).

5 Numerical Experiments with Figures

In order to demonstrate effectivity and validity of our theorems we shall present some numerical experiments. From these results we can declare that our theory works well on inverse problem of the wave equation.

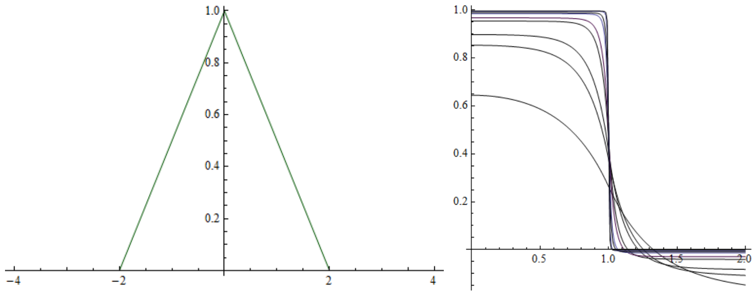

Figure 1. For $F(x) = \chi_{[-1,1]}(x)$ on $\mathbf{R}$, the figures of $g(x)$ on $t = 1$ and $F_{g,\lambda,0}(x)$ for $c = 1, \lambda = 10^{-1}, 5 \cdot 10^{-2}, 10^{-2}, 5 \cdot 10^{-3}, 10^{-3}, 5 \cdot 10^{-4}, 10^{-4}, 5 \cdot 10^{-5}, 10^{-5}$

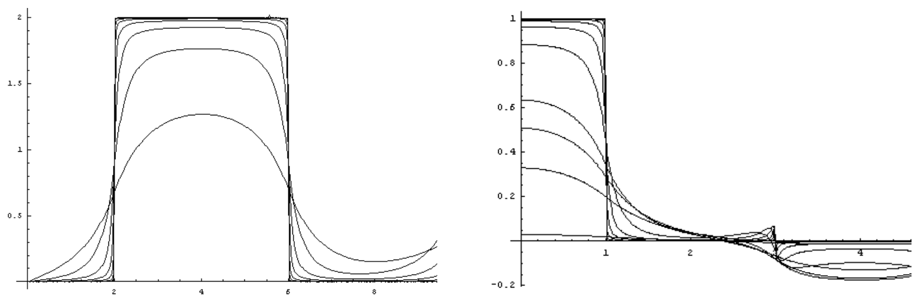

Figure 2. For $g(x) = \chi_{[-1,1]}(x)$ on $\mathbf{R}$, the figures of $F^*_{g,\lambda,t}(x)$ and $\partial_x u_{F^*_{g,\lambda,t}}(x,t)$ for $t = 1, c = 2, \lambda = 100, 10, 1, 0.5, 0.2, 0.1, 0.01, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}$

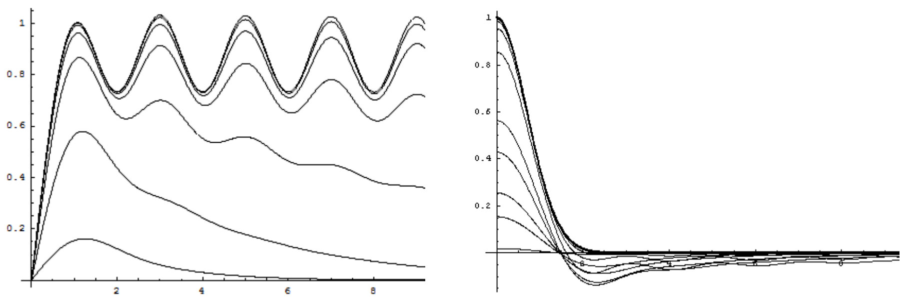

Figure 3. For $g(x) = e^{-x^2}$ on $\mathbf{R}$, the figures of $F^*_{g,\lambda,t}(x)$ and $\partial_x u_{F^*_{g,\lambda,t}}(x,t)$ for $t = 1, c = 1, \lambda = 1, 0.1, 0.01, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$

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FRACTIONAL DIFFUSION EQUATION: NOTE ON CATTANEO TYPE HEAT CONDUCTION EQUATION

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Key words: Caputo fractional derivative, Cattaneo heat conduction (diffusion) model, Laplace and Fourier transformation, generalized functions

AMS Mathematics Subject Classification: 26A33, 35Q79, 44A10, 46F12, 35D30, 35G16

Abstract. Recently, in [1], the classical heat conduction (diffusion) equation is generalized using a generalized heat conduction law. In particular, the space-time Cattaneo heat conduction law (which contains the Caputo symmetrized fractional derivative instead of gradient ∂_x and fractional time derivative instead of the first order partial time derivative ∂_t) is used. In this note we review motivation, main ideas and results given in [1].

1 Introduction

In this note we consider *the Cattaneo type space-time fractional heat conduction equation* which is of the form

$$(I + {}_0^C D_t^\alpha)\partial_t T = \lambda \partial_x \, \mathcal{E}_x^\beta T,$$

equivalent to

$$\partial_t T = \lambda \mathcal{L}^{-1} \left(\frac{1}{1 + \tau s^\alpha} \right) *_t \partial_x \, \mathcal{E}_x^\beta T. \tag{1}$$

Here, and through the paper, $T = T(x,t)$, $x \in \mathbb{R}$, $t \geqslant 0$, denotes the temperature; constants $\tau \geqslant 0$ and $\lambda > 0$ denote relaxation time and the coefficient of the thermal conductivity; operators ∂_x and ∂_t are usual partial differential operators, while ${}_0^C D_t^\alpha$ and $\mathcal{E}_x^\beta$ denote the left Caputo fractional derivative of order α, and the symmetrized Caputo fractional derivative of order β, defined by (10) and (11) below; constants α and β are assumed to satisfy $0 \leqslant \alpha \leqslant$, $0 \leqslant \beta \leqslant 1$; $\mathcal{L}^{-1}$ denotes an inverse Laplace transform and $*_t$ is (time) convolution.

Note that (1) generalizes classical difusion (or heat conduction) equation, since for $\alpha = 0$ and $\beta = 1$ it becomes

$$\partial_t T = \mathcal{D} \, \partial_x^2 T, \quad \mathcal{D} = const.. \tag{2}$$

Equation (1) is subject to initial condition

$$T(x,0) = T_0(x), \tag{3}$$

where T_0 denotes initial temperature distribution, and to boundary condition

$$\lim_{x\to\pm\infty} T(x,t) = 0. \tag{4}$$

In [1] equivalent system to equation (1) is proposed to model heat conduction (see (8),(9) below), and existence of a unique generalized solution (in the space of tempered distributions) of initial-boundary problem (1), (3), (4) is proven. Also some numerical examples are presented.

Here, in this introductory Section 1 we give motivation and mathematical preliminaries, in Section 2 we list main results, and at the end we give an example. All missing proofs can be found in [1] and references therein.

1.1 Motivation

The classical heat conduction (2), with $\mathcal{D} = \lambda/\rho c$, is obtained from the system consisting of two equations: the energy balance equation in simplified form

$$\rho c\, \partial_t T = -\partial_x q,$$

and the Fourier heat conduction law

$$q = -\lambda\, \partial_x T, \tag{5}$$

where, q denotes the heat flux depending on (x,t), $x \in \mathbb{R}$, $t \geqslant 0$, $\rho > 0$ denotes the density and $c > 0$ the specific heat capacity of the media. Similarly, telegraph equation

$$\tau \partial_t^2 T + \partial_t T = \mathcal{D}\, \partial_x^2 T, \quad \mathcal{D} = \frac{\lambda}{\rho c} \tag{6}$$

is obtained when the constitutive equation (5) is replaced by the Cattaneo heat conduction law (cf. [3, 4])

$$\tau \partial_t q + q = -\lambda \partial_x T. \tag{7}$$

In [1] the Cattaneo type space-time fractional heat conduction equation (1) is obtained from the system

$$\partial_t T = -\, \partial_x q, \tag{8}$$

$$\tau\, {}_0^C D_t^\alpha q + q = -\, \lambda \mathcal{E}_x^\beta T. \tag{9}$$

where (9) is *the space-time fractional Cattaneo heat conduction law* which generalizes both, Fourier (5) and Cattaneo (7) heat conduction laws. Indeed, let $\beta \to 1$. Then ${}_a^C\mathcal{E}_b^\beta \to \partial_x$ (see Subsection 1.2), and if in addition $\alpha \to 1$, then (9) reduces to (7). If $\beta \to 1$ and $\tau = 0$ or $\alpha = 0$, then (9) reduces to (5).

If $T(\cdot, t) = const.$, $t \geqslant 0$, then ${}_a^C\mathcal{E}_b^\beta T = 0$, which further implies $\tau\, {}_0^C D_t^\alpha q + q = 0$. The unique solution of this equation is given by $q(x,t) = q(x,0) \cdot e_\alpha(\frac{1}{\tau}, t)/\tau$, which with initial condition $q(x,0) = 0$ gives $q = 0$. Here, e_α is the Mittag-Leffler type special function (cf. [5]). Therefore $T(\cdot, t) = const.$ implies $q = 0$, hence the space-time fractional Cattaneo heat conduction law (9) describes expected physical fact that if the temperature is constant in space, and there is no sources of the heat, there is no heat flux. Classical heat conduction laws, as well as those with fractional time generalizations also exhibit such physical property.

The use of the fractional gradient in space brings a new quality: even if there is a spatial distribution of temperature, if fractional gradient of temperature vanishes, there is no heat flux. Indeed, if $T(\cdot, t) \neq const.$, $t \geqslant 0$, and $\beta \to 0$, then ${}_a^C\mathcal{E}_b^\beta T(\cdot, t) \to 0$ (see Subsection 1.2), and again $q = 0$. Thus, when β tends to zero system (1) describes an ideal heat isolator. In such a body an initial space distribution of the temperature would not change in time. Indeed, since (8) leads to $\partial_t T = 0$, we have that $T(x,t) = T(x) = T_0(x)$. Therefore materials which are good isolators should be modeled with small β.

For similar generalizations of the heat conduction equation or other possibilities for generalization of the heat conduction equation (2) and the telegraph equation (6), using the integer-order or fractional derivative(s) of constant, distributed or variable order see references in [1].

1.2 Mathematical preliminaries

This subsection serves to recall main definitions and properties of fractional derivatives used in the model, the space of tempered distributions and its particular subspaces, and integral transforms used as a main tool for analysis of the solvability of the problem.

Let $0 \leqslant \alpha < 1$, $-\infty \leqslant a < b \leqslant \infty$. The left and right Caputo derivatives, of order α, of an absolutely continuous function u are defined by

$$ {}_a^C D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{u'(\vartheta)}{(t-\vartheta)^\alpha} d\vartheta, \quad \text{and} \quad {}_t^C D_b^\alpha u(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b \frac{u'(\vartheta)}{(\vartheta - t)^\alpha} d\vartheta, \tag{10} $$

where Γ is Euler's gamma function and $u' := \frac{d}{dt}u$, (cf. [5]). Caputo derivatives generalize integer order derivatives since ${}_a^C D_t^0 u(t) = {}_t^C D_b^0 u(t) = u(t)$, and for continuously differentiable functions and distributions it holds $\alpha \to 1^-$, ${}_a^C D_t^1 u(t) \to u'(t)$, ${}_t^C D_b^1 u(t) \to -u'(t)$.

Let $0 \leqslant \beta < 1$, $-\infty \leqslant a < b \leqslant \infty$. The symmetrized fractional derivative of an absolutely continuous function u is defined as

$$ {}_a^C \mathcal{E}_b^\beta u(x) = \frac{1}{2}\left({}_a^C D_x^\beta - {}_x^C D_b^\beta\right) u(x). $$

The symmetrized fractional derivative generalizes the first derivative of a function, since ${}_a^C \mathcal{E}_b^\beta u(x) \to u'(x)$, as $\beta \to 1$, but it does not generalize derivative of order zero since ${}_a^C \mathcal{E}_b^0 u(x) = 0$. For $u = const.$ we have that ${}_a^C \mathcal{E}_b^\beta u = 0$, and conversely, the fact that $u = const.$ is the unique solution to equation ${}_a^C \mathcal{E}_b^\beta u = 0$ is shown in [2].

For $a = -\infty$ and $b = \infty$ we will write $\mathcal{E}_x^\beta$ instead of ${}_a^C \mathcal{E}_b^\beta$ and then

$$ \mathcal{E}_x^\beta u(x) = \frac{1}{2}\frac{1}{\Gamma(1-\beta)}|x|^{-\beta} * u'(x) = \sin\frac{\beta\pi}{2}\frac{d}{dx}I^{1-\beta}u(x), \tag{11} $$

where I^β is the Riesz potential (cf. [5, §12.1]).

Further, recall that the space of Schwartz test functions is denoted by $\mathcal{S}(\mathbb{R}^n)$ and its dual, the space of Schwartz distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$. By $\mathcal{S}'_+, \mathcal{S}'_-$ we denote a subspaces of $\mathcal{S}'(\mathbb{R})$ which consist of distributions with support in $[0,\infty)$ and $(-\infty, 0]$ respectively.

For fractional operators in the distributional setting, one introduces a family $\{f_\alpha\}_{\alpha\in\mathbb{R}} \in \mathcal{S}'_+$ as

$$ f_\alpha(t) = \begin{cases} H(t)\frac{t^{\alpha-1}}{\Gamma(\alpha)}, & \alpha > 0, \\ \frac{d^N}{dt^N} f_{\alpha+N}(t), & \alpha \leqslant 0, \quad \alpha + N > 0, \quad N \in \mathbb{N} \end{cases}, $$

and $\{\check{f}\}_{\alpha\in\mathbb{R}} \in \mathcal{S}'_-$ as $\check{f}_\alpha(t) = f_\alpha(-t)$, where H is the Heaviside function. Then $f_\alpha *$ and $\check{f}_\alpha *$ are convolution operators and for $\alpha < 0$ they are operators of left and right fractional differentiation and for u absolutely continuous function we have that

$$ {}_0^C D_t^\alpha u = f_{1-\alpha} * u' \quad \text{and} \quad {}_t^C D_a^\alpha u = -\check{f}_{1-\alpha} * u'. $$

For $u \in \mathcal{S}'(\mathbb{R})$ the Fourier transform is defined as $\langle \mathcal{F}u, \varphi\rangle = \langle u, \mathcal{F}\varphi\rangle$, $\varphi \in \mathcal{S}(\mathbb{R})$, where for $\varphi \in \mathcal{S}(\mathbb{R})$ we have $\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) = \int\limits_{-\infty}^{\infty} e^{-i\xi x}\varphi(x)\,dx$, $\xi \in \mathbb{R}$, and the

Laplace transform of $u \in \mathcal{S}'(\mathbb{R})$ is defined by

$$\mathcal{L}u(s) = \widetilde{u}(s) = \mathcal{F}(e^{-\xi t}u)(\eta), \quad s = \xi + i\eta.$$

The Lizorkin space of test functions Φ (cf. [5]) is introduced so that Riesz integro-differentiation is well defined. Let

$$\Psi = \{\psi \in \mathcal{S}(\mathbb{R}), \psi^{(j)}(0) = 0, j = 0, 1, 2...\},$$

and consider the space Φ consisting of Fourier transforms of functions in Ψ, i.e., $\Phi = \mathcal{F}(\Psi)$. The Lizorkin function space is invariant relative to Riesz fractional integro-differentiation and $I^{\beta}(\Phi) = \Phi$. The space Ψ' and the space of Lizorkin generalized functions Φ' are dual spaces of Ψ and Φ respectively, and for $f \in \Phi'$ we have

$$\langle \mathcal{F}f\psi\rangle = \langle f\mathcal{F}\psi\rangle, \quad \psi \in \Psi.$$

Within the space Ψ' multiplication with functions smooth apart from the origin is well defined. The function $|x|^{-\alpha}$ is an element of the space Ψ', for all $\alpha \in \mathbb{R}$, and

$$\mathcal{F}[\,|x|^{-\beta}](\xi) = \sin\frac{\beta\pi}{2}\cdot|\xi|^{\beta-1}, \quad \mathcal{F}[\mathcal{E}_x^{\beta}u](\xi) = i\cdot\sin\frac{\beta\pi}{2}\cdot\frac{\xi}{|\xi|^{1-\beta}}\mathcal{F}(\xi).$$

2 Main results

Now we give main results concerning existence and uniqueness of a generalized solution to the initial-boundary value problem (1), (3), (4).

We set up the problem within the space E, space of all distributions $u \in \mathcal{S}'(\mathbb{R}^2)$ such that for fixed first variable x, $u(x,\cdot) \in \mathcal{S}'_+$ and for fixed second variable t, $u(\cdot, t) \in \Phi'$.

Definition 1. $T \in E$ is called a generalized solution to initial problem (1), (3) if

$$\partial_t T = \mathcal{L}^{-1}\left(\frac{1}{1+\tau s^{\alpha}}\right) *_t \partial_x\,\mathcal{E}_x^{\beta}T + T_0(x)\delta(t)$$

holds in E.

Note that if $T \in E$ is a generalized solution in the sense of the above definition then (4) is automatically satisfied, since $T \in E$ implies (4).

For the proof of main theorem following two lemmas are essential.

Lemma 1. *Let* $u_0 \in \Phi'$ *and* $\omega, \nu \in \mathbb{C}$, $\omega \neq 0$. *Then*

$$\partial_x \mathcal{E}_x^\beta u - \omega\, u = -\nu\, u_0 \tag{12}$$

has a unique solution $u \in \Phi'$ *given by* $u = \dfrac{\nu}{\pi}\, u_0 *_x \displaystyle\int_0^\infty \frac{\cos(\rho x)}{\sin\frac{\beta\pi}{2}\rho^{1+\beta} + \omega} d\rho.$

Lemma 2. *Let* $\tau, \vartheta > 0$ *and* $0 < \alpha < 1$. *Then there exist unique* $r_0 > 0$ *and* $\varphi_0 \in (\frac{\pi}{2}, \pi)$ *(depending on* τ, ϑ *and* α*), such that* $s_0 := r_0 e^{i\varphi_0}$ *and* $\overline{s}_0$*, the complex conjugated number of* s_0*, are zeros of the multiplicity one of the function*

$$f(s) = \tau s^{\alpha+1} + s + \vartheta, \quad f : \mathbb{C} \to \mathbb{C}.$$

The existence of a unique solution to the initial-boundary value problem (1), (3), (4) is given in the following theorem.

Theorem 1. *Let* $T_0 \in \Phi'$. *Then there exists a unique generalized solution* $T \in E$ *to the initial value problem (1), (3).*

Proof. Applying the Laplace transform to (1) with respect to t we obtain

$$\partial_x \mathcal{E}_x^\beta \tilde{T} - \frac{s(1+\tau s^\alpha)}{\lambda}\tilde{T} = -\frac{1+\tau s^\alpha}{\lambda} T_0, \tag{13}$$

which is an equation of type (12) where $\omega = \omega(s) := \dfrac{s(1+\tau s^\alpha)}{\lambda}$, $\quad \operatorname{Re} s > 0$, and $\nu = \nu(s) := \dfrac{1+\tau s^\alpha}{\lambda}$. Lemma 1 gives

$$\begin{aligned}\tilde{T}(x,s) &= \frac{1+\tau s^\alpha}{\lambda\pi} T_0(x) *_x \int_0^\infty \frac{\cos(\rho x)}{\sin\frac{\beta\pi}{2}\rho^{1+\beta} + \omega(s)} d\rho = \\ &= \frac{T_0(x)}{\pi} *_x \int_0^\infty \frac{(1+\tau s^\alpha)\cos(\rho x)}{\lambda \sin\frac{\beta\pi}{2}\rho^{1+\beta} + s + \tau s^{\alpha+1}} d\rho \end{aligned} \tag{14}$$

$T(\cdot, s)$ is unique and belongs to Φ'. Existence of a unique inverse Laplace transform of $\tilde{T}(x,\cdot)$ in $\mathcal{S}'_+$, $T(x,t) = \mathcal{L}^{-1}(\tilde{T}(x,s))$ is guaranteed by Lemma 2. □

An explicit calculation of the inverse Laplace transform $\mathcal{L}^{-1}(\tilde{T}(x,s))$ in the above proof is particularly important for a numerical analysis of the problem. Next theorem gives an explicit formula for the generalized solution T to equation (1).

Theorem 2. *Let $T \in E$ be the generalized solution to initial problem (1), (3) whose existence and uniqueness is guaranteed by Theorem 1. Then*

$$T(x,t) = \frac{1}{\pi} T_0(x) *_x \int\limits_0^\infty S(\rho,t)\cos(\rho x)d\rho, \quad t>0,\, x\in\mathbb{R}, \tag{15}$$

with

$$S(\rho,t) = I(\rho,t) + R(\rho,t), \quad t>0,\, \rho>0, \tag{16}$$

where

$$I(\rho,t) = -\frac{1}{\pi}\int\limits_0^\infty \frac{\tau q^\alpha \vartheta(\rho)\sin(\alpha\pi)}{\tau^2 q^{2(\alpha+1)} + 2\tau q^{\alpha+1}(\vartheta(\rho)-q)\cos((\alpha+1)\pi) + (\vartheta(\rho)-q)^2} e^{-qt}\, dq,$$

$$\vartheta(\rho) := \lambda \sin\frac{\beta\pi}{2}\rho^{1+\beta} > 0 \tag{17}$$

and

$$R(\rho,t) = 2\, Re\left\{ \frac{(\tau s^\alpha + 1)e^{st}}{\tau(\alpha+1)s^\alpha + 1}\,\Big|_{s=s_0(\rho)} \right\},$$

$s_0 = r_0 e^{i\varphi_0}$ *is from Lemma 2.*

In the proof of this theorem fundamental solution to (1) (generalized solution with $T_0(x) = \delta$) is calculated as

$$F(x,t) = \frac{1}{\pi}\int\limits_0^\infty S(\rho,t)\cos(\rho x)d\rho, \quad t>0,\, x\in\mathbb{R}. \tag{18}$$

For the proof of Theorem 2 one starts from the Laplace transform of T given by expression (14) and by setting

$$\tilde{S}(\rho,s) := \frac{\tau s^\alpha + 1}{\tau s^{\alpha+1} + s + \vartheta(\rho)}, \quad \operatorname{Re} s > 0,$$

with ϑ given by (17) comes to

$$\tilde{T}(s,x) = \frac{1}{\pi} T_0(x) *_x \int\limits_0^\infty \tilde{S}(s,x)\cos(\rho x)d\rho,$$

which after inversion of the Laplace transform gives (15). Then one looks for the form (16) of S using the inversion formula for the Laplace transform

$$S(\rho, t) = \frac{1}{2\pi i} \int_{s_0 - i\infty}^{s_0 + i\infty} \tilde{S}(\rho, s) e^{st} ds,$$

and the Cauchy residues theorem.

Example. As an example we present calculated solution (15) for particular values of parameters τ, λ, α and β and initial disturbance. Fix $\tau = 0.1$, $\lambda = 1$, $\alpha = 0.25$ and the time instant $t = 1$, assume that the initial temperature distribution is the delta distribution, i.e., $T_0(x) = \delta(x)$, and consider the behavior of T for various β, versus spatial coordinate x. Figure 1 shows that when β decreases, resistance of material to conduct heat increases. As β decreases the position of the peak tends to the T axis (T tends to δ), and presumably in the limiting case $\beta = 0$ it would be on the T axis, i.e., the initial temperature distribution would not change in time. This indicates that the parameter β in the fractional gradient characterizes the ability of media to conduct heat. The limiting case $\beta = 0$ characterizes the media which we call the ideal heat isolator.

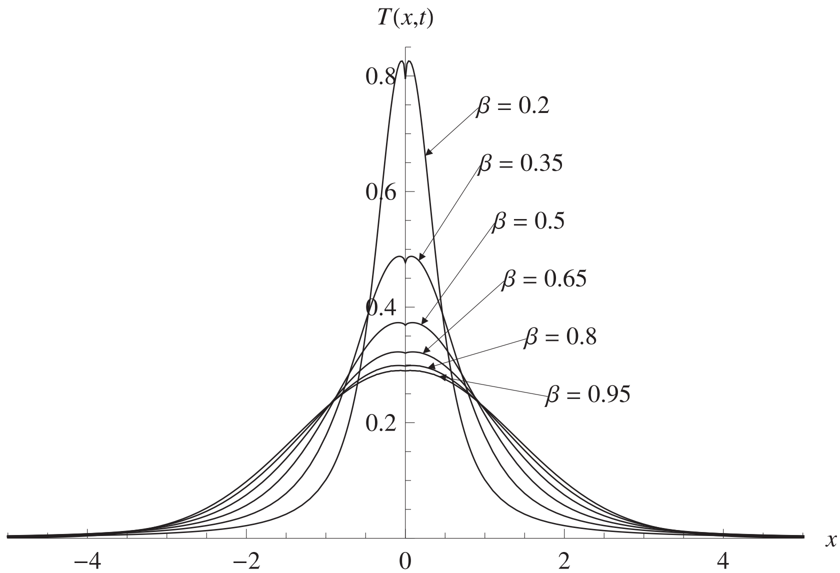

Figure 1. Temperature T as a function of x at $t = 1$ for various β

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A NEW ALGORITHM FOR HANKEL TRANSFORM USING WAVELETS

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Key words: Hankel Transform, Haar Wavelet, Legendre Wavelet, Random Noise

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Abstract. The aim of the present paper is to analyse and compare the efficient algorithms for the numerical evaluation of Hankel transform of order ν, $\nu > -1$ using Legendre and Haar wavelets.The philosophy behind the algorithm is to replace the part $rf(r)$ of the integrand by its wavelet decomposition obtained by using Legendre wavelets and Haar wavelet respectively, thus representing $F_\nu(p)$ as a Fourier–Bessel series with coefficient depending strongly on the input function $rf(r)$ in all cases. Numerical evaluations of test functions with known analytical Hankel transform illustrate the efficiency and stability of the algorithm.

1 Introduction

The Fourier Bessel (Hankel) transform is very useful in analysis of wave fields where it is used in mathematical handling of radiation, diffraction, and field projection. The general Hankel transform (HT) pair with the kernel being J_n is defined as [1]

$$F_n(p) = \int_0^\infty f(r) r J_n(pr)\, dr, \tag{1}$$

$$f(r) = \int_0^\infty F_n(p) p J_n(pr)\, dp \tag{2}$$

and HT being self-reciprocal, its inverse is given by where J_n is the nth-order Bessel function of first kind. Several papers [2-9] have been written to the evaluation of the Hankel transform in general and the zeroth order in particular. Analytical evaluations of (1) and (2) are rare and their numerical computations are difficult because of the oscillatory behavior of the Bessel function and the infinite length of the interval. In 2010, Singh et. al. [8] and Pandey et al. [9] presented a new stable algorithms based on Legendre and Haar wavelets respectively to compute the Hankel transforms.

The purpose of this communication is to represent both the algorithms and compare them in terms of accuracy and stability.

2 Wavelets

Wavelets are a class of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation and translation parameters a and b vary continuously, the following family of continuous wavelets are obtained

$$\psi_{ab}(t) = |a|^{-1/2}\psi\left(\frac{t-b}{a}\right) \quad a, b \in R \quad a \neq 0,$$

when the parameters a and b are restricted to discrete values as $a = 2^{-k}$, $b = n2^{-}k$, then we have the following family of discrete wavelets $\psi_{kn}(t) = 2^{k/2}\psi(2^k t - n)$ $k, n \in Z$, where the function ψ, the mother wavelet, satisfies $\int\limits_a^b \psi(t)\, dt = 0$.

2.1 Legendre wavelets

We define the Legendre wavelets as follows:

$$\psi_{nm}(t) = \begin{cases} \sqrt{2m+1}2^{k/2}P_m(2^k t - \widehat{n}) & \text{for } \dfrac{\widehat{n}-1}{2^k} \leqslant t \leqslant \dfrac{\widehat{n}+1}{2^k}, \\ 0, \quad \text{otherwise} \end{cases}$$

where $P_m(t)$ are the well known Legendre polynomials of order m and defined on $[-1, 1]$ by the following recurrence relation:

$$P_0(t) = 0, \quad P_1(t) = t, \quad P_{m+1}(t) = \frac{2m+1}{m+1} t P_m(t) - \frac{m}{m+1} P_{m-1}(t), \quad m = 1, 2, 3 \cdots$$

The Legendre wavelets $\psi_{nm} = \psi(k, \widehat{n}, m, t)$ have four arguments; $\widehat{n} = 2n-1, n = 0, 1, 2 \cdots 2^{k-1}, k = 1, 2, \cdots m$ is the order of Legendre polynomials and t is the normalised time. The family the family forms $\{\psi_{nm}\}_{n,m}$ forms an orhtonormal basis for $L^2[0, 1]$.

2.2 Haar wavelets

In 1910,Haar developed the Haar wavelets and in recent years,the Haar theory has been innovated and applied to various fields. The scaling function $\varphi^H(t)$ and the fundamental square wave or the mother wavelet $\psi^H(t)$ are defined as,

$$\varphi^H(t) = \begin{cases} 1, 0 \leqslant t < 1 \\ 0, \quad \text{otherwise} \end{cases} \quad \text{and} \quad \psi^H(t) = \begin{cases} 1, 0 \leqslant t < 1/2 \\ -1, 1/2 \leqslant t < 1 \\ 0, \quad \text{otherwise} \end{cases} \tag{3}$$

3 Outline of algorithm

In practical applications, usually the function $f(r)$ has compact support and in many cases,though the support may not be compact, given any $\epsilon > 0$, there exists a compact interval I_ε such that $|f(r)| < \varepsilon$ for r not belongs to interval I_ε. Hence it is more appropriate to consider the finite Hankel transform. Suppose $f(r)$ is supported on $[0, h]$, then (1) reduce to

$$\widehat{F_n(p)} = \int_0^h f(r) r J_n(pr)\, dr \tag{4}$$

known as finite Hankel transform of $f(r)$ where r is replaced by r/h. Writing $g(r) = rf(r)$ in Eq.(4), we get

$$\widehat{F_n(p)} = \int_0^1 g(r) J_n(pr)\, dr. \tag{5}$$

The inverse finite Hankel transform is represented as Fourier–Bessel series [8].

3.1 Algorithm based on Legendre wavelet

As $g(r) \in L^2[0,1]$ and ψ_{nm} are an orthonormal basis for the Hilbert space $L^2[0,1]$, we may expand $g(r)$ as follows:

$$g(r) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(r), \tag{6}$$

where $c_{nm} =< g(r), \psi_{nm}(r) >$ and $<, >$ denotes the standard inner product on the Hilbert space $L^2(R)$. By truncating the infinite series (6) at levels $n = 2^{k-1}$ and

$m = M$, we obtain an approximate representation for $g(r)$ as

$$g(r) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{nm}\psi_{nm}(r) = C^T\psi(r), \tag{7}$$

where the matrices C and ψ are given by

$$C = [c_{10} \cdots c_{1M} \cdots c_{20} \cdots c_{2M} \cdots c_{2^{k-1}0} \cdots c_{2^{k-1}M}]^T \tag{8}$$

and

$$\psi(r) = [\psi_{10}(r) \cdots \psi_{1M}(r) \cdots \psi_{20}(r) \cdots \psi_{2M}(r) \cdots \psi_{2^{k-1}}0(r) \cdots \psi_{2^{k-1}M}(r)]^T \tag{9}$$

substituting (7) in (5), we get

$$\widehat{F_n(p)} \approx C^T \int_0^1 \psi(r) J_n(pr)\, dr, \tag{10}$$

taking $M = 2$ and $k = 2$, Eq.(1) reduced to

$$\widehat{F_n(p)} \approx C^T \left[\int_0^1 \psi_{10}(r) J_n(pr)\, dr, \int_0^1 \psi_{11}(r) J_n(pr)\, dr \cdots \int_0^1 \psi_{22}(r) J_n(pr)\, dr \right]. \tag{11}$$

Integral in Eq.(11) are evaluated using following formulae:

$$\int_0^a J_v(t)\, dt = 2 \sum_{n=0}^{\infty} J_{v+2n+1}(a), \quad \operatorname{Re} v > -1 \quad [9,\text{p.}333]$$

$$\int_0^a t^{1-v} J_n(t)\, dt = \frac{1}{2^{v-1}\Gamma(v)} - a^{1-v} J_{v-1}(a) \quad [9,\text{p.}333]$$

$$\int_0^a t^{\mu} J_v(t)\, dt = \frac{a^{\mu}\Gamma\left(\frac{v+\mu+1}{2}\right)}{\Gamma\left(\frac{v-\mu+1}{2}\right)} \sum_{k=0}^{\infty} \frac{(v+2k+1)\left(\Gamma\left(\frac{v-\mu+1}{2}+k\right)\right)}{\Gamma\left(\left(\frac{v+\mu+3}{2}\right)+k\right)} \times J_{v+2k+1}(a),$$
$$\operatorname{Re}(v+\mu+1) > 0 \quad [10,\text{p.}480]$$

3.2 Algorithm based on Haar wavelet

The Haar wavelet series representation of $g(r) \in L^2(R)$ with Haar bases is given as [6]

$$g(r) = \sum_{k=0}^{\infty} c_k \varphi_k(r) + \sum_{j,k=0}^{\infty} d_{jk} \psi_{jk}(r) \tag{12}$$

where

$$\varphi_k(r) = \varphi^H(r-k), \quad \psi_{jk}(r) = 2^{j/2}\psi^H(2^j r - k)$$

where the wavelet coefficients are evaluated as:

$$c_k = \int_k^{k+1} g(r)\,dr \text{ and } d_{jk} = 2^{j/2}\left(\int_{2^{-j}k}^{2^{-j}(k+1/2)} g(r)\,dr - \int_{2^{-j}(k+1/2)}^{2^{-j}(k+1)} g(r)\,dr \right) \tag{13}$$

From Eq.(5) and (12), we have

$$\widehat{F_n(p)} \approx \int_0^1 \left(\sum_{k=0}^{\infty} c_k \varphi_k(r) + \sum_{j,k=0}^{\infty} d_{jk} \psi_{jk}(r) \right) J_n(pr)\,dr. \tag{14}$$

By change of variable and (14)

$$\begin{aligned} F_n(y) = \frac{2}{p} \lim_{M\to\infty} [\sum_{k=0}^{\infty} \left(\sum_{z=0}^{M} J_{n+2z+1}(y) \right) + \\ + \frac{2}{p} \lim_{M\to\infty} \sum_{j=0}^{\infty} 2^{j/2} \sum_{k=0}^{\infty} d_{jk} \times (2 \sum_{z=0}^{M} J_{n+2z+1}(2^{-j}(k+1/2)p) + \\ + \frac{2}{p} \lim_{M\to\infty} \sum_{z=0}^{M} J_{n+2z+1}(2^{-j}(k+1)p)). \end{aligned} \tag{15}$$

The inverse finite Hankel transform is represented as a Fourier-Bessel series.The corresponding expansion for finite Hankel transform $\widehat{F_n(p)}$ is obtained by putting $k = 0$ in the first part of the summation involving c_k and taking $0 \leqslant k \leqslant 2^j - 1$ in the second part of the summation involving d_{jk} in the above representation of the $\widehat{F_n(p)}$ as given by (15).

4 Numerical results

Now, we discuss, the implementation of our numerical methods and investigate its accuracy and stability by applying it on numerical example with known analytical HT and compare it with each other. In all the examples,the exact data function is denoted by $g(r)$ and noisy data function by $g^{\varepsilon}(r)$ is obtained by adding an ε random error to $g(r)$ such that $g^{\varepsilon}(r_i) = g(r_i) + \varepsilon\vartheta_i$, where $r_i = ih, i = 1, 2 \cdots N, Nh = 40$ and ϑ_i is a uniform random variable with values in $[-1, 1]$ such that Max $0 \leqslant i \leqslant N$ $|g^{\varepsilon}(r_i) - g(r_i)| \leqslant \varepsilon$. The following examples are solved with and without random perturbations to illustrate the efficiency and stability of our method by choosing three different values of randon errors ε as $\varepsilon = 0, 0.0002, 0.0005$ and computing the error $Ej(p)$=Approximate HT obtained from proposed algorithm with random error ε_j-the exact HT, $j = 0, 1, 2$.

We also use the discrete l^2norm and the continuous L^2 norm in $I = [0, P]$ to measure root mean square (RMS) errors as well. These norms are defined as:

$$\|f\|_{2,I} = \left(\frac{1}{N}\sum_{i=1}^{N} |f(r_i)|^2\right)^{1/2} \quad \text{and } \|f\|_2 = \left(\int_0^p |f(r)|^2 \, dr\right)^{1/2}$$

respectively.

Figures (1) and (4) show the comparison between the absolute errors with different perturbation by Haar method.Figures (2) and (5) show the comparison between the absolute errors with different perturbation by Legendre method ,where as figures (3) and (6) show the comparison of absolute error between the exact transform and appropriated transform by Haar method and Legendre method.

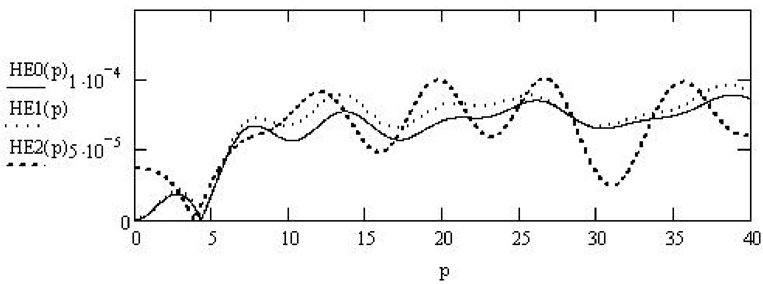

Figure 1. Comparison between exact transform and approximate transform with random perturbations by Haar method

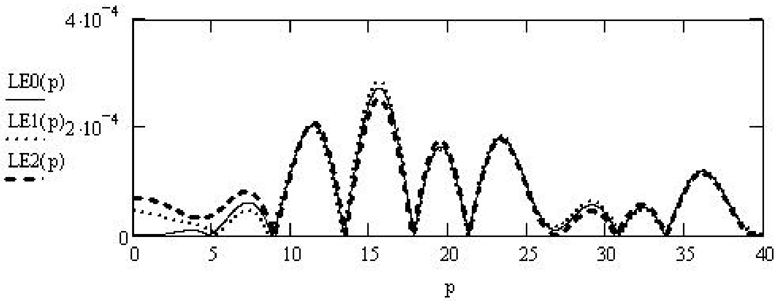

Figure 2. Comparison between exact transform and approximate transform with random perturbations by Legendre method

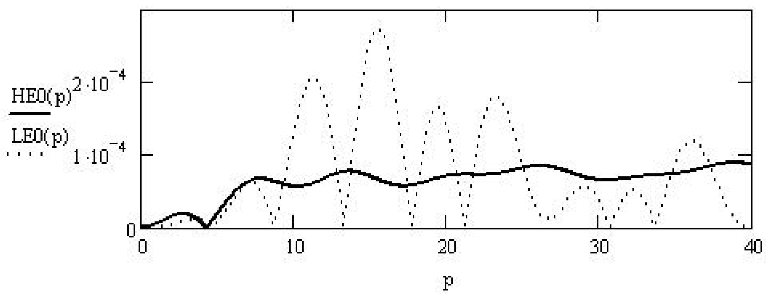

Figure 3. Comparison of absolute error by Haar method HE0(p) and Legendre method LE0(p)

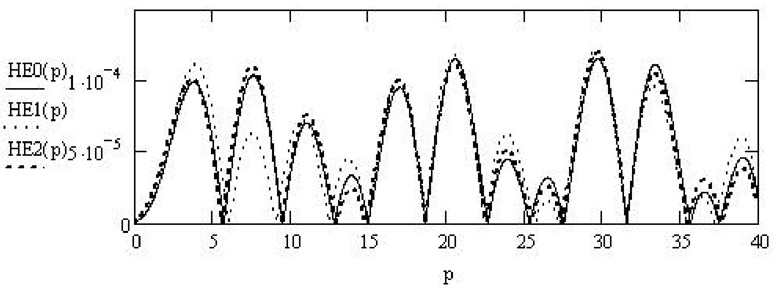

Figure 4. Comparison between exact transform and approximate transform with random perturbations by Haar method

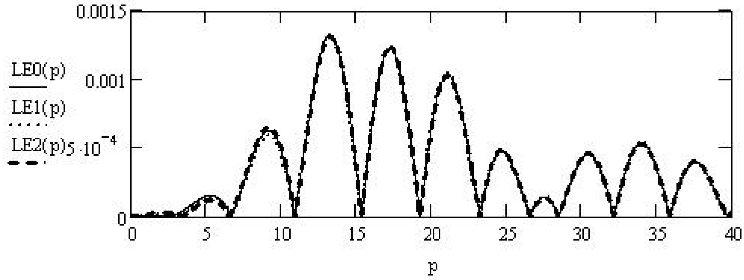

Figure 5. Comparison between exact transform and approximate transform with random perturbations by Legendre method

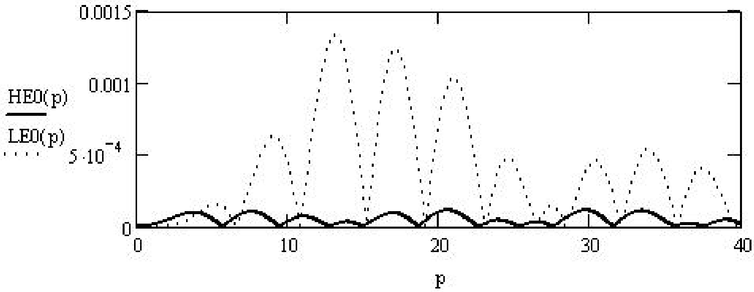

Figure 6. Comparison of absolute error by Haar method(HE0(p)) and Legendre method(LE0(p))

Example 1. Let us consider $f(r) = \frac{2}{\Pi}[\arccos(r) - r(1-r^2)^{1/2}], \quad 0 \leqslant r \leqslant 1$ and $F_0(p) = 2\frac{J_1^2(p/2)}{p^2}$.

Example 2. Let us consider the pair $f(r) = (1-r^2)^{1/2}, \quad 0 \leqslant r \leqslant 1$, and

$$F_1(p) = \begin{cases} \Pi\frac{J_1^2(p/2)}{2p} & 0 < p < \infty \\ 0 & y = 0. \end{cases}$$

5 Conclusion

We notice that in all the cases,the numerical accuracy stability of the Haar wavelet based algorithm is more accurate the Legendre wavelet based algorithm.

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ON SOME INTEGRALS FROM MODIFIED BESSEL FUNCTIONS

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Abstract. Efficient applications of modified BESSEL functions for the solution of some problems of mathematical physics are given. The algorithm of numerical solution of some mixed boundary value problems for the HELMHOLTZ equation in wedge domains by means of dual integral equations method is developed. The integrals from modified BESSEL functions of the second kind with pure imaginary order $K_{i\tau}(x)$ and with complex order $K_{\frac{1}{2}+i\tau}(x)$ are investigated. Analytical considerations and numerical experiments are discussed. Different simplification procedures are used for the evaluation. Examples demonstrate the efficiency of the integral method in the numerical solution of the mixed boundary value problems of elasticity, combustion and electrostatics in the wedge domains.

The definition of two pairs of direct and inverse LEBEDEV–SKALSKAYA integral transforms [5] is cited

$$F_+(\tau) = \int_0^\infty f(x) \operatorname{Re} K_{1/2+i\tau}(x)dx, \quad 0 \leqslant \tau \leqslant \infty,$$

$$f(x) = (4/\pi^2) \int_0^\infty ch(\pi\tau) F_+(\tau) \operatorname{Re} K_{1/2+i\tau}(x)d\tau, \quad 0 < x < \infty,$$

and

$$F_-(\tau) = \int_0^\infty f(x) \operatorname{Im} K_{1/2+i\tau}(x)dx, \quad 0 \leqslant \tau \leqslant \infty,$$

$$f(x) = (4/\pi^2) \int_0^\infty ch(\pi\tau) F_-(\tau) \operatorname{Im} K_{1/2+i\tau}(x)d\tau, \quad 0 < x < \infty.$$

Sufficient conditions for the existence of these transforms and the validity of the inversion formulas are given.

It is shown that the inversion formulas of the LEBEDEV–SKALSKAYA integral transforms can be deduced from the inversion formulas of the KONTOROVITCH–LEBEDEV transforms and the corresponding theorem is proven. For the case of nonnegative finite functions with restricted variation the conditions of present theorem are reduced to one condition, which is then necessary and sufficient [8, 13–15].

The application of the KONTOROVITCH–LEBEDEV integral transforms and dual integral equations to the solution of the mixed boundary value problems are considered. The diffusion and elastic problems reduce to the solution of the proper mixed boundary value problems for the HELMHOLTZ equation.

The mixed boundary value problems for the HELMHOLTZ equation [1, 4, 16]

$$\Delta u - k^2 u = 0 \tag{1}$$

are arised in some fields of mathematical physics.

Let's use the following notations here and further: r, φ — polar coordinates of the point; α - angle of the sectorial domain; u — desired function; η — normal to the boundary.

The numerical solution of some boundary value problems for the equation of the form (1) in arbitrary sectorial domains is considered in our work under the assumption that the function $u|_\Gamma$ is known on the part of the boundary and the normal derivative $\frac{\partial u}{\partial \eta}|_\Gamma$ is known on the other part of the boundary. The KONTOROVITCH–LEBEDEV integral transforms [4] and dual integral equations method [4, 10] are used for finding of the solution.

Let's consider the symmetric case to simplify the calculations

$$\begin{cases} \Delta u - k^2 u = 0, \\ \left.\dfrac{\partial u}{\partial \eta}\right|_{\varphi=\pm\alpha}(r) = g(r), \quad 0 < r < a, \\ u|_{\varphi=\pm\alpha}(r) = f(r), \quad r > a, \\ u|_{r\to 0} \text{ — restricted}, \\ u|_{r\to\infty} \text{ — restricted}. \end{cases} \tag{2}$$

The solution of (2) is determined by the following way in the form of KONTOROVITCH-LEBEDEV integral transforms [4]

$$u(r,\varphi) = \int\limits_0^\infty M(\tau)\frac{\cosh \varphi\tau}{\cosh \alpha\tau} K_{i\tau}(kr)d\tau, \tag{3}$$

where $M(\tau)$ is the solution of dual integral equation

$$\begin{aligned} \int\limits_0^\infty M(\tau)\tau\tanh(\alpha\tau)K_{i\tau}(kr)d\tau &= rg(r), \quad 0<r<a,\\ \int\limits_0^\infty M(\tau)K_{i\tau}(kr)d\tau &= f(r), \quad r>a, \end{aligned} \tag{4}$$

where $g(r)$ and $f(r)$ — given functions and $K_\nu(z)$ — modified Bessel function (Macdonald function) of imaginary order.

The dimension of the problem is lowered one unit by this approach as can be seen easily.

Dual integral equations of this type were considered in [4, 10]. It was shown in [4] that the solutions of these equations may be determined in the form of single quadratures from auxiliary functions satisfying to the second kind Fredholm integral equations with symmetric kernel containing modified Bessel function of the complex order $K_{1/2+i\tau}(x)$.

The general case is reduced to the case $g(r)=0$ as it follows from [4]. For the simplicity, let us consider this case from this point on.

Let us denote

$$\begin{aligned} h(t) &= -\frac{\sqrt{k}e^{kt}}{\pi}\frac{d}{dt}\int\limits_0^\infty \frac{e^{-kr}f(r)}{\sqrt{r-t}}dr,\\ K(s,t) &= \frac{4}{\pi}\int\limits_0^\infty \frac{\sinh[(\pi-\alpha)\tau]}{\sinh(\alpha\tau)}\operatorname{Re}K_{1/2+i\tau}(ks)\operatorname{Re}K_{1/2+i\tau}(kt)d\tau, \end{aligned} \tag{5}$$

where $\operatorname{Re}K_{1/2+i\tau}(z)$ — real part of MacDonald's function of complex order $1/2+i\tau$.

Then we obtain the following procedure for the determination of $M(\tau)$ on the basis of [4]

$$M(\tau) = \frac{2\sqrt{2}\sinh(\pi\tau)\cosh(\alpha\tau)}{\pi\sqrt{\pi}\sinh(\alpha\tau)}\int\limits_a^\infty \psi(t)\operatorname{Re}K_{1/2+i\tau}(kt)dt, \tag{6}$$

where $\psi(t)$ — solution of the integral FREDHOLM equation of the second kind

$$\psi(t) = h(t) - \frac{k}{\pi} \int\limits_a^\infty K(s,t)\psi(s)ds, \quad a \leqslant t < \infty. \tag{7}$$

It is useful under the decision of boundary value problems to find the solution u on the boundary of sectorial domain

$$u|_\Gamma(r) = \int\limits_0^\infty M(\tau) K_{i\tau}(kr) d\tau. \tag{8}$$

Substituting expression (6) for $M(\tau)$ in (8) and transposing the order of the integration we obtain

$$u|_\Gamma(r) = \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int\limits_a^\infty \psi(t) G_r(t) dt, \tag{9}$$

where

$$G_r(t) = \int\limits_0^\infty \frac{\sinh(\pi\tau)\cosh(\alpha\tau)}{\sinh(\alpha\tau)} K_{i\tau}(kr) \mathrm{Re}\, K_{1/2+i\tau}(kt) d\tau. \tag{10}$$

So the numerical solution of the boundary value problem (2) consists from the numerical solution of integral FREDHOLM equation of the second kind with symmetric kernel and from the consequent taking of the quadratures of the solution.

Let us truncate the integral equation (7) in the following way:

$$\psi(t) = h(t) - \frac{k}{\pi} \int\limits_a^b K(s,t)\psi(s)ds, \quad a \leqslant t \leqslant b. \tag{11}$$

Furthermore, the solution of the dual equation was computed by the formulas (6) with the use of codes and routines for the computation of $K_{i\tau}(x)$ and $\mathrm{Re}\, K_{1/2+i\tau}(x)$ [2, 3, 6, 7, 9–12].

Integrals (5), (10) may be expressed through known functions for special values of the angle α, in particular for $\alpha = \frac{\pi}{n}, n = 1, 2, ...$

We obtain for $n = 1$:

$$K(s,t)|_{\alpha=\pi} = 0,$$

for $n = 2$

$$K(s,t)|_{\alpha=\frac{\pi}{2}} = K_0(k(s+t)) + K_1(k(s+t)),$$

for $n = 3$

$$K(s,t)|_{\alpha=\frac{\pi}{3}} = \sqrt{3}K_0(k\sqrt{s^2+t^2+st}) + \frac{\sqrt{3}(s+t)}{\sqrt{s^2+t^2+st}}K_1(k\sqrt{s^2+t^2+st})$$

and so on.

We compute the truncated integrals in fact by the computations of integrals (5), (10) on the computer: the integration is carried on over some interval $[0, B]$. In view of this fact it's important to choose by the correct way the truncation interval $[0, B]$, ensuring the computation of stated integrals with necesssary precision without the expenditure of unnecessary computer time. The estimations of the error

$$K^B(s,t) = \frac{4}{\pi}\int\limits_B^\infty \frac{\sinh[(\pi-\alpha)\tau]}{\sinh(\alpha\tau)}\mathrm{Re}\,K_{1/2+i\tau}(ks)\mathrm{Re}\,K_{1/2+i\tau}(kt)d\tau,$$

$$G_r^B(t) = \int\limits_B^\infty \frac{\sinh(\pi\tau)\cosh(\alpha\tau)}{\sinh(\alpha\tau)}K_{i\tau}(kr)\mathrm{Re}\,K_{1/2+i\tau}(kt)d\tau,$$

arising from the truncation are useful for this purpose. On the basis of inequalities from [4, 8] for functions $K_{i\tau}(x)$ and $\mathrm{Re}\,K_{1/2+i\tau}(x)$ we obtain

$$K^B(s,t) \leqslant c^2k^{-3/2}(st)^{-3/4}\frac{e^{-2\alpha B}}{2\alpha}(B^2 + \frac{B}{\alpha} + \frac{1}{2\alpha^2}), \tag{12}$$

$$G_r^B(t) \leqslant \frac{Ac}{\alpha}k^{-1}r^{-1/4}t^{-3/4}e^{-2\alpha B}, \tag{13}$$

where A and c — some positive constants having the multiplicity of a unit. As it can be seen from (14) and (15) it is necessary to take the extending interval $[0, B]$ for the decreasing values of angle α.

It's necessary to compute N^2 values $K_{ij} = K(s_i, t_j), i = 1, ..., N, j = 1, ..., N$, under the solution of the system of algebraic equations of this form.

It's convenient to use Gauss quadrature formulas by Laguerre polynomials knots and to perform the computations of N integrand for one fixed variable s or t by parallel for the economy of computer time under the integrals computation. Let's note moreover the symmetry $K(s,t) = K(t,s)$ what gives the possibility to decrease the number of computed integrals twice.

Let's describe the conducted evaluations.

Let's represent $K(s,t)$ in the form

$$K(s,t) = -\frac{4k}{\pi^2}\int\limits_0^\infty e^{-2\alpha\tau} P(\tau,s,t)d\tau,$$

$$P(\tau,s,t) = \frac{1-e^{-2(\pi-\alpha)\tau}}{1-e^{-2\alpha\tau}} e^{\frac{\pi\tau}{2}} \operatorname{Re} K_{\frac{1}{2}+i\tau}(ks) e^{\frac{\pi\tau}{2}} \operatorname{Re} K_{\frac{1}{2}+i\tau}(kt).$$

The used quadrature formulas have the form

$$\int\limits_0^\infty e^{-2\alpha\beta} P(\tau,s,t)d\tau = \frac{1}{2\alpha}\sum_{k=1}^n \lambda_k P(\frac{\beta_k}{2\alpha},s,t), \tag{14}$$

where β_k -zeros of LAGUERRE polynomials $L_n^0(x), \lambda_k$ — weights of quadrature formulas.

The computations were conducted simultaneously in loop for all s and all t (for given s.) The representation of the integrand $P(\tau,s,t)$ in the form

$$P(\tau,s,t) = p_1(\tau)p_2(\tau,s)p_2(\tau,t),$$
$$p_1(\tau) = \frac{1-e^{-2(\pi-\alpha)\tau}}{1-e^{-2\alpha\tau}}, \quad p_2(\tau,s) = e^{\frac{\pi\tau}{2}} \operatorname{Re} K_{\frac{1}{2}+i\tau}(ks),$$

allows to decrease significantly the number of handlings to the code of real part of modified BESSEL function $K_{\frac{1}{2}+i\tau}(x)$ computation.

Let's consider the example admitting the complete analytical solution of the problem (2)

$$f(r) = \frac{\sqrt{\pi}}{k\sqrt{2}}(e^{-kr} + e^{kr}[1-\Phi(\sqrt{2k(r+a)})]), \quad g(r) = 0, \quad \alpha = \frac{\pi}{4}.$$

Then we obtain on the basis of relevant calculations [4] that

$$h(t) = e^{-kt} + \frac{1}{\pi}e^{-ka}K_0(k(t+a)), \quad K(s,t) = K_0(k(s+t)) + K_1(k(s+t)), \tag{15}$$

and $\psi(t) = e^{-kt}$.

Substituting (15) in (9) and performing some calculations we obtain for $r < a$

$$u|_\Gamma(r) = \frac{\sqrt{\pi}}{k\sqrt{2}}(e^{-kr}(1-\varphi(\sqrt{2k(a-r)})) + e^{kr}(1-\varphi(\sqrt{2k(a+r)})))$$

and for $r > a$

$$u|_\Gamma(r) = \frac{\sqrt{\pi}}{k\sqrt{2}}(e^{-kr} + e^{kr}(1-\varphi(\sqrt{2k(a+r)})))$$

(verification of the conditions of the problem).

We obtained the precision in 7-8 significant digits under the solution of dual integral equation (computation of the values $(\cosh\frac{\pi\tau}{2})^{-1}M(\tau)$) so for $a = 1.0, k = 1 (\cosh\frac{3\pi}{2})^{-1}M(3) = .9288253_{10} - 01$.

We obtained the precision in 6-7 digits after comma under the calculation of values $u|_\Gamma(r)$ so for $a = 1.0, k = 1 u|_\Gamma(2) = .1745444_{10} + 00$.

The different preliminary procedures of the separation of singularity or transformation of the integral into the integral without the singularity are useful for the computation of integral (9).

It's strongly efficient to use the procedures of numerical integration for the transformed integral [10]. The accuracy of computations is increased and the computer time is shorten by this approach.

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MORREY SPACES FOR NON-DOUBLING MEASURES III

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Abstract. In the present paper the relation between two definitions of Morrey spaces is discussed. This paper will be an announcement of the paper "Liguang Liu, Yoshihiro Sawano and Dachun Yang, Morrey-type Spaces on Gauss Measure Spaces and Boundedness of Singular Integrals, submitted".

1 Introduction

We propose a definition of Morrey spaces when we are given a Radon measure on $\mathbb{R}^n$.

Definition. Let $f \in L^q_{\rm loc}(\mu)$.

$$\|f\|_{\mathcal{M}^p_q(k,\mu)} := \sup_{Q \in \mathcal{Q}(\mu)} \mu(k\,Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}}. \tag{1}$$

Denote by $\mathcal{M}^p_q(k,\mu)$ the set of all $f \in L^q_{\rm loc}(\mu)$ for which $\|f\|_{\mathcal{M}^p_q(k,\mu)} < \infty$.

The parameter $k > 1$ appearing in the definition does not affect the definition of the space. More precisely, we have the following proposition, which will be a key to our arguments throughout this paper.

Proposition. Let $k_1, k_2 > 1$. Then $\mathcal{M}^p_q(k_1,\mu) \approx \mathcal{M}^p_q(k_2,\mu)$, that is, $\mathcal{M}^p_q(k_1,\mu)$ and $\mathcal{M}^p_q(k_2,\mu)$ coincide as a set and their norms are mutually equivalent.

Proof. Let $k_1 \leqslant k_2$. Then the inclusion $\mathcal{M}^p_q(k_1,\mu) \subset \mathcal{M}^p_q(k_2,\mu)$ is obvious by the definition of the norms. Let us show the reverse inclusion. Let $f \in \mathcal{M}^p_q(k_2,\mu)$ and $Q \in \mathcal{Q}(\mu)$. Then we have to estimate

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$$\mu(k_1 Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x)|^q \, d\mu(x) \right)^{\frac{1}{q}}.$$

A simple geometric observation shows that there exist cubes Q_1, Q_2, ..., Q_N with the same sidelength such that

$$Q \subset \bigcup_{i=1}^N Q_i, \quad k_2 Q_i \subset k_1 Q \, (i = 1, 2, \ldots, N) \text{ and } N \leqslant C \left(\frac{k_2 - 1}{k_1 - 1} \right)^n.$$

Using this covering, we easily obtain

$$\mu(k_1 Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x)|^q \, d\mu(x) \right)^{\frac{1}{q}} \leqslant \sum_{i=1}^N \mu(k_1 Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_i} |f(x)|^q \, d\mu(x) \right)^{\frac{1}{q}} \leqslant$$

$$\leqslant \sum_{Q_i \in \mathcal{Q}(\mu)} \mu(k_2 Q_i)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_i} |f(x)|^q \, d\mu(x) \right)^{\frac{1}{q}} \leqslant N \|f\|_{\mathcal{M}^p_q(k_2,\mu)}.$$

Example. Place ourselves in the setting of $\mathbb{R}^2$ with a Radon measure $\mu(x_1, x_2) := e^{2x_2} \, dx_1 \, dx_2$. Then we have

$$\mathcal{M}^2_1(1, \mu) \subset \mathcal{M}^2_1(2, \mu)$$

and the inclusion is strict.

Proof. We take $f := e^{-x_2}$ and for $a, b \in \mathbb{R}$, $h > 0$ we set $Q = Q((a, b), h)$. Then we have

$$\frac{1}{\sqrt{\mu(Q)}} \int_Q f \, d\mu = \frac{\int_Q e^{-x_2} \, dx_1 \, dx_2}{\sqrt{\int_Q e^{-2x_2} \, dx_1 \, dx_2}} = \frac{2h \, e^b (e^h - e^{-h})}{\sqrt{h \, e^{2b} (e^{2h} - e^{-2h})}} =$$

$$= 2 \sqrt{\frac{h \, (e^h - e^{-h})}{e^h + e^{-h}}} \to \infty, \; h \to \infty$$

$$\frac{1}{\sqrt{\mu(2Q)}}\int\limits_Q f\,d\mu = \frac{\int_Q e^{-x_2}\,dx_1\,dx_2}{\sqrt{\int_{2Q} e^{-2x_2}\,dx_1\,dx_2}} = \frac{2h\,e^b(e^h - e^{-h})}{\sqrt{2h\,e^{2b}(e^{4h} - e^{-4h})}} =$$

$$= \sqrt{\frac{2h\,(e^h - e^{-h})}{e^{3h} + e^h + e^{-h} + e^{-3h}}} \to 0,\ h \to \infty$$

showing that $\mathcal{M}_1^2(1,\mu)$ is a proper subset of $\mathcal{M}_1^2(2,\mu)$. $\square$

Example-a special case of the Gauss measure

Here we present an example.

Here and below we define $\gamma = \pi^{-n/2}\exp(-|x|^2)\,dx$ on $\mathbb{R}^n$. The measure γ has a lot to do with the Ornstein-Uhlenbeck process. Denote by

$$\mathcal{B} := \{B(x,r)\,:\, x \in \mathbb{R}^n,\ r \leqslant \min(1, |x|^{-1})\}.$$

One defines

$$\|f\|_{\mathcal{M}_q^p(\mathcal{B},\gamma)} := \sup_{B\in\mathcal{B}} \gamma(k\,B)^{\frac{1}{p}-\frac{1}{q}} \left(\int\limits_B |f(y)|^q\,d\gamma(y)\right)^{\frac{1}{q}}.$$

Denote by $\mathcal{M}_q^p(\mathcal{B},\gamma)$ the set of all $f \in L^q_{\text{loc}}(\gamma)$ for which $\|f\|_{\mathcal{M}_q^p(\mathcal{B},\gamma)} < \infty$.

Theorem. *(i) If $1 \leqslant q = p < \infty$, then $\mathcal{M}_q^p(2,\gamma)$ is a proper subset of $\mathcal{M}_q^p(\mathcal{B},\gamma)$.*

(ii) If $1 \leqslant q < p < \infty$, then $\mathcal{M}_q^p(\mathcal{B},\gamma)$ and $\mathcal{M}_q^p(2,\gamma)$ are isomorphic.

To see (i), for any $t > 0$ and $x \in \mathbb{R}^n$, set $h_t(x) := t^{n/q}e^{(1-t^2)|x|^2/q}$. Then, for all $t > 0$, an easy calculation leads to

$$\|h_t\|^q_{L^q(\gamma)} = \gamma(\mathbb{R}^n) = 1 \tag{2}$$

and that when $t \to 0$,

$$\|h_t\|^q_{\mathcal{M}_q^p(\mathcal{B},\gamma)} = t^n \sup_{B\in\mathcal{B}} \int\limits_B e^{-|tx|^2}\,dx \leqslant t^n \sup_{B\in\mathcal{B}} |B| = t^n|B(0,a)| \to 0. \tag{3}$$

From (2) and (3), it follows that $\mathcal{M}_q^p(\mathcal{B},\gamma)$ can not be embedded into $L^q(\gamma)$.

We outline the proof of (ii).

Let $a \in (0,\infty)$. Any ball $B \in \mathcal{B}$ is said to be *maximal* if $r_B = am(c_B)$. For each maximal ball $B \in \mathcal{B}$ which does not contain the origin, we denote by $M(B)$ the maximal ball in $\mathcal{B}$ with center at a point on the segment $[0, c_B] = \{tc_B : t \in [0,1]\}$ such that the boundary of $M(B)$ contains c_B, and we call $M(B)$ the *mother* of B. In other words, the relation between B and its mother $M(B)$ is as follows:

$$r_{M(B)} = am(c_{M(B)}), \qquad |c_{M(B)}| + r_{M(B)} = |c_B| \quad \text{and} \quad c_{M(B)} = \frac{|c_{M(B)}|}{|c_B|} c_B.$$

For notational convenience, we set $M^0(B) := B$. If $M(B)$ does not contain the origin, then we may consider the *mother of* $M(B)$, which we denote by $M^2(B)$. Therefore, given any maximal ball B in $\mathcal{B}$, we may find a *chain of maximal balls*, $B, M(B), M^2(B), \cdots, M^k(B)$, with the property that $M^j(B)$ is the mother of $M^{j-1}(B)$ for $j \in \{1, \cdots, k\}$, and $M^k(B)$ contains the origin.

We can establish some subtle geometric relations between the maximal admissible balls and their mothers.

Lemma 1. *Let $a \in (0, \infty)$. If the maximal ball $B \in \mathcal{B}$ satisfies that $0 \notin B$, then $B \subset (a+2)M(B)$ and $M(B) \subset (2a+2)B$. Consequently, $\gamma(M(B)) \sim \gamma(B)$ with implicit constants depending only on a and n.*

Proof. Let $r_{M(B)}$ and $c_{M(B)}$ be the *radius* and *center* of the ball $M(B)$, respectively. By the definition of $M(B)$, we have $r_{M(B)} = am(c_{M(B)})$, $r_B = am(c_B)$ and $|c_{M(B)}| + r_{M(B)} = |c_B|$. Using the fact that c_B is on the boundary of $M(B)$, together with the continuity of m, we see that

$$(a+1)^{-1} m(c_{M(B)}) \leqslant m(c_B) \leqslant (a+1) m(c_{M(B)}),$$

and hence $(a+1)^{-1} r_{M(B)} \leqslant r_B \leqslant (a+1) r_{M(B)}$. This implies that for any $z \in B$,

$$|z - c_{M(B)}| \leqslant |z - c_B| + |c_B - c_{M(B)}| < r_B + r_{M(B)} \leqslant (a+2) r_{M(B)},$$

that is, $B \subset (a+2)M(B)$. Meanwhile, for any $z \in M(B)$, we have

$$|z - c_B| \leqslant |z - c_{M(B)}| + |c_{M(B)} - c_B| < 2r_{M(B)} \leqslant 2(a+1) r_B.$$

which implies that $M(B) \subset (2a+2)B$. Furthermore, we conclude that $\gamma(B) \leqslant \gamma((a+1)M(B)) \lesssim \gamma(M(B))$ and $\gamma(M(B)) \leqslant \gamma((2a+2)B) \lesssim \gamma(B)$, which completes the proof of the lemma. □

2 Boundedness of the maximal operator

In this section we shall investigate some maximal inequalities. In proving the maximal inequalities we do not need the growth condition on μ. For $\kappa > 1$ we define the modified maximal operator M_κ by

$$M_\kappa f(x) := \sup_{x\in Q\in\mathcal{Q}(\mu)} \frac{1}{\mu(\kappa Q)} \int_Q |f(y)|\, d\mu(y).$$

The following boundedness of M_κ will be used in the proof of the main theorem of this section.

Lemma 2. *If $\kappa > 1$ and $1 < p \leqslant \infty$, then $\|M_\kappa f\|_{L^p(\mu)} \leqslant C_{d,p,\kappa}\|f\|_{L^p(\mu)}$. If $k, \kappa > 1$ and $1 < q \leqslant p < \infty$, then $\|M_\kappa f\|_{\mathcal{M}^p_q(k,\mu)} \leqslant C_{d,p,q,\kappa,k}\|f\|_{\mathcal{M}^p_q(k,\mu)}$.*

Proof. Fix $Q_0 \in \mathcal{Q}(\mu)$ and put $L := \ell(Q_0)/2$. Let $f_1 := \chi_{\frac{\kappa+7}{\kappa-1}Q_0} f$ and $f_2 := f - f_1$. Then for all $y \in Q_0$ we have

$$M_\kappa f(y) \leqslant M_\kappa f_1(y) + M_\kappa f_2(y). \tag{4}$$

It follows from the definition of M_κ that

$$M_\kappa f_2(y) \leqslant \sup_{\substack{y\in Q\in\mathcal{Q}(\mu)\\ \ell(Q)\geqslant 8L/(\kappa-1)}} \frac{1}{\mu(\kappa Q)} \int_Q |f(x)|\, d\mu(x).$$

Suppose that $y \in Q_0$, $y \in Q \in \mathcal{Q}(\mu)$ and $\ell(Q) \geqslant 8L/(\kappa - 1)$. Then simple calculus yields $Q_0 \subset \dfrac{1+\kappa}{2}Q$. Thus, we obtain

$$M_\kappa f_2(y) \leqslant \sup_{Q_0\subset Q\in\mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{2\kappa}{\kappa+1}Q\right)} \int_Q |f(x)|\, d\mu(x). \tag{5}$$

The lemma above, (1), (2) and Hölder's inequality yield

$$\mu\left(\frac{2\kappa(\kappa+7)}{\kappa^2-1}Q_0\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_0} (M_\kappa f(y)^q\, d\mu(y)\right)^{\frac{1}{q}} \leqslant$$

$$\leqslant \mu\left(\frac{2\kappa(\kappa+7)}{\kappa^2-1}Q_0\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\mathbb{R}^n} M_\kappa f_1(y)^q\, d\mu(y)\right)^{\frac{1}{q}} +$$

$$+ \mu(Q_0)^{\frac{1}{p}-\frac{1}{q}} \cdot \left(\int_{Q_0} M_\kappa f_2(y)^q \, d\mu(y) \right)^{\frac{1}{q}} \leqslant$$

$$\leqslant \mu \left(\frac{2\kappa(\kappa+7)}{\kappa^2-1} \, Q_0 \right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\mathbb{R}^n} M_\kappa f_1(y)^q \, d\mu(y) \right)^{\frac{1}{q}} +$$

$$+ \sup_{\substack{Q_0 \subset Q \\ Q_0 \in \mathcal{Q}(\mu)}} \frac{\mu(Q_0)^{\frac{1}{p}}}{\mu\left(\frac{2\kappa}{\kappa+1}\, Q\right)} \int_Q |f(y)| \, d\mu(y).$$

If we use the Hölder inequality, then we obtain

$$\mu \left(\frac{2\kappa(\kappa+7)}{\kappa^2-1} \, Q_0 \right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_0} (M_\kappa f(y)^q \, d\mu(y) \right)^{\frac{1}{q}} \leqslant$$

$$\leqslant C \, \mu \left(\frac{2\kappa(\kappa+7)}{\kappa^2-1} \, Q_0 \right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\frac{\kappa+7}{\kappa-1} Q_0} |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}} +$$

$$+ C' \sup_{Q_0 \subset Q \in \mathcal{Q}(\mu)} \mu \left(\frac{2\kappa}{\kappa+1} \, Q \right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}} \leqslant C \, \|f\|_{\mathcal{M}^p_q(2\kappa/(\kappa+1),\mu)}.$$

Hence we have $\|M_\kappa f\|_{\mathcal{M}^p_q(2\kappa(\kappa+7)/(\kappa^2-1),\mu)} \leqslant C\|f\|_{\mathcal{M}^p_q(2\kappa/(\kappa+1),\mu)}$. Using the fact that $\mathcal{M}^p_q(\kappa,\mu)$ does not depend on $\kappa > 1$, we obtain the conclusion of the theorem. $\square$

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FAST LAGUERRE PROJECTION METHOD FOR FINITE HANKEL TRANSFORM OF ARBITRARY ORDER

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Key words: Laguerre functions, Hankel transform, finite Hankel transform, Laguerre projection method, fast Laguerre projection method

AMS Mathematics Subject Classification: 11B39

Abstract. The fast Laguerre projection methods for the inversion of finite Hankel transform of arbitrary order and Hankel transform of arbitrary order are introduced. The proposed effective technique for projection coefficients computation is based on the Gauss-Laguerre quadrature. The test results showed the effectiveness of proposed computational algorithm as well as its good approximation quality.

1 Introduction

The general scheme of the projection method for solving type I linear equations $Az = u$ in Hilbert space is based on expanding the solution in a series of the eigenfunctions of self-adjoint operator A^*A [1]. Projection method is well applicable for image processing tasks where it was used for filtration and parametrization of data [2].

Let us consider the following equation

$$Az = \int_0^a z(x) J_\alpha(kx)\sqrt{kx}\, dx = u(k), \tag{1}$$

$$A\colon L_2[0,a] \to L_2[0,a], \quad 0 < a < \infty, \tag{2}$$

where the right part is given approximately and $J_\alpha(x)$ is Bessel function of the first kind of order α. There is a classic projection method for this kind of equation presented in [1]. It is based on the expansion of the solution into the set of eigenfunctions of operator A^*A. However there is a special feature of (1). The operator A has computationally multiple eigenvalues for sufficiently large a. They are equivalent for calculations with computer precision. This fact can lead to the loss of the approximation quality of the solution in the case when the number of

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eigenfunctions corresponding to this eigenvalue used in the projection method is less then the computational multiplicity of the eigenvalue.

The modification of this projection method for (1) was presented in [3]. This modification is based on the replacement of the eigenfunctions which correspond to multiple eigenvalue by Laguerre functions.

Laguerre functions are defined as:

$$\psi_n^\alpha(x) = \frac{1}{\sqrt{n!\Gamma(n+\alpha+1)}} x^{\alpha/2} e^{-x/2} L_n^\alpha(x),$$

where $L_n^\alpha(x)$ are Laguerre polynomials: $L_n^\alpha(x) = (-1)^n x^{-\alpha} e^x \dfrac{d}{dx^n}(x^{n+\alpha}e^{-x})$. They form an orthonormal system in $L_2[0,\infty)$. At the same time, from a computational point of view, each of these functions has a finite support.

The functions $\Psi_n^\alpha(x) = \sqrt{2x}\psi_n^\alpha(x^2)$ are the eigenfunctions of Hankel transform of order α (operator A for $a=\infty$).

In [4] the projection method for the inversion of the following integral transform was presented:

$$Hz = \int_0^\infty z(x) J_\alpha(kx)\sqrt{kx}\,dx = u(k), \tag{3}$$

$$H\colon L_2[0,\infty] \to L_2[0,a], \quad 0 < a < \infty, \tag{4}$$

where the right part is given approximately and $J_\alpha(x)$ is Bessel function of the first kind of order α. This method is based on the expansion of the right part of (3) into the set of Laguerre functions being eigenfunctions of operator $H\colon L_2[0,\infty] \to L_2[0,\infty]$. The theorem which gives the criterion of choosing the number of functions for the projection method was proved in [4].

Computing the projection coefficients for (1) and (3) the following equations occurres:

$$u_i = \int_0^a u_\delta(k)\Psi_i^\alpha(k)\,dk, \tag{5}$$

where $u_\delta(k)$ is the given approximation of the right part $u(k)$:

$$\|u(k) - u_\delta(k)\|_{L_2[0,a]} \leqslant \delta.$$

In the case of sufficiently large a the norms $\|\Psi_i^\alpha(x)\|_{L_2[a,\infty)}$ are close to the zero [3] and

$$u_i = \int_0^a u_\delta(k)\Psi_i^\alpha(k)\,dk \approx \int_0^\infty u_\delta(k)\Psi_i^\alpha(k)\,dk. \tag{6}$$

In this paper we suggest the fast algorithms for computation of projection coefficients (5). These methods are based on the the Gauss-Laguerre quadrature for integrals computation.

2 Fast algorithm for projection coefficients computation

Let us consider the equation

$$u_n \approx \int_0^\infty u_\delta(x)\Psi_n^\alpha(x)\,dx = \int_0^\infty u_\delta(x)\sqrt{2x}\psi_n^\alpha(x^2)\,dx =$$
$$= \int_0^\infty u_\delta(x)(2x)^{-\frac{1}{2}}\psi_n^\alpha(x^2)\,d(x^2). \tag{7}$$

Denote $x^2 = t$ and $\beta_n^\alpha = \sqrt{n!\Gamma(n+\alpha+1)}$ then

$$\int_0^\infty u_\delta(x)(2x)^{-\frac{1}{2}}\psi_n^\alpha(x^2)\,d(x^2) = \frac{1}{\sqrt{2}}\int_0^\infty u_\delta(\sqrt{t})t^{-\frac{1}{4}}\psi_n^\alpha(t)\,dt =$$
$$= \frac{1}{\sqrt{2}}\int_0^\infty t^\alpha e^{-t}\left(u_\delta(\sqrt{t})\frac{1}{\beta_n^\alpha}t^{-\frac{\alpha}{2}-\frac{1}{4}}e^{\frac{t}{2}}L_n^\alpha(t)\right)dt \tag{8}$$

Considering (8) the initial equation (7) can be rewritten as

$$u_n \approx \frac{1}{\sqrt{2}}\int_0^\infty t^\alpha e^{-t}f(t)\,dt, \tag{9}$$

where

$$f(t) = u_\delta(\sqrt{t})\frac{1}{\beta_n^\alpha}t^{-\frac{\alpha}{2}-\frac{1}{4}}e^{\frac{t}{2}}L_n^\alpha(t). \tag{10}$$

Integral (9) can be approximated using Gauss-Laguerre quadrature [5]:

$$\int_0^\infty t^\alpha e^{-t} f(t)\,dt \approx \sum_{k=1}^{N} A_k f(t_k) + R_N, \tag{11}$$

where

$$A_k = \frac{(\beta_N^\alpha)^2 t_k}{\left(L_{N+1}^\alpha(t_k)\right)^2} \tag{12}$$

and t_k are the zeros of Laguerre polynomial L_N^α.

The A_k coefficient for $\alpha = 0$ and different values of t_k and N are presented below [5] in Tab. 1.

Table 1

The A_k coefficient for $\alpha = 0$ and different values of t_k and N

$N = 1$	
$t_1 = 1.000\ 000\ 000\ 000$	$A_1 = 1.000\ 000\ 000\ 000$
$N = 2$	
$t_1 = 0.585\ 786\ 437\ 627$	$A_1 = 0.853\ 553\ 390\ 593$
$t_2 = 3.414\ 213\ 562\ 373$	$A_2 = 0.146\ 446\ 609\ 407$
$N = 7$	
$t_1 = 0.193\ 043\ 676\ 560$	$A_1 = 0.409\ 318\ 951\ 701$
$t_2 = 1.026\ 664\ 895\ 339$	$A_2 = 0.421\ 831\ 277\ 862$
$t_3 = 2.567\ 876\ 744\ 951$	$A_3 = 0.147\ 126\ 348\ 658$
$t_4 = 4.900\ 353\ 084\ 526$	$A_4 = 0.(1)\ 206\ 335\ 144\ 687$
$t_5 = 8.182\ 153\ 444\ 563$	$A_5 = 0.(2)\ 107\ 401\ 014\ 328$
$t_6 = 12.734\ 180\ 291\ 798$	$A_6 = 0.(4)\ 158\ 654\ 643\ 486$
$t_7 = 19.395\ 727\ 862\ 263$	$A_7 = 0.(7)\ 317\ 031\ 547\ 900$

It can be seen that direct computation of A_k coefficients in (12) leads to the loss in approximation precision and increase of computation complexity. The same problem for Gauss-Hermite quadrature coefficients did not give possibility to W. F. Eberlein [6] to implement his idea of acceleration of Fourier transform.

However, taking into consideration (10) equation (11) can be transformed as

$$\int_0^\infty t^\alpha e^{-t} f(t)\,dt \approx \sum_{k=1}^{N} A_k f(t_k) + R_N = \sum_{k=1}^{N} \frac{(\beta_N^\alpha)^2 t_k u_\delta(\sqrt{t_k}) t_k^{-\frac{\alpha}{2}-\frac{1}{4}} e^{\frac{t_k}{2}} L_n^\alpha(t_k)}{\left(L_{N+1}^\alpha(t_k)\right)^2 \beta_n^\alpha} + R_N =$$

$$= \sum_{k=1}^{N} \frac{u_\delta(\sqrt{t_k}) t_k^{\frac{3}{4}} \psi_n^\alpha(t_k)}{\left(\frac{\beta_{N+1}^\alpha}{\beta_N^\alpha}\right)^2 \left(\psi_{N+1}^\alpha(t_k)\right)^2} + R_N = \sum_{k=1}^{N} \frac{u_\delta(\sqrt{t_k}) t_k^{\frac{3}{4}} \psi_n^\alpha(t_k)}{(N+1)(N+\alpha+1)\left(\psi_{N+1}^\alpha(t_k)\right)^2} + R_N\,. \quad (13)$$

Thus the coefficients u_n can be efficiently calculated as

$$u_n \approx \frac{1}{\sqrt{2}} \sum_{k=1}^{N} u_\delta(\sqrt{t_k}) \mu_{N+1}^n\,, \quad (14)$$

where

$$\mu_N^n = \frac{t_k^{\frac{3}{4}} \psi_n^\alpha(t_k)}{N(N+\alpha)\left(\psi_N^\alpha(t_k)\right)^2}\,. \quad (15)$$

It is important to mention that the approximation of $u_\delta(\sqrt{t_k})$ can be performed using linear interpolation for sufficiently smooth data as t_k are not very dense distributed.

The presented computation technique accelerates the computation in several times. If the number of points in the initial uniform grid is M then the number of multiplications to compute coefficient u_i using quadrature formula on the uniform grid is proportional to M. Using the proposed fast algorithm the number of multiplications to compute coefficient u_i can be reduced to N. The value of N can be chosen according to the needed approximation quality.

3 Fast projection method for finite Hankel transform of arbitrary order

We used the suggested fast method to accelerate the scheme of the modified projection method for (1) proposed in [3]. The modified projection method looks as:

1. Compute the eigenfunctions $\{\varphi_i\}$ and the eigenvalues $\{\lambda_i\}$ of operator A^*A:

$$A^*A\varphi_i = \lambda_i\varphi_i, \quad \lambda_i \geqslant \lambda_j > 0, \text{ for } i < j, \quad i = 0, 1, \ldots,\ j = 0, 1, \ldots.$$

2. Compute the functions:

$$\hat{\varphi}_i = \frac{A\varphi_i}{\|A\varphi_i\|}\,.$$

3. Compute the Fourier coefficients:

$$u_i = \int_0^a u_\delta(k)\Psi_i^\alpha(k)\,dk \quad \text{for } i = 0, ..., N_1$$

where $N_1 = \min(N_\delta, N_\lambda)$:

$$N_\delta > 0 : \ \| \sum_{i=0}^{N_\delta} u_i \Psi_i^\alpha - u_\delta \| \leqslant q\delta, \quad q > 1$$

and N_λ is the number of $\lambda_i \approx 1$.

4. Compute the Fourier coefficients:

$$\hat{u}_i = \int_0^a u_\delta(k) \hat{\varphi}_i(k)\, dk \quad \text{for } i = N_1 + 1, ..., N_2$$

where

$$N_2 > 0 : \| \sum_{i=0}^{N_1} u_i \Psi_i^\alpha + \sum_{i=N_1+1}^{N_2} \hat{u}_i \hat{\varphi}_i - u_\delta \| \leqslant q\delta, \quad q > 1. \tag{16}$$

5. Compute the solution as a partial sum of the Fourier series

$$z_\delta = \sum_{i=0}^{N_1} z_i \Psi_i^\alpha + \sum_{i=N_1+1}^{N_2} z_i \hat{\varphi}_i, \quad z_i = \begin{cases} (-1)^i u_i, & \text{for } \ i = 0, ..., N_1 \\ \dfrac{1}{\sqrt{\lambda_i}} \hat{u}_i, & \text{for } \ i = N_1 + 1, ..., N_2 \end{cases}$$

To compare the results obtained by projection method and its fast modification we performed the calculations for (1) with the model function shown in Fig. 1 (a). The calculations were performed for $a = 10$ and $\alpha = 0$. To model the real data situation the uniformly distributed noise with $\delta = 0.49$ was added to the right part $u(x)$ of the equation (1). The comparison between standard method and projection method was given in [3] and it was shown that the projection method allows to achieve more accurate approximation results with lower number of functions in the expansion. The comparison of solutions obtained by projection method and fast projection method is given in Fig. 1 (a). The approximation error for the solution obtained by projection method is 0.180. The approximation error for the solution obtained by fast projection method is 0.244. The number of Laguerre functions to obtain these results was $N = 9$. The number of computationally multiple eigenvalues was $N_\lambda = 17$. The graphs illustrate that the result obtained by the fast algorithm has higher oscillations on the interval $x \in [0, 2]$ than the result obtained by ordinary projection method. But the computational speed in the case of using fast modification is about 10 times higher.

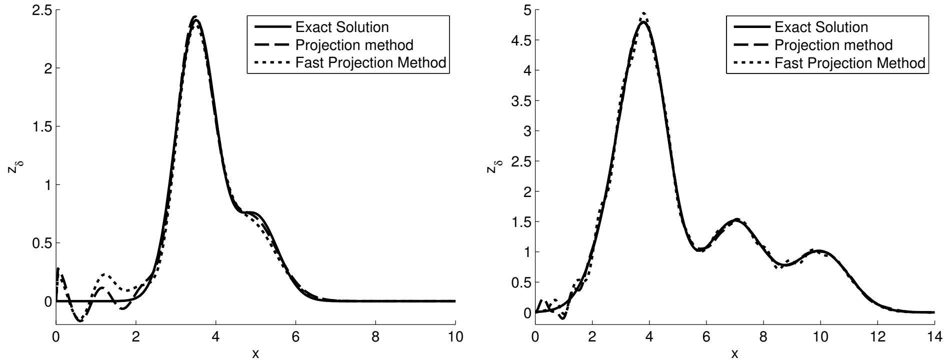

Figure 1. Solution approximation results for model functions for the inversion of finite Hankel transform (a) and Hankel transform (b)

4 Fast projection method for Hankel transform of arbitrary order

Projection method for (3) was proposed in [4]. By this method we choose the number of the Fourier series terms N according to the theorem proposed in [4].

To compare the results obtained by projection method and its fast modification we perform the calculations for (1) with the model function shown in Fig. 1 (b). The calculations were performed for $a = 14$ and $\alpha = 2$. To model the real data situation the uniformly distributed noise with $\delta = 0.53$ was added to the right part $u(x)$ of the equation (3). The comparison of solutions obtained by projection method and fast projection method is given in Fig. 1 (b). The comparison between standard method and projection method was given in [4]. The approximation error for the solution obtained by projection method is 0.172. The approximation error for the solution obtained by fast projection method is 0.261. The number of Laguerre functions to obtain these results was $N = 41$. One can see that the results of the proposed fast projection method are close to the results of the projection method while the computational speed of the fast projection method is about 10 times higher.

5 Conclusion

Fast algorithm for computation of Laguerre projection coefficients has been proposed. The method can be used for inversion of finite Hankel transform of arbitrary order and Hankel transform of arbitrary order. The test results showed the effectiveness of proposed computational algorithm as well as its good approximation quality. The future work will include the comparison of proposed technique with the method based on the Gauss-Hermite quadrature which can be used in the case of less smooth initial data.

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GENERALIZED CONVOLUTIONS OF THE INTEGRAL TRANSFORM OF FOURIER TYPE AND APPLICATIONS

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Key words: Hermite functions, generalized convolution, normed rings

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Abstract. In this report we construct the infinitely many generalized convolutions related to the Hermite functions, and consider an application for structureing the normed rings on $L^1(\mathbb{R}^d)$. There is a fact (interesting, perhaps) most of those normed rings are commutative.

1 Introduction

The theory of integral transforms has been developed for a long time, and applied to many fields of mathematics. In recent years, many papers and books devoting to applications of integral transforms have been published (see [2, 3, 5, 8–11, 11, 13–16] and references therein). Among those studied intensively are the integral transforms of Fourier type.

Having considered recently vigorous discussion about integral transforms, we hereinafter present the following transform

$$(\mathcal{F}f)(x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int\limits_{\mathbb{R}^d} (2\cos xy + \sin xy) f(y)dy, \tag{1}$$

and its inverse transform

$$(\mathcal{F}^{-1}f)(x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int\limits_{\mathbb{R}^d} \left(\frac{1}{2}\cos xy + \sin xy\right) f(y)dy. \tag{2}$$

The main aim of this paper is to present new generalized convolutions $\mathcal{F}$ related to Hermite functions and considers an application in normed rings of the Banach space $L^1(\mathbb{R}^d)$.

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2 Generalized convolutions

To begin with, we formulate the theorem relating to Hermite functions. The multi-dimensional Hermite functions are defined by $\Phi_\alpha(x) := (-1)^{|\alpha|} e^{\frac{1}{2}|x|^2} D_x^\alpha e^{-|x|^2}$ (see [12]).

Theorem 1. *The following formula holds*

$$\mathcal{F}\Phi_\alpha = \begin{cases} (-1)^{\frac{|\alpha|}{2}} 2\Phi_\alpha, & if \quad |\alpha| \quad is\ even \\ (-1)^{\frac{|\alpha|-1}{2}} \Phi_\alpha, & if \quad |\alpha| \quad is\ odd. \end{cases} \tag{3}$$

Proof. By considering $L^1(\mathbb{R}^d)$ as a domain of $\mathcal{F}$, $\mathcal{F}^{-1}$ and $\mathcal{F}$ the following identity holds

$$\mathcal{F} = \frac{2+i}{2}\mathcal{F} + \frac{2-i}{2}\mathcal{F}^{-1}. \tag{4}$$

Since $\mathcal{F}\Phi_\alpha = (-i)^{|\alpha|}\Phi_\alpha$ and $\mathcal{F}^{-1}\Phi_\alpha = (i)^{|\alpha|}\Phi_\alpha$, we have

$$\mathcal{F}\Phi_\alpha = \left[\frac{2+i}{2}(-i)^{|\alpha|} + \frac{2-i}{2}(i)^{|\alpha|}\right]\Phi_\alpha.$$

Calculating the coefficient in the right-side of this equality we obtain (3). The theorem is proved. □

For given $f \in L^1(\mathbb{R}^d)$ define the norm $\|f\|_0 := \frac{1}{(2\pi)^{\frac{d}{2}}} \int\limits_{\mathbb{R}^d} |f(x)|dx$. Put $N_\alpha := \|\Phi_\alpha\|_0$ for given Hermite function Φ_α. It is known that $L^1(\mathbb{R}^d)$ becomes a Banach space by the norm $\|\cdot\|_0$. Let $|\alpha| = r \pmod 4$ where $r \in \{0,1,2,3\}$.

Theorem 2 (main theorem). *If $f, g \in L^1(\mathbb{R}^d)$, then each of the transforms below defines a generalized convolution followed by its normed inequality and factorization identity.*

- *The case $r \in \{0,2\}$.*

$$\left(f \overset{\Phi_\alpha}{\underset{\mathcal{F}}{*}} g\right)(x) = \frac{(-1)^{\frac{r}{2}}}{8(2\pi)^d} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(u)g(v)\left[11\Phi_\alpha(x-u-v) - 5\Phi_\alpha(x+u+v) + \right.$$

$$\left. + 5\Phi_\alpha(x+u-v) + 5\Phi_\alpha(x-u+v)\right] dudv, \tag{5}$$

$$\left\| f \overset{\Phi_\alpha}{\underset{\mathcal{F}}{*}} g \right\|_0 \leqslant \frac{13N_\alpha}{4}\|f\|_0 \cdot \|g\|_0,$$

$$\mathcal{F}\left(f \overset{\Phi_\alpha}{\underset{\mathcal{F}}{*}} g\right)(x) = \Phi_\alpha(x)(\mathcal{F}f)(x)(\mathcal{F}g)(x) \left(f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F},\mathcal{F}^{-1}}{*}} g\right)(x) =$$
$$= \frac{(-1)^{\frac{r}{2}}}{8(2\pi)^d} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(v)g(u)\left[5\Phi_\alpha(x-u-v) - 5\Phi_\alpha(x+u+v) - \right.$$
$$\left. - \Phi_\alpha(x+u-v) + 5\Phi_\alpha(x-u+v)\right] dudv, \quad (6)$$

$$\left\| f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F},\mathcal{F}^{-1}}{*}} g \right\|_0 \leqslant 2N_\alpha \|f\|_0 \|g\|_0,$$
$$\mathcal{F}\left(f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F},\mathcal{F}^{-1}}{*}} g\right)(x) = \Phi_\alpha(x)(\mathcal{F}f)(x)(\mathcal{F}^{-1}g)(x) \left(f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F}^{-1},\mathcal{F}}{*}} g\right)(x) =$$
$$= \frac{(-1)^{\frac{r}{2}}}{8(2\pi)^d} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(v)g(u)\left[5\Phi_\alpha(x-u-v) - 5\Phi_\alpha(x+u+v) + \right.$$
$$\left. + 5\Phi_\alpha(x+u-v) - \Phi_\alpha(x-u+v)\right] dudv, \quad (7)$$

$$\left\| f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F}^{-1},\mathcal{F}}{*}} g \right\|_0 \leqslant 2N_\alpha \|f\|_0 \|g\|_0,$$
$$\mathcal{F}\left(f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F}^{-1},\mathcal{F}}{*}} g\right)(x) = \Phi_\alpha(x)(\mathcal{F}^{-1}f)(x)(\mathcal{F}g)(x) \left(f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F}^{-1},\mathcal{F}^{-1}}{*}} g\right)(x) =$$
$$= \frac{(-1)^{\frac{r}{2}}}{32(2\pi)^d} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(v)g(u)\left[5\Phi_\alpha(x-u-v) - 11\Phi_\alpha(x+u+v) + \right.$$
$$\left. + 5\Phi_\alpha(x+u-v) + 5\Phi_\alpha(x-u+v)\right] dudv, \quad (8)$$

$$\left\| f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F}^{-1},\mathcal{F}^{-1}}{*}} g \right\|_0 \leqslant \frac{13N_\alpha}{16} \|f\|_0 \|g\|_0,$$
$$\mathcal{F}\left(f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F}^{-1},\mathcal{F}^{-1}}{*}} g\right)(x) = \Phi_\alpha(x)(\mathcal{F}^{-1}f)(x)(\mathcal{F}^{-1}g)(x). \quad (9)$$

- *The case $r \in \{1, 3\}$.*

$$\left(f \overset{\Phi_\alpha}{\underset{\mathcal{F}}{*}} g\right)(x) = \frac{(-1)^{\frac{r-1}{2}}}{4(2\pi)^d} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(v)g(u)\left[\Phi_\alpha(x-u-v) + 5\Phi_\alpha(x+u+v) + \right.$$
$$\left. + 5\Phi_\alpha(x+u-v) + 5\Phi_\alpha(x-u+v)\right] dudv, \quad (10)$$

$$\left\| f \overset{\Phi_\alpha}{\underset{\mathcal{F}}{*}} g \right\|_0 \leqslant 4N_\alpha \|f\|_0 \|g\|_0,$$

$$\mathcal{F}\left(f \overset{\Phi_\alpha}{\underset{\mathcal{F}}{*}} g \right)(x) = \Phi_\alpha(x)(\mathcal{F}f)(x)(\mathcal{F}g)(x) \left(f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F},\mathcal{F}^{-1}}{*}} g \right)(x) =$$

$$= \frac{(-1)^{\frac{r-1}{2}}}{16(2\pi)^d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} f(v) g(u) \left[-5\Phi_\alpha(x-u-v) + 5\Phi_\alpha(x+u+v) + \right.$$

$$\left. + 11\Phi_\alpha(x+u-v) + 5\Phi_\alpha(x-u+v) \right] dvdu, \quad (11)$$

$$\left\| f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F},\mathcal{F}^{-1}}{*}} g \right\|_0 \leqslant \frac{13N_\alpha}{8} \|f\|_0 \|g\|_0,$$

$$\mathcal{F}\left(f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F},\mathcal{F}^{-1}}{*}} g \right)(x) = \Phi_\alpha(x)(\mathcal{F}f)(x)(\mathcal{F}^{-1}g)(x) \left(f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F}^{-1},\mathcal{F}}{*}} g \right)(x) =$$

$$= \frac{(-1)^{\frac{r-1}{2}}}{16(2\pi)^d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} f(v) g(u) \left[-5\Phi_\alpha(x-u-v) + 5\Phi_\alpha(x+u+v) + \right.$$

$$\left. + 5\Phi_\alpha(x+u-v) + 11\Phi_\alpha(x-u+v) \right] dvdu, \quad (12)$$

$$\left\| f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F}^{-1},\mathcal{F}}{*}} g \right\|_0 \leqslant \frac{13N_\alpha}{8} \|f\|_0 \|g\|_0,$$

$$\mathcal{F}\left(f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F}^{-1},\mathcal{F}}{*}} g \right)(x) = \Phi_\alpha(x)(\mathcal{F}^{-1}f)(x)(\mathcal{F}g)(x) \left(f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F}^{-1},\mathcal{F}^{-1}}{*}} g \right)(x) =$$

$$= \frac{(-1)^{\frac{r-1}{2}}}{16(2\pi)^d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} g(u) \left[-5\Phi_\alpha(x-u-v) - \Phi_\alpha(x+u+v) + \right.$$

$$\left. + 5\Phi_\alpha(x+u-v) + 5\Phi_\alpha(x-u+v) \right] dudv, \quad (13)$$

$$\begin{gathered} \left\| f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F}^{-1},\mathcal{F}^{-1}}{*}} g \right\|_0 \leqslant \|f\|_0 \|g\|_0, \\ \mathcal{F}\left(f \overset{\Phi_\alpha}{\underset{\mathcal{F},\mathcal{F}^{-1},\mathcal{F}^{-1}}{*}} g \right)(x) = \Phi_\alpha(x)(\mathcal{F}^{-1}f)(x)(\mathcal{F}^{-1}g)(x). \end{gathered} \quad (14)$$

Proof. By estimating the integral inequalities we can prove the normed inequality in (5) as: $\|f \overset{\Phi_\alpha}{\underset{\mathcal{F}}{*}} g\|_0 \leqslant \frac{13N_\alpha}{4} \|f\|_0 \cdot \|g\|_0$. We now prove the factorization identity. Using simultaneously two identities: $\Phi_\alpha = (-1)^{\frac{r}{2}} \mathcal{F}\Phi_\alpha = (-1)^{\frac{r}{2}} \mathcal{F}^{-1}\Phi_\alpha$, we have

$$\frac{\Phi_\alpha(x)}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}[2\cos x(u+v)+\sin x(u+v)]f(u)g(v)dudv=$$
$$=\frac{\Phi_\alpha(x)}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\left[\frac{2+i}{2}e^{-i\langle x,u+v\rangle}+\frac{2-i}{2}e^{i\langle x,u+v\rangle}\right]f(u)g(v)dudv=$$
$$=\frac{(-1)^{\frac{r}{2}}}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\left[\frac{2+i}{2}e^{-i\langle x,u+v+t\rangle}+\frac{2-i}{2}e^{i\langle x,u+v+t\rangle}\right]\Phi_\alpha(t)f(u)g(v)dudv=$$
$$=\frac{(-1)^{\frac{r}{2}}}{(2\pi)^{\frac{3d}{2}}}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\left[\frac{2+i}{2}e^{-i\langle x,y\rangle}+\frac{2-i}{2}e^{i\langle x,y\rangle}\right]\Phi_\alpha(y-u-v)f(u)g(v)dudvdy=$$
$$=\frac{(-1)^{\frac{r}{2}}}{(2\pi)^{\frac{3d}{2}}}\int\limits_{\mathbb{R}^d}(2\cos xy+\sin xy)\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\Phi_\alpha(y-u-v)f(u)g(v)dudvdy. \quad (15)$$

Replacing u with $-u$, v with $-v$, and afterward $f(-u)$ with $f(u)$, $g(-v)$ with $g(v)$, we obtain

$$\frac{\Phi_\alpha(x)}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}[2\cos x(u-v)+\sin x(u-v)]f(u)g(v)dudv=$$
$$=\frac{(-1)^{\frac{r}{2}}}{(2\pi)^{\frac{3d}{2}}}\int\limits_{\mathbb{R}^d}(2\cos xy+\sin xy)\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\Phi_\alpha(y-u+v)f(u)g(v)dudvdy; \quad (16)$$

$$\frac{\Phi_\alpha(x)}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}[2\cos x(-u+v)+\sin x(-u+v)]f(u)g(v)dudv=$$
$$=\frac{(-1)^{\frac{r}{2}}}{(2\pi)^{\frac{3d}{2}}}\int\limits_{\mathbb{R}^d}(2\cos xy+\sin xy)\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\Phi_\alpha(y+u-v)f(u)g(v)dudvdy; \quad (17)$$

$$\frac{\Phi_\alpha(x)}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}[2\cos x(-u-v)+\sin x(-u-v)]f(u)g(v)dudv=$$

$$= \frac{(-1)^{\frac{r}{2}}}{(2\pi)^{\frac{3d}{2}}} \int\limits_{\mathbb{R}^d} (2\cos xy + \sin xy) \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \Phi_\alpha(y+u+v) f(u) g(v) du dv dy. \quad (18)$$

On the other hand,

$$(2\cos xv + \sin xv)(2\cos xu + \sin xu) = \frac{11}{8}[2\cos x(u+v) + \sin x(u+v)] +$$
$$+ \frac{5}{8}[2\cos x(u-v) + \sin x(u-v)] + \frac{5}{8}[2\cos x(-u+v) + \sin x(-u+v)] -$$
$$- \frac{5}{8}[2\cos x(-u-v) + \sin x(-u-v)]. \quad (19)$$

Using (15)–(18), and (19) we get

$$\Phi_\alpha(x)(\mathcal{F}f)(x)(\mathcal{F}g)(x) = \frac{(-1)^{\frac{r}{2}}}{8(2\pi)^{\frac{3d}{2}}} \int\limits_{\mathbb{R}^d} (2\cos xy + \sin xy) \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \Big[11\Phi_\alpha(y-u-v) +$$
$$+ 5\Phi_\alpha(y-u+v) + 5\Phi_\alpha(y+u-v) - 5\Phi_\alpha(y+u+v) \Big] f(u) g(v) du dv dy = \mathcal{F}\left(f \overset{\Phi_\alpha}{\underset{\mathcal{F}}{*}} g \right)(x).$$

The factorization identity of (5) is proved.
We have to prove the theorem for the other convolutions. However, due to the limitted of the paper we leave the proof to the readers as those convolutions may be proved in the same way as the proof of (5). The theorem 2 is proved. □

3 Normed ring structures

In this section, we shall prove that *a half* of convolution multiplications in Theorem 1 consists of commutative convolutions. Moreover, all the above-mentioned convolution operators are continuous maps from $L^1(\mathbb{R}^d)$ into itself whenever f (or g) is fixed. Therefore, those convolutions are useful for constructing normed rings on $L^1(\mathbb{R}^d)$ that could be applied to theories of Banach algebras (see [1,2,4,6,7,16]).

Theorem 3. *The space* $X := L^1(\mathbb{R}^d)$, *equipped with each one of convolution multiplications* (5) *and with the norm* $\|f\|_\alpha = \frac{13N_\alpha}{4}\|f\|_0$, *or with each one of those* (6), (7) *and the norm* $\|f\|_\alpha = 2N_\alpha\|f\|_0$, *or with* (8) *and the norm* $\|f\|_\alpha = \frac{13}{16}N_\alpha\|f\|_0$, *or with each one of those* (10) *and the norm* $\|f\|_\alpha = 4N_\alpha\|f\|_0$, *or with each one of those* (11), (12) *and the norm* $\|f\|_\alpha = \frac{13}{8}N_\alpha\|f\|_0$, *or with* (13) *and the norm* $\|f\|_\alpha = N_\alpha\|f\|_0$, *becomes a normed ring. Moreover, for convolutions* (5), (8), (10), (13), *X is commutative.*

Proof. Obviously, X, equipped with one of the convolution multiplications (5)–(13) and with the above appropriate norm is a commutative ring. The multiplicative inequalities follow directly from the norm inequalities in (5)–(13). Therefore, X is a normed ring. We have to prove that X has no unit.

We shall prove the assertion for the convolution multiplication (5), and those for the others may be proved analogously.

Suppose that there exists an element $e \in X$ such that $f = f \overset{\Phi_0}{\underset{\mathcal{F}}{*}} e = e \overset{\Phi_0}{\underset{\mathcal{F}}{*}} f$ for every $f \in X$. As $\Phi_0 \in X$, $\Phi_0 = \Phi_0 \overset{\Phi_0}{\underset{\mathcal{F}}{*}} e = e \overset{\Phi_0}{\underset{\mathcal{F}}{*}} \Phi_0$. By using the factorization identity and Theorem 1, we obtain $\Phi_0 = \Phi_0^2 \mathcal{F} e$. Since $\Phi_0(x) \neq 0$ for every $x \in \mathbb{R}^d$, we have $\Phi_0(x)(\mathcal{F}e)(x) = 1$ for every $x \in \mathbb{R}^d$. But, this contradicts to the identity $\lim_{x\to\infty} \big[\Phi_0(x)(\mathcal{F}e)(x)\big] = 0$ that deduced from the Riemann-Lebesgue lemma and the fact $\Phi_0 \in \mathcal{S}$. Hence, X has no unit.

The commutativity of the convolutions (5), (8), (10), (13) are clearly. The theorem is proved. □

Remark. (1) Actually, we can prove the non-commutativity of convolution multiplications (6), (7), (11), (12). However, it does not need in any sense.

(2) Due to the limited pages of this report, this report does not present neither other applications of $\mathcal{F}$ nor of the constructed convolutions such as those in operator equations, eigen-functions, spectral radius, and in integral transforms of convolution type. Therefore, those applications will be addressed in another paper.

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CHARACTERIZATION OF INTEGRALS WITH RESPECT TO ALL RADON MEASURES ON AN ARBITRARY HAUSDORFF SPACE AS LINEAR FUNCTIONALS

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Key words: Radon measure, regular measure, Radon integral, symmetrizable functions, uniform functions

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Abstract. The problem of characterization of integrals as linear functionals is considered in the paper. It starts from the familiar results of F. Riesz (1909) and J. Radon (1913) on integral representation of bounded linear functionals by Riemann – Stiltjes integrals on a segment and by Lebesgue integrals on a compact in $\mathbb{R}^n$, respectively. After works of J. Radon, M. Fréchet, and F. Hausdorff the problem of characterization of integrals as linear functionals took the particular form of the *problem of extension of Radon's theorem* from compact subspaces of $\mathbb{R}^n$ to more general topological spaces with Radon measures. Therefore it may be naturally called the *Riesz – Radon – Fréchet problem of characterization of integrals*. The important stages of its solving are connected with such mathematicians as S. Banach, S. Saks, S. Kakutani, P. Halmos, E. Hewitt, R. E. Edwards, N. Bourbaki, V. K. Zakharov, A. V. Mikhalev, et al. In this paper the Riesz – Radon – Fréchet problem is solving for the general Radon measures on arbitrary Hausdorff spaces.

1 Lattice linear spaces of functions and linear functionals on them

Let $(T, \mathcal{G})$ be a Hausdorff topological space with the ensembles $\mathcal{G}$, $\mathcal{F}$, $\mathcal{C}$, and $\mathcal{B}$ of all open, closed, compact, and Borel subsets. Denote the family of all real-valued functions on a set T by $F(T)$.

Let $A(T) \subset F(T)$ be a lattice linear space of functions on T. Its subfamilies of all nonnegative and all bounded functions is denoted by $A(T)_+$ and $A_b(T)$, respectively. The subspace of $A(T)$ consisting of all functions with a compact support is denoted by $A_c(T)$.

A function family $A(T)$ is said to be *truncatable* or to have *the Stone property* if the condition $f \in A(T)$ implies $f \wedge \mathbf{1} \in A(T)$ (see [3, 71D] and [5, I.7.2]). For

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example, the space $C(T,\mathcal{G}) = M(T,\mathcal{G})$ of all continuous functions and its subspaces $C_b(T,\mathcal{G})$ and $C_c(T,\mathcal{G})$ have this property.

A functional $\varphi : A(T) \to \mathbb{R}$ is called *bounded* if $\sup\{|\varphi f| \mid f \in A(T) \wedge |f| \leqslant g\} < \infty$ for every $g \in A(T)_+$. The set of all bounded linear functionals on $A(T)$ is denoted by $A(T)^\sim$; it is lattice and linear.

Define for an arbitrary linear functional $\varphi : A(T) \to \mathbb{R}$ the (*lower* and *upper*) *boundedness indices of a functional*:

$$\underline{b}(\varphi) \equiv \inf\{\varphi f \mid f \in A(T)_+ \wedge f \leqslant \mathbf{1}\},$$
$$\bar{b}(\varphi) \equiv \sup\{\varphi f \mid f \in A(T)_+ \wedge f \leqslant \mathbf{1}\}.$$

It is clear that $-\infty \leqslant \underline{b}(\varphi) \leqslant \varphi(0) = 0$ and $0 = \varphi(0) \leqslant \bar{b}(\varphi) \leqslant \infty$. A linear functional $\varphi : A(T) \to \mathbb{R}$ is said to be *natural* if at least one of its boundedness indices is finite.

A functional φ on $A(T)$ is called *pointwise σ-continuous* if for every monotone sequence $(f_m \in A(T) \mid m \in M \subset \mathbb{N})$ and every function $f \in A(T)$ the convergence $(f_m(t) \mid m \in M) \to f(t)$ in all points $t \in T$ implies $(\varphi f_m \mid m \in M) \to \varphi f$ [5, ch. I, 8.1]. Every σ-pointwise continuous functional is bounded.

A functional φ on $A(T)$ is said to be *tight* or to have *the Prokhorov property* [3, 73G(e)] if for every $\varepsilon > 0$ there is a compact set $C \subset T$ such that the conditions $f \in A(T)$ and $|f| \leqslant \chi(T \setminus C)$ imply $|\varphi f| < \varepsilon$. The set of all tight bounded linear functionals on $A(T)$ will be denoted by $A(T)^\pi$.

A functional φ on $A(T)$ will be called *locally tight* or *with the local Prokhorov property*) [10] if for every $G \in \mathcal{G}$, $u \in A(T)_+$, and $\varepsilon > 0$ there is a compact subset $C \subset G$ such that the conditions $f \in A(T)$ and $|f| \leqslant \chi(G \setminus C) \wedge u$ imply $|\varphi f| < \varepsilon$.

A functional φ on $A(T)$ is said to be *σ-exact* if it is pointwise σ-continuous and locally tight. All σ-exact functionals are bounded in virtue of their σ-continuity. The set of all σ-exact linear functionals on $A(T)$ will be denoted by $A(T)^\triangle$. It is clear that this space is lattice and linear. The family of all natural σ-exact functionals on $A(T)$ will be denoted by $(A(T)^\triangle)_{\mathrm{nat}}$.

2 Radon measures on a Hausdorff space

Let $(T,\mathcal{G})$ be a Hausdorff space. A Borel measure $\mu : \mathcal{B} \to [-\infty,\infty[$ or $]-\infty,\infty]$ is called a *Radon measure on* $(T,\mathcal{G})$, if the following conditions are fulfilled:

1. $\mu C \in \mathbb{R}$ for every $C \in \mathcal{C}$;
2. for every $B \in \mathcal{B}$ such that $\mu B \in \mathbb{R}$ and every $\varepsilon > 0$ there is $C \in \mathcal{C}$ such that $C \subset B$ and $|\mu B - \mu C| < \varepsilon$;

3. for every $B \in \mathcal{B}$ such that $\mu B = \infty$ [respectively, $\mu B = -\infty$] and every $a \in \mathbb{R}$ there is $C \in \mathcal{C}$ such that $C \subset B$ and $\mu C > a$ [respectively, $\mu C < a$].

For the first time this definition of a general Radon measure appeared in V. K. Zakharov's paper [10]. In the case of positive measures the joint property 2)&3) is equivalent to the property of inner $\mathcal{C}$-regularity (*compact regularity*), which was used earlier in the definition of a positive Radon measure (see, e. g., [3, 73A] and [5, ch. V, 1.2]).

Property 2) does lie in the base of the definition of finite Radon measure on a compact space used by J. Radon, S. Banach, S. Saks, and S. Kakutani (see [8, 18.2.1]).

Generalizing their definition for the case of finite measure on a noncompact space it is naturally to replace the approximation by closed subsets by the approximation by compact subsets.

The family of all Radon measures will be denoted by $\mathfrak{RM}(T, \mathcal{G})$. Unfortunately, it is not a linear space because $\infty - \infty$ is not defined in $\overline{\mathbb{R}}$. The subscripts 0 and b are used to denote subfamilies of positive and bounded measures. We use the subscript 0 here because the subscript + is reserved for cones of positive elements of lattice linear spaces.

If $\mu \in \mathfrak{RM}(T, \mathcal{G})$, then the integral functional $i_\mu : f \mapsto \int f \, d\mu$ is called the *Radon integral.*

3 Properties of envelopment for symmetrizable functions

Let $(T, \mathcal{G})$ be a topological space. Consider the multiplicative ensemble $\mathcal{K} \equiv \{G \cap F \mid G \in \mathcal{G} \wedge F \in \mathcal{F}\}$ of all *symmetrizable sets* $K \equiv G \cap F$ [9].

A function $f \in F(T)$ will be called *symmetrizable* if for every $\varepsilon > 0$ there exists a finite cover $(K_i \in \mathcal{K} \mid i \in I)$ of the set T such that the oscillation $\omega(f, K_i) \equiv \sup\{|f(s) - f(t)| \mid s, t \in K_i\} < \varepsilon$ for every $i \in I$. Symmetrizable functions are uniform functions (for these functions see, e. g., [12]) as a particular case: they are uniform with respect to the ensemble $\mathcal{K}$.

The space $S(T, \mathcal{G})$ of all symmetrizable functions on $(T, \mathcal{G})$ is linear and lattice [14], contains the unit function $\mathbf{1}$, and, therefore, it is truncatable. It is clear that $S_b(T, \mathcal{G}) = S(T, \mathcal{G})$.

We will say that a family $A(T)$ *envelopes* [*σ-envelopes*] *from above* a function $h \in F(T)$ if there is a net $(f_m \in A(T) \mid m \in M)$ [a sequence $(f_m \in A(T) \mid m \in M \subset \mathbb{N})$] such that $(f_m(t) \mid m \in M) \downarrow h(t)$ in each point $t \in T$. Similarly, we will say that $A(T)$ *envelope* [*σ-envelope*] *from below* a function $g \in F(T)$ if $(f_m(t) \mid m \in M) \uparrow g(t)$ in each point $t \in T$.

A family $A(T)$ is said to have *Dini property* (D) if the pointwise convergence of a net $(f_m \in A(T) \mid m \in M)$ to a function $f \in A(T)$ implies its uniform convergence on every compact subset $C \subset T$.

By the Dini theorem if $A(T)$ is contained in the lattice linear space $C(T, \mathcal{G})$ of all continuous functions on a Hausdorff space $(T, \mathcal{G})$, then $A(T)$ has property (D).

A family $A(T)$ is said to *have property* (E) [respectively, (E_σ)] if the following three conditions are fulfilled:

(i) for every $G \in \mathcal{G}$ and every $u \in A(T)_+$ the family $A(T)$ envelopes [σ-envelopes] from below the function $\chi(G) \wedge u$,

(ii) for all $F \in \mathcal{F}$, $C \in \mathcal{C}$, and $u \in A(T)_+$ the family $A(T)$ envelopes [σ-envelopes] from above the functions $\chi(F) \wedge u$ and $\chi(C)$,

(iii) for every $G \in \mathcal{G}$ and every compact subset $C \subset G$ there is a function $v \in A(T)$ such that $\chi(C) \leqslant v \leqslant \chi(G)$.

It is clear that property (E_σ) is stronger than property (E). The spaces $S(T, \mathcal{G})$, $S_c(T, \mathcal{G})$, $C_b(T, \mathcal{G})$, and $C_c(T, \mathcal{G})$ used in the last section have these properties.

4 Construction of the representing Radon measure for a given positive σ-exact linear functional

Using a simplified variant of Daniell's method of functionals extension the following theorem can be proved [11]. Put $B(T) \equiv \{f \in S(T) \mid \exists\, u \in A(T)\,(|f| \leqslant u)\}$.

Theorem 1. *1. For every σ-exact linear functional φ on $A(T)$ there is the unique σ-exact linear functional φ_S on $B(T)$ extending the functional φ.*

2. The mapping $Q : \varphi \mapsto \varphi_S$ is an isomorphism of lattice linear spaces $A(T)^\triangle$ and $B(T)^\triangle$.

The representing measure μ is constructed from a positive σ-exact functional φ_S in the following way. Consider the ensemble $\mathcal{R}$ of all sets $R \subset T$ such that $B(T)$ σ-envelopes from above the function $\chi(R)$. Define on $\mathcal{R}$ the evaluation $\lambda : \mathcal{R} \to \mathbb{R}_+$ setting $\lambda R \equiv \inf\{\varphi_S f \mid f \in B(T)_+ \wedge f \geqslant \chi(R)\}$. Further, define the evaluation $\nu : \mathcal{P} \to \overline{\mathbb{R}}_+$ setting $\nu E \equiv \sup\{\lambda R \mid R \in \mathcal{R} \wedge R \subset E\}$. Consider the ensemble $\mathcal{M} \equiv \{M \in \mathcal{P} \mid \forall\, L \in \mathcal{R}\,(\lambda L \leqslant \nu(L \cap M) + \nu(L \setminus M))\}$ and the evaluation $\mu_0 \equiv \nu|\mathcal{M}$. Then $\mathcal{M}$ is a σ-algebra and $\mu_0 : \mathcal{M} \to \overline{\mathbb{R}}_+$ is a measure extending λ and possessing the property of *$\mathcal{R}$-regularity*: $\mu_0 M = \sup\{\mu_0 R \mid R \in \mathcal{R} \wedge R \subset M\}$ for every $M \in \mathcal{M}$. Moreover, $\mu \equiv \mu_0|\mathcal{B}$ is a positive Radon measure.

Theorem 2. *The functional φ_S is integrally representable with respect to the measure μ, i. e., $\varphi_S f = i_\mu f$ for all $f \in B(T)$.*

The following characterization of positive Radon integrals [11] is based on these two technical theorems.

Theorem 3. *Let $(T, \mathcal{G})$ be a Hausdorff space, $A(T)$ be a truncatable lattice linear subspace in $S(T, \mathcal{G})$. Let $A(T)$ has either property (E_σ) or property $(E)\&(D)$. Then for every positive σ-exact linear functional φ there is a unique positive Radon measure μ such that φ is integrally representable with respect to the measure μ and $\mu C = \inf\{\varphi f \mid f \in A(T) \wedge f \geqslant \chi(C)\}$ for every compact set C.*

5 Characterization of integrals with respect to all Radon measures as linear functionals

Theorem 3 gives a key to the characterization of integrals with respect to all Radon measures.

Theorem 4. *Let $(T, \mathcal{G})$ be a Hausdorff space and $A(T)$ be a truncatable lattice linear subspace in the space $S(T, \mathcal{G})$ possessing property (E_σ) or property $(E)\&(D)$. Suppose $\varphi \in A(T)^\triangle$. Then the functional φ is natural if and only if there exists the (unique) Radon measure μ such that $\varphi f = \int f\, d\mu$ for all $f \in A(T)$.*

Moreover, the injection $\varphi \mapsto \mu$ of $(A(T)^\triangle)_{\mathrm{nat}}$ into $\mathfrak{RM}(T, \mathcal{G})$ preserves all linear and lattice structures.

This *general parametric theorem* with the parameter $A(T)$ gives for concrete functional families $A(T)$ all the earlier obtained well known characterization theorems and also some new ones. Put $I(A(T), RM(T)) \equiv \{i_\mu | A(T) \mid \mu \in RM(T)\}$.

Corollary 1 (the Zakharov – Mikhalev – Rodionov theorem). Let $(T, \mathcal{G})$ be a Hausdorff space. Then $I(S_c(T, \mathcal{G}), \mathfrak{RM}(T, \mathcal{G})) = (S_c(T, \mathcal{G})^\triangle)_{\mathrm{nat}}$.

Corollary 2 (the Zakharov – Mikhalev theorem). Let $(T, \mathcal{G})$ be a Hausdorff space. Then $I(S(T, \mathcal{G}), \mathfrak{RM}_b(T, \mathcal{G})) = S(T, \mathcal{G})^\triangle$ and the mapping $\mu \mapsto i_\mu | S(T, \mathcal{G})$ is an isomorphism of the lattice linear spaces *[13]*.

Corollary 3 (the Bourbaki – Prokhorov theorem). Let $(T, \mathcal{G})$ be a Tychonoff space. Then $I(C_b(T, \mathcal{G}), \mathfrak{RM}_b(T, \mathcal{G})) = C_b(T, \mathcal{G})^\pi$ and the mapping $\mu \mapsto i_\mu | C_b(T, \mathcal{G})$ is also an isomorphism *(see [1, ch. IX, § 5, no. 2] and [3, 73G(e)])*.

Corollary 4 (the generalized Halmos – Hewitt – Edwards theorem). Let $(T, \mathcal{G})$ be a locally compact space. Then $I(C_c(T, \mathcal{G}), \mathfrak{RM}(T, \mathcal{G})_0) = (C_c(T, \mathcal{G})^\sim)_+$ *(see [2, 4])* and $I(C_c(T, \mathcal{G}), \mathfrak{RM}(T, \mathcal{G})) = (C_c(T, \mathcal{G})^\sim)_{\mathrm{nat}}$.

Corollary 5 (the Radon – Banach – Saks – Kakutani theorem). Let $(T, \mathcal{G})$ be a compact space. Then $I(C(T, \mathcal{G}), \mathfrak{RM}(T, \mathcal{G})) = C(T, \mathcal{G})^\sim$ and the mapping $\mu \mapsto i_\mu | C(T, \mathcal{G})$ is also an isomorphism *(see [6, 7])*.

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CHROMATIC EXPANSIONS IN REPRODUCING-KERNEL HILBERT SPACES

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Abstract. Chromatic series expansions of bandlimited functions have recently been introduced in signal analysis with promising results as a tool for signal processing. Chromatic series share similar properties with Taylor series insofar as the coefficients of the expansions, which are called chromatic derivatives, are based on the ordinary derivatives of the function, but unlike Taylor series, chromatic series have better rate of convergence and more practical applications. The purpose of this paper is to show that chromatic series expansions can be used to characterize some reproducing-kernel Hilbert spaces. We show that functions in the Paley-Wiener space PW_σ and the Bargmann-Segal-Foch space $\mathfrak{F}$ can be characterized by their chromatic series expansions that use chromatic derivatives associated with the Legendre and Hermite polynomials, respectively.

1 Introduction

Chromatic derivatives and series expansions have recently been introduced in signal analysis by A. Ignjatovic in [8, 9] as an alternative representation to Taylor series for bandlimited functions and they have been shown to be more useful in practical applications than Taylor series; see [1–3, 5–7, 13, 14].

Recall that a function f is bandlimited to $[-\sigma, \sigma]$ if it can be represented as

$$f(t) = \int_{-\sigma}^{\sigma} e^{-ixt} g(x)\, dx, \quad t \in \mathbb{R}\,, \tag{1}$$

for some function $g \in L^2(-\sigma, \sigma)$.

The space of bandlimited functions, which is also known as the Paley–Wiener space of bandlimited functions, consists of entire functions of exponential type that are square integrable on the real axis. The Paley–Wiener space of functions bandlimited to $[-\sigma, \sigma]$ will be denoted by PW_σ. The Whittaker-Shannon-Kotel'nikov (WSK) sampling theorem states that if $f \in PW_\sigma$, then it can be reconstructed from its samples, $f(k\pi/\sigma)$. The construction formula is

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\sigma}\right) \frac{\sin(\sigma t - k\pi)}{(\sigma t - k\pi)} = \sum_{k=-\infty}^{\infty} f(t_k) \frac{\sin[\sigma(t-t_k)]}{\sigma(t-t_k)}, \quad t \in \mathbb{R}, \tag{2}$$

where $t_k = k\pi/\sigma$, and the series being absolutely and uniformly convergent on compact subsets of $\mathbb{R}$. See, e.g., [17, p. 16].

The WSK theorem plays an important role in communication engineering because it enables engineers to reconstruct an analogue signal from its samples at a discrete set of points. The WSK expansion may be viewed as a global expansion because it uses function values at infinitely many points uniformly distributed on the real line.

On the other hand, as an entire function, f has a Taylor series expansion of the form $f(t) = \sum\limits_{n=0}^{\infty} \left(f^{(n)}(0)/n!\right) t^n$, which may be viewed as a local expansion since it uses the values of f and all its derivatives at a single point.

While the Whittaker–Shannon-Kotel'nikov sampling expansion has played an important role in digital signal processing applications, the Taylor series has limited practical applications. This can be attributed to two facts.

— The *Sinc* function, Sinc $x = \sin \pi x/\pi x$ belongs to PW_π, and its translates $\{\operatorname{sinc}(t-n)\}_{n\in\mathbb{Z}}$, form an orthogonal basis for PW_π; hence any $f \in PW_\pi$ can be approximated by truncating its sampling series, while a truncated Taylor series, i.e., a polynomial of degree n, is not bandlimited.

— Function evaluation is easier to compute and more numerically stable than numerical evaluation of derivatives.

Another representation of bandlimited functions is provided by the chromatic series expansions which are based on the notion of chromatic derivatives.

2 Chromatic Derivatives

The n-th chromatic derivative $K^n[f](t_0)$ of an analytic function $f(t)$, at t_0, whose formal definition will be given below, is a linear combination of the ordinary derivatives $f^{(k)}(t_0)$, $0 \leqslant k \leqslant n$, where the coefficients of the combination are based on systems of orthogonal polynomials. However, unlike the ordinary derivatives, the chromatic derivatives can be obtained more accurately in a noise robust way. Although there are several notions of derivatives available in the literature, such as symmetric, Peano, L^p, symmetric L^p, quantum, Fréchet, and Gateaux derivatives, they are all mathematical generalizations of the ordinary derivative and they are used to describe the local behavior of a function in a neighborhood of the point of differentiation. In contrast, chromatic derivatives arose from real world applications and analogous to ordinary derivatives they can be used to describe the global behavior of analytic functions.

Although chromatic series are like Taylor series, locally in nature, they provide more numerically robust expansions than their Taylor counterparts. The transfer functions of ordinary derivatives cluster tightly together and obliterate all but the edges of the spectrum of a bandlimited signal. On the contrary, the transfer functions of chromatic derivatives form a family of well separated, interleaved refined comb filters.

We will briefly describe how chromatic series are constructed. Let $W(\omega)$ be a non-negative weight function such that all of its moments are finite, i.e., such that

$$\mu_n = \int_{-\infty}^{\infty} \omega^n W(\omega) d\omega < \infty.$$

Let $\{P_n(\omega)\}_{n=0}^{\infty}$ be the family of polynomials orthonormal with respect to $W(\omega)$:

$$\int_{-\infty}^{\infty} P_n(\omega) P_m(\omega) W(\omega) d\omega = \delta_{m,n},$$

and let $K^n(f) = P_n(i\frac{d}{dt})(f)$ be the corresponding linear differential operator obtained from $P_n(\omega)$ by replacing ω^k $(0 \leqslant k \leqslant n)$ with $i^k \frac{d^k}{dt^k}$. These differential operators are called *chromatic derivatives* associated with the family of orthogonal polynomials $\{P_n(\omega)\}$ because they preserve the spectral features of band-limited signals. They can be evaluated with high accuracy and in a noise robust way from samples of the signal taken at a small multiple of the usual Nyquist rate; see [5, 6] for details.

Let $\psi(z)$ be the Fourier transform of the weight function $W(\omega)$,

$$\psi(z) = \int_{-\infty}^{\infty} e^{i\omega z} W(\omega) d\omega.$$

Because $\psi(z)$ will be used in a Taylor-type expansion of functions analytic in a domain around the origin, we shall assume that $\limsup(\mu_n/n!)^{1/n} < \infty$, where, $\psi^{(n)}(0) = i^n \mu_n$. This condition implies that $\psi(z)$ is analytic around the origin. As shown in [4], this condition holds if and only if

$$\int_{-\infty}^{\infty} e^{c|\omega|} W(\omega) d\omega < \infty$$

for some $c > 0$, and in this case $\psi(z)$ is analytic in the strip $S(c/2) = \{z \,:\, \mathrm{Im}(z) < c/2\}$.

The chromatic series expansion of $f \in C^{\infty}(\mathbb{R})$ is given by the following formal series.

$$f(z) \sim \sum_{n=0}^{\infty} K^n(f)(0) K^n(\psi)(z). \tag{3}$$

3 Chromatic Expansions in the Paley–Wiener Space

It has been shown in [4] that if $f(z)$ is analytic in the strip $S(c/2)$ and $\sum_{n=0}^{\infty} |K^n(f)(0)|^2$ converges, then the series (3) converges to $f(z)$, uniformly in every strip $\{z \,:\, |\mathrm{Im}(z)| < c/2 - \varepsilon\}$, for any $\varepsilon > 0$. Here it should be emphasized that although chromatic series were originally introduced for bandlimited functions, the theory now applies to a much larger class of functions.

In the particular case, where $W(\omega) = \chi_{(-1,1)}$, the chromatic series associated with the Legendre polynomials converge in the whole complex plane, i.e., the strip $S(c/2)$ is $\mathbb{C}$, and the set of entire functions for which $\sum_{n=0}^{\infty} |K^n(f)(0)|^2$ converges is precisely the set of L^2 functions whose Fourier transforms are finitely supported, i.e., the set of bandlimited functions.

More precisely, let $P_n(t)$ be the Legendre polynomial of degree n normalized so that

$$\int_{-1}^{1} P_n(t) P_m(t) dt = 1,$$

then if we define the chromatic derivative $K^n[F](z)$ of F order n to be

$$K^n\left[F\right](t) = P_n\left(-i\frac{d}{dt}\right) F(t),$$

the for any bandlimited function $F(z) = \int_{-1}^{1} f(\omega) e^{iz\omega} d\omega$, we have

$$F(z) = \sum_{n=0}^{\infty} K^n(F)(0)K^n(\psi)(z),$$

where $\psi(z) = 2\text{Sinc}(z)$.

For such functions the chromatic expansions converge uniformly on $\mathbb{R}$, and their truncated series are themselves bandlimited which is analogous to the Whittaker-Shannon sampling series [17]. This is in contrast to Taylor series whose truncated series are not bandlimited. For this reason chromatic series have more practical applications in signal processing than Taylor series.

To extend chromatic expansions to other function spaces, we need to generalize the notion of chromatic derivatives. In two recent papers [15,16] we introduced more general types of chromatic derivatives and series that are better suited to handle integral transforms other than the Fourier transform. In [16] we presented two different methods to construct a differential operator L that gives rise to generalized chromatics derivatives and their associated integral transform. In the first method the operator L arises from certain Sturm-Liouville boundary-value problems, while in the second, it arises from initial-value problems involving differential operators of order n.

Consider the singular Sturm–Liouville boundary-value problem on the half line

$$Ly = -y'' + q(x)y = \lambda y, \quad 0 \leqslant x < \infty, \tag{4}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad -\pi < \alpha \leqslant \pi \tag{5}$$

where $q(x) \in L^1(\mathbb{R}^+)$ is real valued. It is known that the condition $q \in L^1(\mathbb{R}^+)$ implies that the problem is in the limit point case at ∞ and that the spectrum is continuous [11]. In fact, there exists a non-decreasing function $\rho(\lambda)$ such that for all $f \in L^2(\mathbb{R}^+)$

$$\hat{f}(\lambda) = \int_0^{\infty} f(x)\varphi(x,\lambda)dx, \tag{6}$$

exists in the mean and defines a function $\hat{f}(\lambda)$ such that

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\lambda)\varphi(x,\lambda)d\rho(\lambda) \tag{7}$$

where $\varphi(x,\lambda)$ is a solution of the differential equation (4) that satisfies the initial condition

$$\varphi(0,\lambda) = \sin\alpha, \varphi'(0,\lambda) = -\cos\alpha. \tag{8}$$

We call the integral transform (6) the φ-transform of f. Fix $0 < a < \infty$, and let $K^2(a)$ denote the set of all functions with supports in $[0,a]$ that are square integrable with respect to $d\rho$. In most cases of interest $d\rho$ is supported on a half-line which, without loss of generality, we may take as $[0,\infty)$. For sufficient conditions for this to hold see [10, p.128]. The main result can be summarized in the following theorem whose proof can be found in [16]

Theorem 1. *Consider the boundary-value problem (4) and (5) and let $CK^2(a)$ denote the image of $K^2(a)$ under the transformation (7). Then there exists a sequence of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ that are orthonormal with respect to $d\rho$ on $[0,a]$ and $p_n(\lambda)$ is of exact degree n. Furthermore, for any $f \in CK^2(a)$ we have for $\alpha \neq 0,\ \pi$*

$$f\,(x) = \frac{1}{(\sin\alpha)}\sum_{n=0}^{\infty}\left[p_n(L)f\right](0)\psi_n(x), \tag{9}$$

where$\psi_n(x) = \int_0^{\infty} p_n(\lambda)\varphi(x,\lambda)d\rho(\lambda)$, and the series converges to f in the mean. Similar expressions exist for $\alpha = 0$ or π. The functions $\{\psi_n(x)\}$ are orthonormal on $[0,\infty)$ and satisfy the initial condition (8). The series (9) converges to $f(x)$ pointwise for $0 \leqslant x < \infty$. In fact, the series converges to f uniformly on compact subsets of $(0,\infty)$.

Definition 1. The n-th generalized chromatic derivative of a function f associated with the differential operator L at $x = 0$ is defined as

$$K^n[f](0) = \langle \hat{f}, p_n\rangle_{d\rho} = \frac{1}{(\sin\alpha)}\left[p_n(L)f\right](0), \text{ for } \alpha \neq 0, \pi$$

where $\hat{f}$ is the φ transform of f.

Analogous to (3), we define the generalized chromatic series expansion of f as

$$\sum_{n=0}^{\infty} K^n[f](0)K^n[\psi](x), \quad \text{where} \quad \psi_n(x) = K^n[\psi](x),$$

and

$$\psi(x) = \int_0^a \varphi(x,\lambda)d\rho(\lambda).$$

4 Chromatic Expansions in the Bargmann–Segal–Foch Space

In this section we show that functions in the Bargmann-Segal-Foch space can be characterized by their chromatic series expansion.

Definition 2. Let $d\mu(z) = \rho dxdy,\ \rho = (\pi)^{-1}\exp\left(-|z|^2\right)$, where $z = x + iy$. The Bargmann-Segal-Foch space, $\mathfrak{F}$ consists of all entire functions $F(z)$ in $\mathbb{C}$ such that

$$\|F\|_{\mathfrak{F}}^2 = \int_{\mathbb{C}} |F(z)|^2 d\mu(z) < \infty.$$

It is a Hilbert space with inner product defined by

$$\langle F, G\rangle_{\mathfrak{F}} = \int_{\mathbb{C}} F(z)\overline{G}(z)d\mu(z) < \infty,$$

and hence with norm

$$\|F\|_{\mathfrak{F}}^2 = \int_{\mathbb{C}} |F(z)|^2 d\mu(z).$$

It is known that $\mathfrak{F}$ is a reproducing-kernel Hilbert space. Since $\left\{u_m(z) = z^m/\sqrt{m!}\right\}_{m=0}^{\infty}$ is an orthonormal basis of $\mathfrak{F}$, the reproducing kernel can be found explicitly. In fact, since $\{u_m(z)\}_{m=0}^{\infty}$ is an orthonormal basis of $\mathfrak{F}$, the reproducing kernel is readily seen to be

$$K(z, w) = \sum_{m=0}^{\infty} \frac{z^m w^m}{m!} = e^{zw}.$$

Thus, for any $F \in \mathfrak{F}$, we have

$$\langle F(z), K(z, w)\rangle_{\mathfrak{F}} = \int F(z)e^{wz}d\mu(z) = \sum_{n=0}^{\infty} n!a_n \frac{w^n}{n!} = F(w). \tag{10}$$

Definition 3. The Bargmann transform $\mathcal{A}[f]$ of a function $f : \mathbb{R} \to \mathbb{C}$ is defined by

$$\mathcal{A}[f](z) = F(z) = (\pi)^{-1/4} \int_{\mathbb{R}} f(q)\exp\left\{-(|z|^2 + |q|^2)/2 + \sqrt{2}\,(zq)\right\} dq \tag{11}$$

which can be written as

$$F(z) = \langle k(z,q), \overline{f}(q)\rangle_{L^2(\mathbb{R})}, \qquad \text{whenever } f \in L^2(\mathbb{R}),$$

where $z = x + iy$, and

$$k(z,q) = (\pi)^{-1/4} \exp\left\{-(|z|^2 + |q|^2)/2 + \sqrt{2}\ (zq)\right\}\ , \tag{12}$$

$q \in \mathbb{R}$ and $z \in \mathbb{C}$.

Let $\tilde{H}_m(x)$ be the Hermite polynomial of degree m defined by

$$\tilde{H}_m(x) = (-1)^m e^{x^2} \left(\frac{d}{dx}\right)^m e^{-x^2},\ 0 \leqslant m.$$

We define the normalized Hermite polynomials by

$$H_m(x) = \frac{1}{\sqrt[4]{\pi}2^{m/2}\sqrt{m!}}\tilde{H}_m(x)$$

so that

$$\int\limits_{\mathbb{R}} H_k(x)H_m(x)e^{-x^2}dx = \delta_{k,m}$$

or

$$\int\limits_{\mathbb{R}} h_k(x)h_m(x)dx = \delta_{k,m},$$

where $h_k(x) = H_k(x)e^{-x^2/2}$ are the normalized Hermite functions, which are an orthonormal basis of $L^2(\mathbb{R})$.

Let

$$L = \frac{1}{\sqrt{2}}\left(\frac{d}{dz} + z\right).$$

It is easy to see that

$$LF(z) = \int\limits_{\mathbb{R}} f(q)qk(z,q)dq,$$

hence

$$H_m\left(L\right)F(z) = \int\limits_{\mathbb{R}} f(q)H_m(q)k(z,q)dq.$$

Definition 4. We define the m-th chromatic derivative of $F(z)$ with respect to the operator L and the Hermite polynomials as

$$K^m F(z) = H_m(L)F(z).$$

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing functions consisting of all $\varphi \in C^\infty(\mathbb{R})$ such that

$$\gamma_{l,m}(\varphi) = \sup_{q\in\mathbb{R},\beta\leqslant l,\alpha\leqslant m} \left| q^\beta \frac{d^\alpha \varphi(q)}{dq^\alpha} \right| < \infty,$$

where $l, m, \alpha, \beta \in \mathbb{N}$.

Finally, we are ready to state the main result of this section whose proof will be published somewhere else.

Theorem 2. *There exists a function* $\varphi \in \mathcal{S}(\mathbb{R})$ *whose Bergmann transform* $\mathcal{A}(\varphi) = \psi(z) \in \mathfrak{F}$ *has the property that its chromatic derivatives* $\{K^m\psi(z)\}$ *are an orthogonal basis of* $\mathfrak{F}$*. Hence, any* $F \in \mathfrak{F}$ *can be written in the form*

$$F(z) = \sum_{m=0}^{\infty} K^m F(0) K^m \psi(z). \tag{13}$$

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II.3. Spaces of Differentiable Functions of Several Variables and Applications

(Sessions organizers: V.I. Burenkov, M.L. Goldman, V.D. Stepanov)

CHARACTERIZATION OF SPACES OF FUNCTIONS OF ZERO SMOOTHNESS

O. V. Besov

Key words: function spaces, embedding theorems

AMS Mathematics Subject Classification: 46E35

Abstract. For functions of many variables in the spaces $B^s_{p,\theta}(\mathbb{R}^n)$, $0 \leqslant s < 1$, we study new difference characteristics which define equivalent norms for $0 < s < 1$, as well as for $s = 0$ in some cases.

For functions of many variables in the spaces $B^s_{p,\vartheta}(\mathbb{R}^n)$, $0 \leqslant s < 1$, we study new difference characteristics which define equivalent norms for $0 < s < 1$, as well as for $s = 0$ in some cases. Using the averaged difference of a function f,

$$\delta(h)f(x) := (2h)^{-2n} \int\limits_{[-h,h]^n} \int\limits_{[-h,h]^n} |f(x+y) - f(x+z)|\, dy\, dz,$$

we construct Banach spaces $\overline{B}^s_{p,\vartheta}(\mathbb{R}^n)$, $0 \leqslant s < 1$.

Here $\mathbb{R}^n$ is the Euclidean n-space of points $x = (x_1, \dots, x_n)$, $1 \leqslant p \leqslant \infty$.

Let L_p be the Lebesgue space with the norm $\|f|L_p\| = \big(\int\limits_{\mathbb{R}^n} |f(x)|^p\, dx\big)^{1/p}$.

Consider the Banach space $B^s_{p,\vartheta} = B^s_{p,\vartheta}(\mathbb{R}^n)$ of generalized functions $f \in S'$ with the norm

$$\|f|B^s_{p,\vartheta}\| = \left\{ \sum_{j=0}^{\infty} 2^{s\vartheta} \|a_j|L_p\|^{\vartheta} \right\}^{1/\vartheta}, \quad s \in \mathbb{R}, \quad 1 \leqslant p \leqslant \infty, \tag{1}$$

where

$$a_j(x) = F^{-1}\varphi_j F f, \quad \varphi_0(x) = \varphi(x), \quad \varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x),$$
$$\varphi \in C_0^\infty(\mathbb{R}^n), \quad \varphi(x) = 1 \quad \text{for} \quad |x| \leqslant 1, \quad \varphi(x) = 0 \quad \text{for} \quad |x| \geqslant 2,$$

so that

$$\operatorname{supp}\varphi_0 \subset \{x : |x| \leqslant 2\},$$
$$\operatorname{supp}\varphi_j \subset \{x : 2^{j-1} \leqslant |x| \leqslant 2^{j+1}\} \quad \text{for} \quad j \in \mathbb{N},$$

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \forall x \in \mathbb{R}^n,$$

$$\varphi_j(x) = \varphi_1\Big(\frac{x}{2^{j-1}}\Big), \quad \varphi_1(x) = \varphi\Big(\frac{x}{2}\Big) - \varphi(x).$$

We compare the space $B^s_{p,\vartheta}$, $0 \leqslant s < 1$, with four other Banach function spaces defined for $0 \leqslant s < 1$, $1 \leqslant p \leqslant \infty$, and $1 \leqslant \vartheta \leqslant \infty$ as the spaces of locally integrable functions on $\mathbb{R}^n$ with the norms

$$\|f|^{\Delta}B^s_{p,\vartheta}\| := |||f|||_{p,s} + \left\{ \int\limits_{|y|<1} \left(\frac{\|\Delta(y)f|L_p\|}{|y|^s} \right)^{\vartheta} \frac{dy}{|y|^n} \right\}^{1/\vartheta}, \tag{2}$$

$$\|f|^{\overline{\Delta}}B^s_{p,\vartheta}\| := |||f|||_{p,s} + \left\{ \int\limits_{0}^{1} \left(\frac{\|\overline{\Delta}(h)f|L_p\|}{h^s} \right)^{\vartheta} \frac{dh}{h} \right\}^{1/\vartheta}, \tag{3}$$

where $\Delta(y)f(x) = f(x+y) - f(x)$, $\overline{\Delta}(h)f(x) = h^{-n} \int\limits_{|y|<h} |\Delta(y)f(x)|\, dy$, $Q = [-1,1]^n$,

$$|||f|||_{p,s} := \begin{cases} \|f|L_p\| & \text{for } 0 < s < 1,\ 1 \leqslant p \leqslant \infty; \\ \left\| \int\limits_{Q} |f(\cdot + y)|\, dy \Big| L_p \right\| & \text{for } s = 0,\ 1 \leqslant p \leqslant \infty. \end{cases}$$

It is known that for $0 < s < 1$ the norms (1), (2) and (3) are equivalent.

$$\|f|\overline{B}^s_{p,\vartheta}\| := |||f|||_{p,s} + \left\{ \int\limits_{0}^{1} \left(\frac{\|\delta(h)f|L_p\|}{h^s} \right)^{\vartheta} \frac{dh}{h} \right\}^{1/\vartheta}, \tag{4}$$

where

$$\delta(h)f(x) := (2h)^{-2n} \int\limits_{[-h,h]^n} \int\limits_{[-h,h]^n} |f(x+y) - f(x+z)|\, dy\, dz,$$

$$\|f|\widetilde{B}^s_{p,\vartheta}\| := |||f|||_{p,s} + \left\{ \int\limits_{0}^{1} \left(\frac{\|v_h * f|L_p\|}{h^s} \right)^{\vartheta} \frac{dh}{h} \right\}^{1/\vartheta}, \tag{5}$$

where

$$v_h = h^{-n} v\Big(\frac{x}{h}\Big), \quad v(x) = 2\omega\Big(\frac{x}{2}\Big) - \omega(x), \quad \omega \in C_0^\infty(\mathbb{R}^n),$$

$$\mathrm{supp}\omega \subset Q, \quad \omega(x) = \overline{\omega}(|x|), \quad \int \omega(x)\,dx \neq 0.$$

Theorem 1. *For* $0 < s < 1$

$$^{\Delta}B^s_{p,\vartheta} = {}^{\overline{\Delta}}B^s_{p,\vartheta} = \overline{B}^s_{p,\vartheta} = \widetilde{B}^s_{p,\vartheta} = B^s_{p,\vartheta}.$$

For $s = 0$

$$^{\Delta}B^0_{p,\vartheta} \subset {}^{\overline{\Delta}}B^0_{p,\vartheta} \subset \overline{B}^0_{p,\vartheta} \subset \widetilde{B}^0_{p,\vartheta} \subset B^0_{p,\vartheta},$$

$$\overline{B}^0_{p,\vartheta} \neq \widetilde{B}^0_{p,\vartheta} \quad \textit{for} \quad 1 \leqslant p < \infty, \quad 1 \leqslant \vartheta < \infty.$$

For comparison, consider the space $\mathrm{bmo}(\mathbb{R}^n) = F^0_{\infty,2}(\mathbb{R}^n)$ of locally integrable functions with the norm

$$\|f|\mathrm{bmo}(\mathbb{R}^n)\| = \sup_{x\in\mathbb{R}^n} \int\limits_Q |f(x+y)|\,dy + \\ + \sup_{h>0,\, x\in R^n} (2h)^{-n} \int\limits_{hQ} |f(x+y) - f_h(x)|\,dy, \quad (6)$$

where $f_h(x) = (2h)^{-n} \int\limits_{hQ} f(x+y)\,dy$, $hQ = [-h,h]^n$.

Lemma 1. $\overline{B}^0_{\infty,\infty}(\mathbb{R}^n) = \mathrm{bmo}(\mathbb{R}^n) = F^0_{\infty,2} \subset \widetilde{B}^0_{\infty,\infty}(\mathbb{R}^n)$.

It is interesting to find out when $\tilde{B}^0_{p,\vartheta}$ and $B^0_{p,\vartheta}$ coincide.

This problem is solved by using the following Sickel–Triebel result (1995):

Let $1 \leqslant p \leqslant \infty$ and $1 \leqslant \vartheta \leqslant \min\{p,2\}$. Then

$$B^0_{p,\vartheta} \subset \begin{cases} L_p & \text{for } 1 \leqslant p < \infty, \\ \mathrm{bmo} & \text{for } p = \infty. \end{cases}$$

Theorem 2. *For* $1 \leqslant p \leqslant \infty$ *and* $1 \leqslant \vartheta \leqslant \min\{p,2\}$ $\widetilde{B}^0_{p,\vartheta}(\mathbb{R}^n) = B^0_{p,\vartheta}(\mathbb{R}^n)$.

In $\mathbb{R}^n$, consider domains satisfying the flexible cone condition. The definition of the space $\overline{B}^0_{p,\vartheta}(\mathbb{R}^n) = \overline{B}^0_{p,\vartheta}$ extends naturally to such domains.

On a domain G satisfying the flexible cone condition, consider the spaces $B^s_{p,\vartheta}(G)$, $s > 0$, and the Sobolev spaces $W^s_p(G)$, $s \in \mathbb{N}$,

$$\|f|W^s_p(G)\| = \|f|L_p\| + \sum_{i=1}^{n} \|D^s_i f|L_p(G)\|.$$

Theorem 3. *Let G be a domain satisfying the flexible cone condition, $1 \leqslant p < q \leqslant \infty$, $s = \frac{n}{p} - \frac{n}{q}$ and $1 \leqslant \vartheta \leqslant \infty$. Then*

$$B^s_{p,\vartheta}(G) \subset \overline{B}^0_{q,\vartheta}(G).$$

Theorem 4. *Let G be a domain satisfying the flexible cone condition, $1 \leqslant p < q \leqslant \infty$, and $s = \frac{n}{p} - \frac{n}{q}$. Then*

$$W^s_p(G) \subset \overline{B}^0_{q,p}(G) \quad \textit{for} \ \ s \in \mathbb{N}, \quad p > 1.$$

Remark 1. If $G = \mathbb{R}^n$, $1 < p \leqslant 2$, $2 \leqslant q \leqslant \infty$ the statement of Theorem 4 strengthens the known embeddings

$$W^s_p \subset L_q \qquad \text{for} \quad q < \infty,$$
$$W^s_p \subset \mathrm{bmo} \qquad \text{for} \quad q = \infty.$$

The proofs are based on integral representations of functions in terms of their derivatives and differences.

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NEW DESCRIPTION OF UNIFORMLY CONTINUOUS BOUNDED FUNCTIONS ON TOPOLOGICAL GROUPS

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AMS Mathematics Subject Classification: 11B39

Abstract. In the paper the new description of uniformly continuous bounded functions on topological groups is introduced and studied. In the case of the additive group of real numbers with the natural topology the description is eguivalent to the classical definition of a uniformly continuous function in mathematical analysis.

Let $\mathcal{G} = \langle G, \cdot, \tau \rangle$ be a topological group with the unity e and T_1-topology τ, Σ_e — a base for the topological space (G, τ) at the point e.

Definition 1. Let $f : \mathcal{G} \to [a, b]$ be a continuous bounded nonconstant function from the topological group $\mathcal{G}$ to the interval $[a, b]$ of the real line, $a = \inf\{fx \mid x \in G\}$, $b = \sup\{fx \mid x \in G\}$. The family $\mathrm{Im} = \{F_r = f^{-1}[a, r) \mid r \in D\}$ of the inverse images of the subsets $[a, r) \subset [a, b]$, where D — an everywhere dense subset of $(a, b]$, is called a basis of the function f.

Definition 2. A basis Im of the function $f : \mathcal{G} \to [a, b]$ is called a left multiplicative basis or, shortly, m-basis if for every $r, s \in D$, $r < s$, there exists a neighbourhood $T = T(r, s)$ of e such that $T \cdot F_r \subseteq F_s$.

Note. The definitions of a basis and m-basis for a special case of a continuous function $f : \mathcal{G} \to [0, 1]$, $fe = 0$, were given in [1].

The following properties of the basis Im are obvious.

1. For any $r \in D$ F_r is an open neighbourhood of a.
2. For any $r, s \in D$ if $r < s$, then $\bar{F_r} \subseteq F_s$ where $\bar{F_r}$ is the closure of F_r.

Proposition 1. If a continuous function $f : \mathcal{G} \to [a, b]$ has m-basis $\mathrm{Im} = \{F_r \mid r \in D\}$, then any basis $\mathcal{H} = \{H_s = f^{-1}[a, s) | s \in E\}$ of the function f is also m-basis.

Proof. Let $p, t \in E$, $p < t$. D is an everywhere dense subset of $(a, b]$ that is why there exist $r, s \in D$ such that $p \leqslant r < s \leqslant t$. $H_p \subseteq F_r \subseteq F_s \subseteq H_t$ and for some $T \in \Sigma_e$ $T \cdot F_r \subseteq F_s$. The last relation shows that $T \cdot H_p \subseteq H_t$. Hence $\mathcal{H}$ — m-basis of f. □

The following definition of a uniformly continuous function from a topological group will be used.

Definition 3. A continuous function $f : \mathcal{G} \to \mathrm{Re} = \langle R, +, natural\rangle$ from a topological group $\mathcal{G}$ to the additiveve group of real numbers is a uniformly continuous function with respect to right uniformity on $\mathcal{G}$ if for every $\varepsilon > 0$ there exists a neighbourhood $T = T(\varepsilon)$ such that for every $x, y \in G$ if $y \in T\cdot x$ we have $|fx - fy| < \varepsilon$ or, equivalent, for every $x \in G$ we have $\sup\{|fx - f(t\cdot x)| \mid x \in G, t \in T\} < \varepsilon$.

Theorem. *A continuous bounded nonconstant function $f : \mathcal{G} \to [a, b]$ from a topological group $\mathcal{G}$ to an interval $[a, b]$ of the real line is a uniformly continuous function with respect to right uniformity on $\mathcal{G}$ if and only if the function f has a left multiplicative basis.*

Proof. Let $f : \mathcal{G} \to [a, b]$ be a uniformly continuous bounded function with a basis $\mathrm{Im} = \{F_r \mid r \in D\}$. We have to show that Im – m-basis. Let us take arbuitrary elements $r, s \in D$, $r < s$. For these r, s there exists a number $\varepsilon > 0$ such that $r + \varepsilon < s$. The function f is uniformly continuous relative to right uniformity on $\mathcal{G}$. Thus for this ε there exists a neighbourhood T of e such that for every $x \in G$ $\sup\{|fx - f(t\cdot x)| \mid t \in T\} < \varepsilon$, in particular, for every $x \in F_r$ and every $t \in T$ $|fx - f(t\cdot x)| < \varepsilon$. The last unequality shows that for every $x \in F_r$ $f(T\cdot x) \subseteq [a, r+\varepsilon) \subseteq [a, s)$ from where for every $x \in F_r$ $T\cdot x \subseteq F_s$. Hence $T\cdot F_r \subseteq F_s$ and the basis Im of a uniformly continuous function f is m-basis.

Conversely, let a continuous function $f : \mathcal{G} \to [a, b]$ have a left multiplicative basis $\mathrm{Im} = \{F_r \mid r \in D\}$. For simplicity of notation we consider $D = (a, b]$. Suppose f is not a uniformly continuous function: there exists $\varepsilon > 0$ such that for any neighbourhood T of e there exists $x, y \in G$ such that $y \in T\cdot x$ but $|fx - fy| > \varepsilon$. We have to show that under this assumption the basis Im will be not m-basis.

Because ε — the fixed positive number there exists the finite cover of the space (G, τ) by disjoint sets

$$F_{\frac{\varepsilon}{2}}, F_{\frac{2\cdot\varepsilon}{2}} \setminus F_{\frac{\varepsilon}{2}}, F_{\frac{3\cdot\varepsilon}{2}} \setminus F_{\frac{2\cdot\varepsilon}{2}}, \dots, F_{\frac{i\cdot\varepsilon}{2}} \setminus F_{\frac{(i-1)\cdot\varepsilon}{2}}, \dots, F_b \setminus F_{\frac{k\cdot\varepsilon}{2}}, F_b' \tag{1}$$

where F_b' is the complement of F_b.

We shall index the elements of the base Σ_e by the partial ordered set Γ so that $\Sigma_e = \{T_\alpha | \alpha \in \Gamma\}$ and $\alpha \leqslant \beta \iff T_\alpha \supseteq T_\beta$. Γ is a right directed set. Let us assume also that the elements of Σ_e are symmetrical neighbourhoods of e: $T_\alpha^{-1} = T_\alpha$ for every $\alpha \in \Gamma$. On the assumption that f is not a uniformly continuous function it follows that for any $\alpha \in \Gamma$ there exist $x_\alpha, y_\alpha \in G$ such that $y_\alpha \in T_\alpha\cdot x_\alpha$, $x_\alpha \in T_\alpha\cdot y_\alpha$ but $|fx_\alpha - fy_\alpha| > \varepsilon$. The last unequality means that for every $\alpha \in \Gamma$ points x_α and y_α do not belong to the same or to the neighbouring sets of the cover (1).

The cover (1) is finite, therefore there exist two nonneighbouring sets of the cover (1) each of which contains infinite and confinal with respect to the index set Γ subset of the sets $\{x_\alpha \mid \alpha \in \Gamma\}$, $\{y_\alpha \mid \alpha \in \Gamma\}$. Moreover, because the relations $y_\alpha \in T_\alpha \cdot x_\alpha$, $x_\alpha \in T_\alpha \cdot y_\alpha$, $|fx_\alpha - fy_\alpha| > \varepsilon$ are symmetric with respect to x_α and y_α we can consider that there exist integers i, j such that $j > i + 1$ and confinal with respect to indexes subset of $\{x_\alpha \mid \alpha \in \Gamma\}$ is contained in $F_{\frac{i\cdot\varepsilon}{2}} \setminus F_{\frac{(i-1)\cdot\varepsilon}{2}}$, confinal with respect to indexes subset of $\{y_\alpha \mid \alpha \in \Gamma\}$ is contained in $F_{\frac{j\cdot\varepsilon}{2}} \setminus F_{\frac{(j-1)\cdot\varepsilon}{2}}$. For simplisity of notation we assume that these confinal subsets coincide with the sets $\{x_\alpha \mid \alpha \in \Gamma\}$ and $\{y_\alpha \mid \alpha \in \Gamma\}$, that is $\{x_\alpha \mid \alpha \in \Gamma\} \subseteq F_{\frac{i\cdot\varepsilon}{2}} \setminus F_{\frac{(i-1)\cdot\varepsilon}{2}}$, $\{y_\alpha \mid \alpha \in \Gamma\} \subseteq F_{\frac{j\cdot\varepsilon}{2}} \setminus F_{\frac{(j-1)\cdot\varepsilon}{2}}$ and $j > i + 1$.

Now for any $T_\alpha \in \Sigma_e$ there exist x_α, y_α, for which $y_\alpha \in T_\alpha \cdot x_\alpha$ and $y_\alpha \in T_\alpha \cdot (F_{\frac{i\cdot\varepsilon}{2}} \setminus F_{\frac{(i-1)\cdot\varepsilon}{2}}) \subseteq T_\alpha \cdot F_{\frac{i\cdot\varepsilon}{2}}$. But $y_\alpha \in F_{\frac{j\cdot\varepsilon}{2}} \setminus F_{\frac{(j-1)\cdot\varepsilon}{2}}$. The last two relations show that for any $T_\alpha \in \Sigma_e$

$$T_\alpha \cdot F_{\frac{i\cdot\varepsilon}{2}} \nsubseteq F_{\frac{(i+1)\cdot\varepsilon}{2}} \tag{2}$$

because $y_\alpha \in T_\alpha \cdot F_{\frac{i\cdot\varepsilon}{2}}$, $y_\alpha \in F_{\frac{j\cdot\varepsilon}{2}} \setminus F_{\frac{(j-1)\cdot\varepsilon}{2}}$ and $j > i + 1$, hence $y_\alpha \notin T_\alpha \cdot F_{\frac{(i+1)\cdot\varepsilon}{2}}$. The obtained relation (3) demonstrates that Im is not m-basis. The achieved contradiction with the assumption that the basis Im is m-basis completes the proof of the theorem. □

Example 1. Let $\mathrm{Re} = \langle R, +, \tau \rangle$ be an additive group of real numbers with the natural topology on real line. The continuous function $f : \mathrm{Re} \to [0, 1]$ is defined by the formula $\forall x \in R\ fx = |\sin(x^2)|$. The function f is not uniformly continuous. We have to show that any basis Im of the function f is not m-basis.

Let r, s be arbitrary elements of D and $r < s \leqslant 1$. For simplicity we consider f for $x \geqslant 0$ (f — the even function). The element $F_r = f^{-1}[0, r)$ of Im contains all points $x_k = \sqrt{\pi k}$, $k = 0, 1, 2, \ldots, n, \ldots$, in which the value of f is equal to zero. Therefore

$$F_r \supset \{x_0, x_1, x_2, \ldots, x_n, \ldots\}. \tag{3}$$

It is easy to show that if $k \to +\infty$ the distance $d(x_k, x_{k+1}) = x_{k+1} - x_k$ between neighbouring zeros of f monotonically approaches to zero. For arbitrary $\varepsilon > 0$ there exists a natural number $n > 0$ for which $d(x_n, x_{n+1}) < 2\varepsilon$ and this unequality is true for all natural numbers grater n. Therefore for any ε-neighbourhood $T_\varepsilon = (-\varepsilon, +\varepsilon)$ of 0 there exists a real number $a > 0$ such that

$$T_\varepsilon + \{x_0, x_1, x_2, \ldots, x_n, \ldots\} \supset [a, +\infty), \tag{4}$$

we can take as a any real number greater than $x_n - \varepsilon$. From (4), (5) we have

$$[a, +\infty) \subset T_\varepsilon + F_r. \tag{5}$$

Note that the set F_s does not include any ray $[a, +\infty)$

$$[a, +\infty) \not\subseteq F_s \tag{6}$$

because $F_s = f^{-1}[0, s)$ does not contain the set of points $\{\sqrt{\pi/2 + \pi k} \mid k = 0, 1, 2, \ldots, n, ...\}$ in which $fx = 1$.

The relation (7), along with the relation (6), shows that for any $\varepsilon > 0$ $T_\varepsilon + F_r \not\subseteq F_s$. Hence any basis Im of the function $fx = |\sin(x^2)|$ is not m-basis.

Example 2. Let $\mathcal{Q} = \langle Q, +, \tau \rangle$ be an additive group of rational numbers with the natural topology τ induced by the natural topology on real line. For an irrational number $\alpha > 0$ we define the function $f : \mathcal{Q} \to [0, 1]$ by the formula

$$fx = \begin{cases} 0 & for \;\; x \in Q \cap [-\alpha, \alpha], \\ 1 & for \;\; x \in Q \setminus [-\alpha, \alpha]. \end{cases}$$

The function f is continuous, but f is not a uniformly continuous function. Any basis Im of the function f is not m-basis because for every $r \in D$ $F_r = Q \cap [-\alpha, \alpha]$.

The importance of the proven theorem for topological algebra lies in the fact that for topological-algebraic systems, more general than topological groups (that are paratopological and semitopological groups, topological monoids and topological loops [1, 2]), which have not uniform structures, continuous functions with m-basises are the natural generalization of uniformly continuous functions. Application of functions with m-basises permits to solve some problems, which concern the existance of embeddings such topological-algebraic systems in topological groups [1].

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ON THE SPECTRUM OF THE STÜRM–LIOUVILLE PROBLEM ON FINITE BINARY TREES

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Key words: finite binary regular trees, periodic potentials, spectrum

AMS Mathematics Subject Classification: 05C05, 34B24

Abstract. The paper deals with investigation of spectral properties of the Stürm–Liouville operators on finite binary trees. They have direct applications to many areas of Engineering because Stürm–Liouville equations are commonly used to model elastic structures, and their eigenvalues correspond to the natural frequencies of vibrations. The structures are often complex and modeled by graphs, finite trees in particular. We mainly discuss the case of periodic potentials on finite regular trees and, in particular, we present the complete description of the spectrum of the Dirichlet Laplacian on a binary regular tree of an arbitrary height.

1 Introduction

Consider a general binary tree T of height h with the nodes

$$N_{-1};\quad N_0;\quad N_1,\ N_2;\quad N_{11},\ N_{12},\ N_{21},\ N_{22};\quad \ldots\quad ; N_{k_1,\ldots,k_{h-1}},$$

$k_j = 1, 2,\quad j = 1, \ldots, h-1$. Let

$$b_0 = [N_{-1}, N_0];\quad b_1 = [N_0, N_1],\ b_2 = [N_0, N_2];\quad \ldots\quad ;$$

$$b_{k_1,\ldots,k_{h-2},k_{h-1}} = [N_{k_1,\ldots,k_{h-2}}, N_{k_1,\ldots,k_{h-2},k_{h-1}}];\quad k_j = 1, 2,\ j = 1, \ldots, h-1,$$

be the branches of the tree T:

$$T = \bigcup_{m=0}^{h-1} \bigcup_{\substack{k_j = 1, 2,\\ j = 1, \ldots, m}} b_{k_1,\ldots,k_m}\,.$$

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It is assumed that the interiors of the intervals $b_{k_1,\dots,k_m}$ are disjoint. (If $m=0$ it is meant that $b_{k_1,\dots,k_m} \equiv b_0$.) Moreover, let

$$l_0;\ \ l_1,\ l_2;\ \ l_{11},\ l_{12},\ l_{21},\ l_{22};\ \ \dots\ \ ; l_{k_1,\dots,k_{h-1}},\ \ k_j = 1,2,\ j=1,\dots,h-1,$$

be the lengths of the branches and let

$$x_{-1}=0;\ \ x_0 = l_0;\ \ x_1 = l_0 + l_1,\ x_2 = l_0 + l_2;\ \ \dots\ \ ;$$

$$x_{k_1,\dots,k_{h-1}} = \sum_{m=0}^{h-1} l_{k_1,\dots,k_m},\ \ k_j = 1,2,\ j=1,\dots,h-1\,.$$

Given a point $P \in T$, $x(P)$ is the distance of P to N_{-1} along the tree. Say, if $P \in b_{k_1,\dots,k_m}$, then $x(P) = x_{k_1,\dots,k_{m-1}} + |N_{k_1,\dots,k_m-1}P|$.

Let a real-valued function y be defined on T. Then $y = \{y_{k_1,\dots,k_m}\}_{m=0,\dots,h-1}$; $k_j = 1,2$, $j=1,\dots,h-1$, where the functions $y_{k_1,\dots,k_m}$ are defined on the branches $b_{k_1,\dots,k_m}$.

On T we consider the operator $Ly \equiv -y'' + q(x)y$, where $q(x) \geqslant 0$ is the potential, satisfying the Dirichlet boundary conditions. The domain $D(L)$ is defined in the following way: $y \in D(L)$ if and only if

$$\begin{cases}
y_{k_1,\dots,k_m} \in C^2[x_{k_1,\dots,k_{m-1}}, x_{k_1,\dots,k_m}], & \\
m = 0,\dots,h-1, & \text{smoothness conditions,} \\[2ex]
y_0(0) = 0, & \text{Dirichlet boundary condition,} \\[2ex]
y_{k_1,\dots,k_m}(x_{k_1,\dots,k_m}) = y_{k_1,\dots,k_m,1}(x_{k_1,\dots,k_m}) & \\
= y_{k_1,\dots,k_m,2}(x_{k_1,\dots,k_m}) & \\
m = 0,\dots,h-2, & \text{continuity conditions,} \\[2ex]
y'_{k_1,\dots,k_m}(x_{k_1,\dots,k_m}) = y'_{k_1,\dots,k_m,1}(x_{k_1,\dots,k_m}) & \\
+y'_{k_1,\dots,k_m,2}(x_{k_1,\dots,k_m}) & \\
m = 0,\dots,h-2, & \text{Kirchhoff conditions,} \\[2ex]
y_{k_1,\dots,k_{h-1}}(x_{k_1,\dots,k_{h-1}}) = 0, & \text{Dirichlet boundary conditions.}
\end{cases}$$

Note that $D(L)$ is dense in $L^2(T)$. For $y \in D(L)$

$$(Ly)(x) = \{(Ly_{k_1,\ldots,k_m})(x)\}_{m=0,\ldots,h-1,\ k_j=1,2} = \{-y''_{k_1,\ldots,k_m}(x) + q(x)y_{k_1,\ldots,k_m}(x),$$

$$x \in (N_{k_1,\ldots,k_{m-1}}, N_{k_1,\ldots,k_m})\}_{m=0,\ldots,h-1,\ k_j=1,2}\,.$$

(At the nodes of the tree $(Ly)(x)$ may not be defined.)

The operator L is symmetric, non-negative, and its eigenvalues are positive. Throughout the paper it is assumed that the potential q is such that the spectrum of the operator L is discrete and consists of countable number of eigenvalues of finite multiplicity.

2 Simultaneous equations for eigenvalues and eigenfunctions

2.1 The case of a general potential for a tree of an arbitrary height h

Consider a particular branch $b_{k_1,\ldots,k_m} = [N_{k_1,\ldots,k_{m-1}}, N_{k_1,\ldots,k_m}]$ of a general tree T of height h.

Moreover, let $\varphi_{k_1,\ldots,k_m}(x,\lambda)$ and $\psi_{k_1,\ldots,k_m}(x,\lambda)$ be the solutions of the equation

$$-y'' + q(x)y = \lambda y, \quad x_{k_1,\ldots,k_{m-1}} \leqslant x \leqslant x_{k_1,\ldots,k_m}\,,$$

satisfying

$$\begin{cases} \varphi_{k_1,\ldots,k_m}(x_{k_1,\ldots,k_{m-1}},\lambda) = 0, & \varphi'_{k_1,\ldots,k_m}(x_{k_1,\ldots,k_{m-1}},\lambda) = 1, \\ \psi_{k_1,\ldots,k_m}(x_{k_1,\ldots,k_{m-1}},\lambda) = 1, & \psi'_{k_1,\ldots,k_m}(x_{k_1,\ldots,k_{m-1}},\lambda) = 0. \end{cases}$$

Throughout the paper it is assumed that the potential q is such that the solutions $\varphi_{k_1,\ldots,k_m}(x,\lambda)$ and $\psi_{k_1,\ldots,k_m}(x,\lambda)$ exist, are defined uniquely, and also are analytic in λ.

The general solution of this equation has the form

$$y_{k_1,\ldots,k_m} = A_{k_1,\ldots,k_m}\varphi_{k_1,\ldots,k_m}(x,\lambda) + B_{k_1,\ldots,k_m}\psi_{k_1,\ldots,k_m}(x,\lambda)\,,$$

$$m = 0,\ldots,h-1,\ k_j = 1,2\,,\ j = 1,\ldots,m\,,$$

and $y = \{y_{k_1,\ldots,k_m}\}_{m=0,\ldots,h-1,\ k_j=1,2}$ is the general solution of the equation

$$Ly \equiv -y'' + q(x)y = \lambda y$$

on the tree T.

In order that $y \in D(L)$ it is necessary and sufficient that

$$
\begin{cases}
B_0 = 0 \\
\\
A_{k_1,\dots,k_m}\varphi_{k_1,\dots,k_m}(x_{k_1,\dots,k_m},\lambda) + B_{k_1,\dots,k_m}\psi_{k_1,\dots,k_m}(x_{k_1,\dots,k_m},\lambda) \\
= B_{k_1,\dots,k_m,1} = B_{k_1,\dots,k_m,2}\,, \\
m = 0,\dots,h-2,\ k_j = 1,2\,, \\
\\
A_{k_1,\dots,k_m}\varphi'_{k_1,\dots,k_m}(x_{k_1,\dots,k_m},\lambda) + B_{k_1,\dots,k_m}\psi'_{k_1,\dots,k_m}(x_{k_1,\dots,k_m},\lambda) \\
= A_{k_1,\dots,k_m,1} + A_{k_1,\dots,k_m,2}\,, \\
m = 0,\dots,h-2,\ k_j = 1,2\,, \\
\\
A_{k_1,\dots,k_{h-1}}\varphi_{k_1,\dots,k_{h-1}}(x_{k_1,\dots,k_{h-1}},\lambda) + B_{k_1,\dots,k_{h-1}}\psi_{k_1,\dots,k_{h-1}}(x_{k_1,\dots,k_{h-1}},\lambda) = 0, \\
k_j = 1,2\,,
\end{cases}
$$

This is a system of $2^{h+1}-2$ equations in $2^{h+1}-1$ unknowns $A_{k_1,\dots,k_m}, B_{k_1,\dots,k_m}$, $m = 0,\dots,h-1,\ k_j = 1,2$, and λ, defining the eigenvalues and eigenfunctions of the problem.

2.2 The case of a periodic potential

In this case all branches have the same length l, $x_{k_1,\dots,k_m} = (m+1)l,\ m = 1,\dots,h-1$, and $q(x) = q(x-ml)$ for $ml \leqslant x \leqslant (m+1)$.

Let $\varphi(x,\lambda) \equiv \varphi_0(x,\lambda)$ and $\psi(x,\lambda) \equiv \psi_0(x,\lambda)$, i. e. $\varphi(x,\lambda))$ and $\psi(x,\lambda)$ are the solutions of the equation

$$-y'' + q(x)y = \lambda y, \quad 0 \leqslant x \leqslant l\,,$$

satisfying

$$
\begin{cases}
\varphi(0,\lambda) = 0, & \varphi'(0,\lambda) = 1, \\
\psi(0,\lambda) = 1, & \psi'(0,\lambda) = 0.
\end{cases}
$$

Then

$$y_{k_1,\dots,k_m} = A_{k_1,\dots,k_m}\varphi(x-ml,\lambda) + B_{k_1,\dots,k_m}\psi(x-ml,\lambda)\,,$$

$$m = 0,\dots,h-1,\ k_j = 1,2\,,\ j = 1,\dots,m\,,$$

and the system of equations defining eigenvalues and eigenfunctions takes the form

$$\begin{cases} B_0 = 0 \\ \\ A_{k_1,\dots,k_m}\varphi(l,\lambda) + B_{k_1,\dots,k_m}\psi(l,\lambda) = B_{k_1,\dots,k_m,1} = B_{k_1,\dots,k_m,2}\,, \\ m = 0,\dots,h-2,\ k_j = 1,2\,, \\ \\ A_{k_1,\dots,k_m}\varphi'(l,\lambda) + B_{k_1,\dots,k_m}\psi'(l,\lambda) = A_{k_1,\dots,k_m,1} + A_{k_1,\dots,k_m,2}\,, \\ m = 0,\dots,h-2,\ k_j = 1,2\,, \\ \\ A_{k_1,\dots,k_{h-1}}\varphi(l,\lambda) + B_{k_1,\dots,k_{h-1}}\psi(l,\lambda) = 0,\ k_j = 1,2\,. \end{cases}$$

Lemma 1. *Let* $A_0 \in \mathbb{R}$*,*

$$W = \begin{pmatrix} \varphi'(l,\lambda) & \psi'(l,\lambda) \\ 2\varphi(l,\lambda) & 2\psi(l,\lambda) \end{pmatrix}, \quad W_0 = \begin{pmatrix} 0 & 0 \\ 2\varphi(l,\lambda) & 2\psi(l,\lambda) \end{pmatrix}.$$

Then all eigenvalues of the problem are defined by the equation

$$W_0 W^{h-1} \begin{pmatrix} A_0 \\ 0 \end{pmatrix} = 0\,.$$

Lemma 2. *Let* $\alpha(l,\lambda) = \varphi'(l,\lambda) + 2\psi(l,\lambda)$*, then this equation is equivalent to the equation* $A_0\varphi(l,\lambda)P_h(\alpha) = 0$*, where*

$$P_h(\alpha) = \sum_{i=0}^{[\frac{h-1}{2}]} (-1)^i 2^i \binom{h-1-i}{i} \alpha^{h-1-2i}\,.$$

Remark 1. Note that

$$P_h(\alpha) = 0 \Longleftrightarrow \begin{cases} \alpha = 0 \ \text{ or } \ B_h(\frac{2}{\alpha^2}) = 0, & h \ \text{ even,} \\ B_h(\frac{2}{\alpha^2}) = 0, & h \ \text{ odd,} \end{cases}$$

where

$$B_h(x) = \sum_{i=0}^{[\frac{h-1}{2}]} \binom{h-1-i}{i} (-1)^i x^i\,.$$

These polynomials are independent of the potential q, and in that sense they are universal. In the sequel they will be called the branch polynomials.

3 Structure of the spectrum of the Dirichlet Laplacian on a finite regular tree

For the Dirichlet Laplacian $q \equiv 0$, hence $\alpha(l, \lambda) = 3 \cos l\sqrt{\lambda}$.

Theorem 1. *The spectrum S_h of the Dirichlet Laplacian on a regular tree of height h consists of*

1) *the eigenvalues*

$$\lambda_n^{(1)} = \left(\frac{\pi n}{l}\right)^2, \quad n \in \mathbb{N},$$

of multiplicity 2^{h-1}.

2) *the eigenvalues*

$$\lambda_n^{(2)} = \left(\frac{\frac{\pi}{2} + \pi n}{l}\right)^2, \quad n \in \mathbb{N}_0,$$

of multiplicity $\frac{1}{3}(2^{h-1} + (-1)^h)$.

3) *the eigenvalues*

$$\lambda_{m,s,n}^{(3)} = \left(\frac{\arccos \frac{1}{3}\sqrt{\frac{2}{x_{m,s}}} + \pi n}{l}\right)^2, \quad n \in \mathbb{N}_0,$$

where $x_{m,s}$ are all roots of the branch polynomial B_m, and

4) *the eigenvalues*

$$\lambda_{m,s,n}^{(4)} = \left(\frac{\pi - \arccos \frac{1}{3}\sqrt{\frac{2}{x_{m,s}}} + \pi n}{l}\right)^2, \quad n \in \mathbb{N}_0.$$

If $\nu_n^{(3)}$ is one of the eigenvalues of the third group, then its multiplicity is equal to

$$\sum_{m:\, \lambda_{m,s,n}^{(3)} = \nu_n^{(3)}} 2^{(h-m-1)_+},$$

where $a_+ = a$ if $a \geqslant 0$ and $a_+ = 0$ if $a < 0$. The multiplicities of the eigenvalues of the fourth group are defined similarly.

The total number of all eigenvalues, which do not exceed $(\frac{\pi}{l})^2$, counted together with their multiplicities, is equal to $2^h - 1$.

Remark 2. Note that for all four series of the eigenvalues the expressions $\sqrt{\lambda_n^{(1)}}$, $\sqrt{\lambda_n^{(2)}}$, $\sqrt{\lambda_{m,s,n}^{(3)}}$, $\sqrt{\lambda_{m,s,n}^{(3)}}$ are periodic functions of n of period $\frac{\pi}{l}$.

Remark 3. One can expect that the spectrum S_∞ of the Dirichlet Laplacian on an infinite regular tree can be obtained from the case of finite regular trees of height h by passing to the limit as $h \to \infty$. If so, then it follows that

$$S_\infty = \bigcup_{n=0}^{\infty} \left\{ \left[\left(\frac{\vartheta + \pi n}{l} \right)^2, \left(\frac{\vartheta + \pi n}{l} \right)^2 \right] \bigcup \left\{ \left(\frac{\pi + \pi n}{l} \right)^2 \right\} \right\},$$

where $\vartheta = \arccos \frac{2\sqrt{2}}{3}$. Moreover,

$$\bigcup_{n=0}^{\infty} \left[\left(\frac{\vartheta + \pi n}{l} \right)^2, \left(\frac{\vartheta + \pi n}{l} \right)^2 \right]$$

should be a continuous spectrum, and $(\frac{\pi + \pi n}{l})^2$, $n \in \mathbb{N}_0$, should be eigenvalues of infinite multiplicity. This conforms with the result for infinite regular trees established by A. Sobolev and M. Solomyak [4].

4 The cases of special periodic potentials

Let q be the periodic function on the tree T defined by

$$q(x) = \begin{cases} a, & \text{for } \ 0 \leqslant x < m, \\ b, & \text{for } \ m \leqslant x \leqslant l, \end{cases}$$

where $0 < m < l, a, b \geqslant 0$, or by

$$q(x) = \frac{2}{\cos^2 x}, \quad 0 \leqslant x \leqslant l \ \left(0 < l < \frac{\pi}{2} \right).$$

In these cases one can find the explicit solutions to the equation

$$Ly = -y'' + q(x)y = \lambda y, \quad 0 \leqslant x \leqslant l.$$

In the first case direct calculations show that

$$y = \begin{cases} A \sin x\sqrt{\lambda - a} + B \cos x\sqrt{\lambda - b}, & 0 \leqslant x < m, \\ C \sin x\sqrt{\lambda - a} + D \cos x\sqrt{\lambda - b}, & m \leqslant x \leqslant l, \end{cases}$$

where A, B, C, D are arbitrary real numbers satisfying the conditions $y(m-0) = y(m+0)$, $y'(m-0) = y'(m+0)$.

In the second case

$$y = c_1(-\sqrt{\lambda}\sin x\sqrt{\lambda} + \tan x\cos x\sqrt{\lambda}) + c_2(\sqrt{\lambda}\cos x\sqrt{\lambda} + \tan x\sin x\sqrt{\lambda})\,,$$

where c_1 and c_2 are arbitrary real numbers. (See E. Kamke [3], pp. 504-505.)

This allows to explicitly compute $\alpha(L,\lambda)$ and to write out all the eigenvalues in the spirit of Theorem 1. In both cases there are also 4 groups of the eigenvalues $\lambda_n^{(1)}$, $\lambda_n^{(2)}$, $\lambda_{m,s,n}^{(3)}$, $\lambda_{m,s,n}^{(4)}$, but, in contrast to the case of the Dirichlet Laplacian, the square roots of these eigenvalues are not periodic in n. However for large n they are close to being periodic of period $\frac{\pi}{l}$, namely

$$\lim_{n\to\infty}\left(\sqrt{\lambda_{n+1}^{(k)}} - \sqrt{\lambda_n^{(k)}}\right) = \frac{\pi}{l}\,, \quad k = 1, 2\,,$$

and similar equalities hold for the eigenvalues of the third and the fourth groups. This follows since in both cases $\alpha(l,\lambda) \sim 3\cos l\sqrt{\lambda}$ as $\lambda \to \infty$. (Much better $\alpha(l,\lambda)$ is approximated by the expression $3\cos l\sqrt{\lambda - \bar{q}}$, where $\bar{q} = \frac{1}{l}\int_0^l q(x)dx$; in the first case $\bar{q} = \frac{m}{l}a(1-\frac{m}{l})b$, in the second case $\bar{q} = 2\frac{\tan l}{l}$.)

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ON APPLICATION OF EMBEDDING THEOREMS TO THE STUDY OF OSCILLATION OF SOLUTIONS OF THE SOBOLEV-TYPE

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Abstract. The paper analyzes functions of several variables oscillating on a half-axle on one selected variable. Using the weighted Sobolev space, the functional space is constructed for which embedding of functions into the space containing oscillating functions is proved. The results allow describing the behavior of solutions of Sobolev-type equations over time.

1 Introduction

The objective of this report is to generalize the work methods and deliverables [1–3] for a case of Sobolev weighted spaces. The study plan is similar to [3]: firstly, on the basis of a certain functional space (in this case, the weighted Sobolev space $W^N_{p,\alpha}(\mathbb{R}^+)$) to construct the space of functions of one variable to contain oscillating functions, then, based on appropriate Sobolev spaces, to determine the relevant spaces of functions of several variables, then to prove the local embedding theorem (the theorem of traces) for the two types of spaces.

Behavior of solutions of the first boundary-value problem for Sobolev-type equations is studied as an example of application of this method, in particular, to prove that eventually solutions cannot grow monotonically. The method establishes that behavior of such solutions in each point of the spatial domain depends on the scope of growth of the solution norm in certain Sobolev space. Meanwhile the requirement for that growth not exceeding the power growth value, is, due to the relevant embedding theorems, a sufficient condition for the solution belonging to a certain space with oscillating functions. At the same time, the specified condition is not necessary, therefore in this case we are forced to combine oscillating and summable functions in one class.

The basic data from the embedding theorems used in this report can be found in [4, 5], and from the theory of Sobolev-type differential equations — in [6, 7].

2 Background Information. Problem Statement

In 1943-1945, while looking for solutions of important applied problems S.L. Sobolev considered the problem of small-amplitude oscillations of a rotating ideal liquid about the dynamic equilibrium position [8–10]. The assumption is made that the volume Ω is filled up with the ideal liquid, which has the density $\rho = 1$ and rotates with a constant angular velocity ω as a solid body. The liquid does not move in the rotating coordinate system. That is a position of equilibrium about which the liquid creates small amplitude oscillations. Linearization of the Euler system of equations results in the following system of equations

$$\begin{cases} D_t\vec{v} + 2\omega[\vec{k} \times \vec{v}] + \nabla p = 0, \\ \operatorname{div}\vec{v} = 0 \quad (\operatorname{div} D_t\vec{v} = 0) \end{cases} \tag{1}$$

with two unknowns: velocity: $\vec{v} = \vec{v}(t, x_1, x_2, x_3) = (v_1, v_2, v_3)$ and pressure $p = p(x_1, x_2, x_3)$.

The Cauchy problem was set for the system of equations (1): $\vec{v}\,|_{t=0} = \vec{v}_0$. Please note that the system of equations (1) from the Cauchy problem is not a Cauchy–Kovalevskaya-type system.

By introducing into review the scalar potential function $u = u(t, x_1, x_2, x_3)$ by which the unknown velocity $\vec{v} = \nabla D_t^2 u + [\nabla D_t u, \vec{k}] + \vec{k}(\nabla u, \vec{k})$ and pressure $p = -D_t^3 u - D_t u$ are expressed, S.L. Sobolev moved from (1) to the equation

$$D_t^2 \Delta u + D_{x_3}^2 u = 0, \tag{2}$$

subsequently named the Sobolev equation. The equation (2) is one of the first examples of a partial differential equation not solved with respect to the higher-order derivative.

In [8] it was demonstrated that in $W_2^1(\mathbb{R}^3)$ the Cauchy problem

$$\begin{cases} u\,|_{t=0} = u_0(x), \\ D_t u\,|_{t=0} = u_1(x) \end{cases} \tag{3}$$

for the equation (2) was formulated correctly (the solution exists, it is unique and depends continuously upon the initial conditions), and also, the explicit problem solution is presented.

In [8] for the equation (2) the first initial-boundary problem (3)–(4) was set, where

$$u\,\Big|_{[0,T]\times\partial\Omega} = 0 \quad \forall T \in \mathbb{R}^+, \tag{4}$$

in case the domain Ω is homeomorphic to a ball.

It was also shown in [8] that in the $W_2^1(\Omega)$ the solution of the problem (2)–(4) exists, is unique and depends continuously on the initial conditions.

Dependence of the solution of the first initial-boundary problem (2)–(4) upon the boundary of the domain in a case where $n = 2$, $\Phi = 0$; u_0, $u_1 \in C^\infty(\Omega)$ was considered in [10, 11]. In particular, it was shown that in the circle $r < a$, in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$, and also in the rectangle $\{(x, y) \in \mathbb{R}^2 \,||x - x_0| < a,\, |y - y_0| < b\}$ the solution is almost periodical (has a discrete spectrum) while at any $\varepsilon > 0$ in the domain $r < 1 - \varepsilon \sin 4\varphi$ the solution is not almost periodical (has a continuous spectrum).

The similar problem for the case of $n = 3$, $\Phi = 0$; u_0, $u_1 \in W_2^1(\Omega)$ was studied in [12] where it was established that in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1$ and in the cylinder with a generator, which is parallel to the axis Oz, the solution of the first initial-boundary problem for the Sobolev equation $D_t^2 \Delta u + D_z^2 u = 0$ is almost periodical.

The studies performed demonstrated that "the solution behavior depends on the domain . . . , but the dependence is much more complicated than in the classical case" [10].

As you know, the energy integral always exists in the classical problems of mathematical physics. The conservation laws stipulate continuous dependence of the solution upon the initial and boundary conditions, and upon the right part of the domain. In particular, if the closure $\bar{\Omega}$ of the domain Ω is compact, the solution is almost periodical.

The problem (2)–(4) features a conservation law

$$\int\limits_{\Omega} D_t \operatorname{div} u dx dy dz = const,$$

but it is impossible to construct [11] integrals of energy of the type of

$$\left\| u(t, x), W_2^2(\Omega) \right\| \leqslant const, \tag{5}$$

therefore traditional methods of solution are not applicable to the problem (2)–(4).

Works of Sobolev and his followers [8–12] were a starting point of the systematic study of the equation (2) and its numerous generalizations. In particular, in [13] the Sobolev equation was studied in a case when the spatial variable belongs to the $n-$ dimensional space; in [14] generalization of the Sobolev equation was studied in a case of a compressible liquid; in [15] the following equation was considered

$$D_t^2 \Delta u + (D_{x_1}^2 + D_{x_2}^2)u = 0,$$

which was modeling small-amplitude nonlinear waves on shallow water.

By the end of the 20th century, equations of the type of

$$D_t^2 \sum_{i=1}^{n} \sum_{j=1}^{n} D_i(a_{ij}(t,x) \cdot D_j u) + \sum_{k=1}^{n-1} \sum_{l=1}^{n-1} D_k(b_{kl}(t,x) \cdot D_l u) = f(t,x) \tag{6}$$

were named the Sobolev-type equations. The extensive bibliography on that subject can be found in [16–19]. The study of the Sobolev-type equations was the first in-depth study of the following equations

$$D_t^l L_0(D_x)u + \sum_{k=0}^{l-1} D_t^{k-1} L_{l-k}(D_x)u = F,$$

not solved with respect to the higher-order time derivative. One of the fullest descriptions of the theory of equations of this type is presented in [6].

Let's recall that it is impossible to construct an integral of energy of the type (5) for the first boundary-value problem even for the equation (2). However in [20] it was proved for the first boundary-value problem for the equation (6) that if Ω is the bounded domain, and u_0, $u_1 \in W_2^m(\Omega)$, then for any sub-domain Ω' lying in the domain Ω together with its closure, the estimate

$$\left\| D_t^k u, W_2^m(\Omega') \right\| \leqslant C(t^{m-1} + 1). \tag{7}$$

is correct.

In [1, 2, 20, 21] the study was initiated of the problem on relationship between estimate (7) with an asymptotic behavior of a solution of the first boundary-value problem (6)–(3)–(4) in case of arbitrary domain. The present paper follows up on those studies, using embedding theorems of functional spaces.

3 Embedding Theorems and Oscillation of Functions of Several Variables on One Selected Variable

Let $\alpha \in \mathbb{R}$ and $N \in N_0$, $p \in \mathbb{R}$, $p \geqslant 1$.

The weighted Sobolev space is considered

$$W_{p,\alpha}^N(\mathbb{R}^+) = \left\{ \frac{D^k f}{(1+t)^\alpha} \in L_p^{loc}(\mathbb{R}^+) \forall k \in \{0,1,...,N\} \right\}$$

with norm $\left\| f, W_{p,\alpha}^N(\mathbb{R}^+) \right\| = \sum\limits_{k=0}^{N} \left\| \frac{D^k f}{(1+t)^\alpha}, L_p(\mathbb{R}^+) \right\|$.

For functions (and their derivatives) from space $W_{p,\alpha}^N(\mathbb{R}^+)$ the Laplace transform is defined as $\hat{f}(\gamma) = \int\limits_0^{+\infty} f(t) e^{-\gamma t} dt$ which we will consider at real values of parameter γ, where $\gamma > 0$.

Lemma 1. *Let* $\dfrac{1}{p} + \dfrac{1}{q} = 1$*,* $\alpha q + 1 > 0$*. If* $f \in W_{p,\alpha}^N(\mathbb{R}^+)$*, then* $\hat{D}^k f$ *is a function analytical on the interval* $(0, +\infty)$ $\forall k \in \{0,1,...,N\}$.

Remark. Lemma 1 represents an analogue of the known Paley–Wiener theorem for Fourier transform (about isometric homeomorphism of space $L_2(-a,a)$ and the whole analytic functions space).

Let's consider the following subspace in space $W_{p,\alpha}^N(\mathbb{R}^+)$

$$QW_{p,\alpha}^N(\mathbb{R}^+) = \left\{ f \in W_{p,\alpha}^N(\mathbb{R}^+) \,\middle|\, \exists \sup_{(0,1]} \left| D^k f(\gamma) \right| \in \mathbb{R} \right\}$$

with norm

$$\left\| f, QW_{p,\alpha}^N(\mathbb{R}^+) \right\| = \left\| f, W_{p,\alpha}^N(\mathbb{R}^+) \right\| + \sum_{k=0}^{N} \sup_{(0,1]} \left| \hat{D}^k f(\gamma) \right|.$$

According to the known Sobolev embedding theorem, if $f \in W_{p,\alpha}^N(\mathbb{R}^+)$, then $D^\beta f \in C(\mathbb{R}^+)$ at $\beta < N - \frac{1}{p}$.

It is interesting to find out the properties which the corresponding continuous derivative $D^\beta f$ of function f from space $QW_{p,\alpha}^N(\mathbb{R}^+)$ possesses.

Definition (see [6]). The determined and continuous function $y = f(t)$ on $\mathbb{R}^+$ is called oscillated on $\mathbb{R}^+$ if at any neighborhood of infinity it changes its sign: $\forall t \in \mathbb{R}^+ \quad \exists t_1$ and $t_2 \in \mathbb{R}^+ : t_1 > t$, $t_2 > t$, $t_1 \neq t_2$ and $f(t_1) \cdot f(t_2) < 0$.

Lemma 2. *Let* $\dfrac{1}{p} + \dfrac{1}{q} = 1$*,* $\alpha q + 1 > 0$*. If* $f \in QW_{p,\alpha}^N(\mathbb{R}^+)$*, then* $D^k f$ *is either a summable function, or a function oscillating on the half-axle* $\mathbb{R}^+$ *at every* $k \in \{0,1,...,N-1\}$.

Proof. It follows from the embedding theorem $D^\rho W_p^N(\mathbb{R}^+) \subset C(\mathbb{R}^+)$ that derivatives $D^k f$ of $k \in \{0, 1, ..., N-1\}$ order of functions from space $W_{p,\alpha}^N(\mathbb{R}^+)$ are continuous on the half-axle $\mathbb{R}^+$.

Subsequent proof is proved from the contrary. □

It is assumed that the function $D^k f$ is not oscillating on the half-axle $\mathbb{R}^+$. Let's consider the sequence $\{D^k f(t) e^{-\frac{t}{n}}\}_{n\in N}$ of nonnegative summables on the interval $[t_0, +\infty)$ of functions where t_0 is determined by the fact that at $[t_0, +\infty)$ the function $D^k f$ does not change its sign.

Moreover, $D^k f(t) e^{-\frac{t}{n}} \underset{n\to\infty}{\rightrightarrows} f(t)$, therefore under the Fatou theorem the $D^k f$ function is summable on $\mathbb{R}^+$ which contradicts the lemma condition.

The space $QW_{p,\alpha}^N(\mathbb{R}^+)$ contains a wide class of oscillating functions, in particular, $P_n(t)\sin t(P_n(t)$ — a polynomial of n order), the Bessel functions and others. At the same time, there exist oscillating functions from the space $W_{p,\alpha}^N(\mathbb{R}^+)$, not belonging to $QW_{p,\alpha}^N(\mathbb{R}^+)$, for example.

Make a note that the space $QV_{1,\alpha}^0(\mathbb{R}^+)$ contains not all oscillating functions from space $V_{1,\alpha}^0(\mathbb{R}^+)$. For example, convolution $\Delta_t * y$, where Δ_t — function from δ-shaped sequence of functions, and

$$y(t) = \begin{cases} a\sin t, & \text{if } t \in [2\pi n, \pi(2n+1)), \\ b\sin t, & \text{if } t \in [\pi(2n+1), 2\pi(n+1)), \end{cases}$$

$$n \in N_0, \quad a, b \in \mathbb{R}, \quad a > 0, \quad b > 0, \quad a \neq b.$$

Let's move over to consideration of functions of several variables.

Let's define space $W_{1,\beta}^N(\mathbb{R}^+, W_r^m(\Omega))$ as the space of the functions $f : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ on cylinder set $\mathbb{R}^+ \times \Omega$, where Ω is a spatial domain with a minimally smoothed boundary and the following properties:

1. $\forall k \in \{0, 1, ..., N\}$ and $\forall\,|\beta| \leqslant m \quad D_t^k D_x^\beta f \in L_1^{loc}([0, T] \times \Omega)$,
2. $\forall k \in \{0, 1, ..., N\} \quad D_t^k f\,|_\Omega \in W_r^m(\Omega)$,
3. $\left\|D_t^k f, W_r^m(\Omega)\right\| \in L_{1,\beta}(\mathbb{R}^+)$.

Let's define the norm of function f from space $W_{1,\beta}^N(\mathbb{R}^+, W_r^m(\Omega))$ as follows:

$$\left\|f, W_{1,\beta}^N(\mathbb{R}^+, W_r^m(\Omega))\right\| = \sum_{k=0}^N \left\| \frac{\left\|D_t^k f, W_r^m(\Omega)\right\|}{(1+t)^\beta}, L_1(\mathbb{R}^+) \right\|.$$

Theorem 1. *We have the following embedding:*

$$D_t^k D_x^\beta W_{1,\alpha}^N(\mathbb{R}^+, W_2^m(\Omega)) \Big|_{\mathbb{R}^+\times(x_1^0,x_2^0,\dots,x_n^0)} \subset QW_{1,\alpha}^{N-k}(\mathbb{R}^+)$$

for all $\rho = (\rho_1, \rho_2, ..., \rho_n)$ *so that* $\sum\limits_{i=1}^{n} \rho_i < m - \frac{n}{2}$ *and*

1) for all $k \in \{0, 1, ..., N\}$ *at* $\alpha \leqslant 1$,

2) for all $k \in \{[\alpha], [\alpha] + 1, ..., N\}$ *at* $1 < \alpha \leqslant N + 1$.

The embedding is understood to mean that

1. $\forall f \in W_{1,\alpha}^N(\mathbb{R}^+, W_2^m(\Omega))\ \exists \varphi = f \Big|_{\mathbb{R}^+\times(x_1^0,x_2^n,\dots,x_n^0)} : \ D_t^k D_x^\rho \varphi \in QW_{1,\alpha}^{N-k}(\mathbb{R}^+)$ *and*
2. $\left\| D_t^k D_x^\rho \varphi, QW_{1,\alpha}^{N-k}(\mathbb{R}^+) \right\| \leqslant C \left\| f, W_{1,\alpha}^N(\mathbb{R}^+, W_2^m(\Omega) \right\|.$

The proof is similar to the proof of theorem 1 [3].

4 The First Initial Boundary Value Problem for Sobolev-Type Equations with Constant Coefficients

The first initial boundary-value problem for the following Sobolev-type equation is under consideration

$$D_t^2 \sum_{i=1}^{n}\sum_{j=1}^{n} D_i(a_{ij}(x)\cdot D_j u(t,x)) + \sum_{k=1}^{n-1}\sum_{l=1}^{n-1} D_k(b_{kl}(x)\cdot D_l u(t,x)) = 0, \tag{8}$$

$$u(t,x)\,|_{t=0} = u_0(x), \quad D_t u(t,x)\,|_{t=0} = u_1(x), \quad u(t,x)\Big|_{[0,T]\times\partial\Omega} = 0 \tag{9}$$

for any $T \geqslant 0$, where the spatial domain Ω is the domain with a minimally smoothed boundary.

Moreover, it is assumed that in closure $\bar{\Omega}$ of domain Ω coefficients (8) have the following properties:

1. $\forall i, j \in \{1, 2, ..., n\}, \quad \forall k, l \in \{1, 2, ..., n-1\} \quad a_{ij}(x), b_{kl}(x) \in C^\infty(\bar{\Omega})$,
2. $\forall i, j \in \{1, 2, ..., n\} \forall k, l \in \{1, 2, ..., n-1\} a_{ij}(x) = a_{ji}(x), \quad b_{kl}(x) = b_{lk}(x)$, and also
3. $\mu_0 \left\|\xi, \mathbb{R}^n\right\|^2 \leqslant \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant \mu_1 \left\|\xi, \mathbb{R}^n\right\|^2$,

$$\nu_0 \sum_{k=1}^{n-1} \xi_k^2 \leqslant \sum_{k=1}^{n-1}\sum_{l=1}^{n-1} b_{kl}(x)\xi_k\xi_l \leqslant \nu_1 \sum_{k=1}^{n-1} \xi_k^2 \quad \forall x \in \Omega\ \forall \xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n.$$

Solution of the problem (8)–(9) lies in the space $\{C_t^N(\mathbb{R}^+)\,\big|\exists m\in N: D_t^k u\in W_2^m(\Omega)$ at $t\in\mathbb{R}^+\}$, with $N\geqslant 3$. In this case, the solution exists and it is unique [6].

Lemma 3. *For the problem* (8)–(9) *there exist first integrals*

$$I_K(t)=\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}(x)D_t^K D_j u(t,x), D_t^K D_i u(t,x))+$$
$$+\sum_{k=1}^{n-1}\sum_{l=1}^{n-1}(b_{kl}(x)D_t^{K-1}D_l u(t,x), D_t^{K-1}D_k u(t,x))=const$$

at $t\in\mathbb{R}^+$ *for all* $K\in\{1,2,...,N-1\}$.

Lemma 4. *If* $\Omega'\subset\bar{\Omega}'\subset\Omega$, *then* $D_t u\in W_{1,2}^{N-2}(\mathbb{R}^+, W_2^1(\Omega'))$, $D_l u\in W_{1,2}^{N-2}(\mathbb{R}^+, W_2^0(\Omega'))\quad\forall l\in\{1,2,...,n-1\}$.

Theorem 2. *Let* Ω *— a spatial domain with a minimally smoothed boundary,* $\Omega'\subset\bar{\Omega}'\subset\Omega$, $N\geqslant 3$. *Then the solution of the problem* (8) - (9) *is characterized by the following properties: functions* $y=D_x^\rho D_t^2 u(t,x)$, $y=D_x^\rho D_t^3 u(t,x)$, $\ldots$, $y=D_x^\rho D_t^{N-1}u(t,x)$, *and also* $y=D_x^\rho D_l D_t^u(t,x)$, $y=D_x^\rho D_l D_t^2 u(t,x)$, $\ldots$, $y=D_x^\rho D_l D_t^{N-2}u(t,x)$ *are either oscillated, or summable on* $\mathbb{R}^+$ *in any point* $x\in\Omega'\forall l\in\{1,2,...,n-1\}$, $\forall\rho=(\rho_1,...,\rho_n)$: $\sum\limits_{i=1}^{n}\rho_i<m-\frac{P}{2}$.

Apparently, the considered method to study the behavior of a solution can be applied to a rather wide class of equations, it allows considering problems whose solutions admit the estimate (7) from a unified point of view. In particular, the approach under consideration can be generalized to Sobolev-type equations with both constant and variable coefficients. Besides, the method is a universal one to the extent the conditions of the spatial domain are determined only by corresponding embedding theorems.

However, the method has some drawbacks. In particular, it does not distinguish almost periodic solutions. In addition, it does not allow obtaining the asymptotics of the solutions up to the boundary of the spatial domain.

By now, a range of further study areas has been identified. It is desirable to find conditions under which the solution of the first initial-boundary value problem for equation (8) is oscillatory. In addition, it is interesting to find out in which cases the solution will be almost periodical. There are many problems in mathematical physics whose solutions satisfy the estimates of the type (7). In this connection, the question of describing classes of equations to which the considered method can be applied is of special interest.

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THE SMALL PARAMETER METHOD FOR LINEAR DIFFERENTIAL REGULAR EQUATIONS IN $\mathbb{R}^N$ AND $\mathbb{R}^N_+$

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Key words: regular operator, hypoelliptic operator, boundary layer, regular degeneration, singular perturbation, uniformly solubility

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Abstract. Algorithms for asymptotic expansion of the solution of Dirichlet problem for regular equation with small parameter ε ($\varepsilon > 0$) at higher derivatives in $\mathbb{R}^n$ and $\mathbb{R}^n_+$ based on solution of the degenerated (under $\varepsilon \to 0$). Dirichlet problem for regular and hypoelliptic equation of the lower order are described. The approximate estimates for remainder terms of those expansions are obtained.

Introduction

The degeneration of the Dirichlet problem $\mathfrak{D}_\varepsilon$ for regular (in the sense of Mikhailov - Nikol'skii [1]– [5]) equation with small parameter ε ($\varepsilon > 0$) at higher derivatives into the Dirichlet problem $\mathfrak{D}_0$ for regular and hypoelliptic (introduced by Hormander [6]) equation in the Sobolev's anisotropic spaces $\mathbb{W}_2^{\mathscr{M}}(G)$ (generated by regular polyhedron $\mathscr{M}$ and by unbounded manifold G) is considered. The methods for constructing asymptotic expansion of the solution of Problem $\mathfrak{D}_\varepsilon$ based on Lindshted-Poincare's method, Prandell's boundary layer method (for references and for more details about those methods see [7]– [11]) and Lyusternik-Vishik's method [12] (as well as with using Newton's polyhedron method [13]) are described.

Note that the degenerated Problem $\mathfrak{D}_0$ can be solved by Bubnov-Galyorkin's method (see Ghazaryan and Karapetyan [14]) by choosing anisotropic B-splines as a base functions (see Tananyan [15]).

1 Basic notation and terminology

Throughout the paper, we will use the following standard notation: $\mathbb{N}$ is a set of natural numbers, $\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, $\mathbb{R}$ is a set of real numbers. For $n \in \mathbb{N}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$, $\mathscr{M} \subset \mathbb{N}_0^n$ and

$\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ we denote

$$|x| = \left(x_1^2 + \ldots + x_n^2\right)^{\frac{1}{2}}, \quad x^{(j)} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \ (1 \leqslant j \leqslant n),$$

$$\alpha! = \alpha_1! \ldots \alpha_n!, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n, \quad \beta \leqslant \alpha \iff \beta_j \leqslant \alpha_j \ (1 \leqslant j \leqslant n),$$

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta! \, (\alpha - \beta)!} \ (\beta \leqslant \alpha), \quad \alpha\beta = \alpha_1\beta_1 + \ldots + \alpha_n\beta_n,$$

$$\mathscr{M}^2 \equiv \mathscr{M} \times \mathscr{M} \equiv \{(\alpha, \beta) : \alpha \in \mathscr{M}, \ \beta \in \mathscr{M}\}$$

$$\xi^\alpha = \xi_1^\alpha \ldots \xi_n^\alpha, \quad D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n},$$

where $D_j = \frac{\partial}{\partial x_j}$ $(1 \leqslant j \leqslant n)$.

Through $\mathbb{C}(G)$ is denoted the space of uniformly continuous on a domain G of functions f with the norm

$$\|f\|_{\mathbb{C}(G)} \equiv \sup_{x \in G} |f(x)|.$$

For a finite set of multi-indices $\mathscr{M} \subset \mathbb{N}_0^n$ and domain $G \subset \mathbb{R}^n$ we denote

$$\mathbb{W}_2^{\mathscr{M}}(G) \equiv \left\{ f \in \mathbb{L}_2(G) : \|f\|_{\mathbb{W}_2^{\mathscr{M}}(G)} \equiv \sum_{\alpha \in \langle \mathscr{M} \cup \{0\} \rangle} \|D^\alpha f\|_{\mathbb{L}_2(G)} < \infty \right\},$$

where $\langle \mathscr{M} \rangle$ is a convex hull of the collection $\mathscr{M}$, and by $\mathring{\mathbb{H}}_{\mathscr{M}}(G)$ is denoted a closure of the set $\mathbb{C}_0^\infty(G)$ with respect to the norm $\|.\|_{\mathbb{W}_2^{\mathscr{M}}(G)}$.

In the Hilbert space $\mathbb{H}$ the inner product is denoted by $(.,.)_{\mathbb{H}}$.

In this work, all functional scapes will be assumed to be real.

2 Problem Statement

Let $\Omega \subseteq \mathbb{R}^n$, $\overline{\varepsilon} \in (0,1)$, $\mathscr{N} \subset \mathbb{N}_0^n$ and $\mathscr{N}_0 \subseteq \mathscr{N}$ be finite collections of multi-indices, ψ is a non-negative function defined on $\mathscr{N} \times \mathscr{N}$, and let

$$L_\varepsilon \equiv L_\varepsilon(x, D) \equiv \sum_{\alpha, \beta \in \mathscr{N}} \varepsilon^{\psi(\alpha,\beta)} D^\alpha \left(\eta_{\alpha,\beta}(x, \varepsilon) D^\beta\right) \quad (\eta_{\alpha,\beta}(x,\varepsilon) \not\equiv 0, \ \alpha, \beta \in \mathscr{N}) \tag{1}$$

and

$$L_0 \equiv L_0(x, D) \equiv \sum_{\alpha, \beta \in \mathscr{N}_0} D^\alpha \left(\eta_{\alpha,\beta}(x, 0) D^\beta\right) \quad (\eta_{\alpha,\beta}(x,0) \not\equiv 0, \ \alpha, \beta \in \mathscr{N}_0) \tag{2}$$

be linear differential operators with real coefficients defined on $\Omega \times [0, \overline{\varepsilon}]$.

Consider the following boundary problems:

Problem $\mathfrak{D}_0$. Find a solution $u \in \mathring{\mathbb{H}}_{\mathscr{N}_0}(\Omega)$ of the equation

$$L_0 u = h, \quad h \in \mathbb{W}_2^{\infty}(\Omega) \equiv \bigcap_{p=1}^{\infty} \mathbb{W}_2^{(p)}(\Omega). \tag{3}$$

Problem $\mathfrak{D}_\varepsilon$. Find a solution $u \in \mathring{\mathbb{H}}_{\mathscr{N}}(\Omega)$ of the equation

$$L_\varepsilon u_\varepsilon = h, \quad h \in \mathbb{W}_2^{\infty}(\Omega). \tag{4}$$

In the further will be used the following notations

$$\varphi(\nu) \equiv \min_{\substack{\alpha,\beta \in \mathscr{N} \\ \alpha+\beta=\nu}} \psi(\alpha, \beta) \quad \nu \in \mathscr{N} + \mathscr{N} \equiv \{\alpha + \beta : \alpha, \beta \in \mathscr{N}\},$$

$$\varphi_{\mathscr{M}}^{opt}\left(\alpha^0\right) \equiv \min \left\{ q \in \mathbb{R} : \forall \varepsilon \in (0, \overline{\varepsilon}], \forall \xi \in \mathbb{R}^n, \xi \geqslant \mathbf{0}, \varepsilon^q \xi^{\alpha^0} \leqslant \sum_{\alpha \in \mathscr{M}} \varepsilon^{\varphi(\alpha)} \xi^{\alpha} \right\} \tag{5}$$

$$\alpha^0 \in \langle \mathscr{M} \rangle, \mathscr{M} \subseteq \mathscr{N} + \mathscr{N}.$$

On the operators L_0 and L_ε we impose the following restrictions:

(A$_1$) a) the functions $\eta_{\alpha,\beta}(x, \varepsilon)$ $(\alpha, \beta \in \mathscr{N})$ are infinitely differentiable on $\overline{\Omega} \times [0, \overline{\varepsilon}]$;

b) for each $\alpha, \beta \in \mathscr{N}_0$ the function $\eta_{\alpha,\beta}(x, \varepsilon)$, as $\varepsilon \to 0$, uniformly with respect to x tends to $\eta_{\alpha,\beta}(x, 0)$;

(A$_2$) there exists the constant $\chi_1 > 0$ such that

$$(L_0 w, w) \geqslant \chi_1 \sum_{\alpha \in \mathscr{N}_0 \cup \{0\}} \|D^\alpha w\|^2 \qquad \forall w \in \mathbb{C}_0^{\infty}(\Omega); \tag{6}$$

(A$_3$) $\{\gamma \in \mathbb{N}_0^n : \gamma \leqslant \alpha\} \subseteq \langle \mathscr{N} \cup \{0\} \rangle$ for all $\alpha \in \mathscr{N}$;

(A$_4$) a) the functions $\eta_{\alpha,\beta}(x, \varepsilon)$ are uniformly continuous with respect to x on $\Omega \times (0, \overline{\varepsilon}]$, for $(\alpha, \beta) \in \mathscr{R} \equiv \left\{(\alpha, \beta) \in \mathscr{N}^2 \backslash \mathscr{N}_0^2 : |\alpha + \beta| \equiv 0 \,(\mathrm{mod} 2)\right\}$;

b) there exists the constant $\kappa_1 > 0$ such that

$$|\eta_{\alpha,\beta}(x, \varepsilon)| \leqslant \kappa_1 \quad \forall x \in \Omega, \forall \varepsilon \in (0, \overline{\varepsilon}], (\alpha, \beta) \in \mathscr{R};$$

c) there exists the constant $\chi_2 > 0$ such that

$$\sum_{(\alpha,\beta)\in\overline{\mathscr{R}}} \varepsilon^{\psi(\alpha,\beta)}\eta_{\alpha,\beta}\left(x,0\right)\left(i\xi\right)^{\alpha+\beta} \geqslant \chi_2 \sum_{\alpha\in\mathscr{B}} \varepsilon^{\varphi^{opt}_{\mathscr{N}+\mathscr{N}}(2\alpha)}\xi^{2\alpha} \quad \forall\xi\in\mathbb{R}^n, \forall\varepsilon\in(0,\overline{\varepsilon}],$$

where

$$\overline{\mathscr{R}} \equiv \left\{(\alpha,\beta)\in\mathscr{R} : \varphi\left(\alpha+\beta\right) = \varphi^{opt}_{\mathscr{N}+\mathscr{N}}\left(\alpha+\beta\right)\right\},$$

$$\mathscr{V} \equiv \left\{\alpha\in\mathscr{N}\backslash\mathscr{N}_0 : \alpha\notin\left\langle(\mathscr{N}\backslash\mathscr{N}_0)\setminus\{\alpha\}\right\rangle\right\},$$

$$\mathscr{B} \equiv \mathscr{V}\cup\left\{\alpha\in(\mathscr{N}\backslash\mathscr{N}_0)\backslash\mathscr{V} : \varphi^{opt}_{(\mathscr{N}\backslash\mathscr{N}_0)\backslash\{\alpha\}}\left(\alpha\right) > \varphi^{opt}_{\mathscr{N}\backslash\mathscr{N}_0}\left(\alpha\right)\right\};$$

d) there exists the constant $\kappa_3 > 0$ such that for every $(\alpha,\beta)\in\mathscr{I}\equiv\left\{(\alpha,\beta)\in\mathscr{N}^2\backslash\mathscr{N}_0^2 : |\alpha+\beta|\equiv 1\,(\mathrm{mod}2)\right\}$ and $\gamma,\delta\in\mathbb{N}_0^n$, if $\gamma\leqslant\alpha$, $\delta\leqslant\beta$ and $\gamma+\delta\neq\alpha+\beta$, then

$$\left|D^{\gamma+\delta}\eta_{\alpha,\beta}\left(x,\varepsilon\right)\right| \leqslant \kappa_3 \quad x\in\Omega, \varepsilon\in(0,\overline{\varepsilon}];$$

(A_5) for every $\alpha,\beta\in\mathscr{N}+\mathscr{N}, \alpha\leqslant\beta, \alpha\neq\beta$

$$\varphi^{opt}_{\mathscr{N}+\mathscr{N}}\left(\alpha\right) < \varphi^{opt}_{\mathscr{N}+\mathscr{N}}\left(\beta\right).$$

(A_6) ψ takes only integer values;

3 Terms of solubility and uniform solubility

Definition 1. Problem $\mathfrak{D}_0$ is said to be solvable if for any $h\in\mathbb{L}_2(\Omega)$ the equation (3) has a solution $w_0\in\mathring{\mathbb{W}}_2^{\mathscr{N}_0}\left(\Omega\right)$ such that

$$\|w_0\|_{\mathbb{W}_2^{\mathscr{N}_0}(\Omega)} \leqslant C\,\|h\|_{\mathbb{L}_2(\Omega)}$$

for some constant $C > 0$ independent of h.

Remark 1. (see [1] and [3]) Let Ω is whole space, half space or strip. Then Problem $\mathfrak{D}_0$ is solvable if Condition (A_2) holds. If $h\in\mathbb{W}_2^\infty(\Omega)$ then the solution w_0 of Problem $\mathfrak{D}_0$ is smooth, i.e. $w_0\in\mathbb{W}_2^\infty(\Omega)$ (see [19]) and hence $D^\alpha w_0\in\mathbb{C}\left(\overline{\Omega}\right)$ for any $\alpha\in\mathbb{N}_0^n$ by the known embedding theorem (see [20], p. 129).

Definition 2. (see [12] and [16]) Problem $\mathfrak{D}_\varepsilon$ is said to be uniformly solvable if there exists a number $\varepsilon_0 > 0$ for which

a) Problem $\mathfrak{D}_\varepsilon$ is solvable for $\varepsilon\in(0,\varepsilon_0]$, i.e., for every $h\in\mathbb{L}_2(\Omega)$ equation (4) has a solution $u_\varepsilon\in\mathring{\mathbb{W}}_2^{\mathscr{N}}\left(\Omega\right)$;

b) there exists the number $C_0 > 0$, and the functional space B_ε $\left(\mathring{\mathbb{H}}_{\mathscr{N}}(\Omega) \subset B_\varepsilon\right)$ with the norm $\|.\|_{B_\varepsilon}$ such that for all $u \in \mathring{\mathbb{H}}_{\mathscr{N}}(\Omega)$

$$\|u\|_{B_\varepsilon} \leqslant C_0 \|h\|_{\mathbb{L}_2(\Omega)}, \quad \varepsilon \in (0, \varepsilon_0].$$

Remark 2. (see [1] and [3]) Let Ω is whole space, half space or strip. Then Problem $\mathfrak{D}_\varepsilon$ is solvable for any fixed $\varepsilon \in (0, \varepsilon_0]$ if Condition $(\mathrm{A}_1) - (\mathrm{A}_4)$ hold.

Theorem 1. *(see [18]) Let $\mathscr{N} \subset \mathbb{N}_0^n$, $\langle\mathscr{N}\rangle$ be a regular polyhedron, $\Omega \subset \mathbb{R}^n$ be a domain satisfying the shift conditions (see [20]), and the operator L_ε satisfy conditions $(A_1) - (A_8)$. Then the problem $\mathfrak{D}_\varepsilon$ is uniformaly solvable, i.e. there exists the constants $\overline{\overline{\varepsilon}} \in (0, \overline{\varepsilon}]$, $C_1 > 0$ and $C_2 > 0$ such that for all $u \in \mathring{\mathbb{H}}_{\mathscr{N}}(\Omega)$ holds*

$$\|u\|_\varepsilon^2 \equiv \sum_{\alpha \in \langle\mathscr{N}\rangle \setminus \langle\mathscr{N}_0\rangle} \varepsilon^{\varphi^{opt}_{\mathscr{N}+\mathscr{N}}(2\alpha)} \|D^\alpha u\|^2 + \sum_{\alpha \in \langle\mathscr{N}_0 \cup \{0\}\rangle} \|D^\alpha u\|^2 \leqslant C_1 (L_\varepsilon u, u) \quad \forall \varepsilon \in (0, \overline{\overline{\varepsilon}}].$$

4 Poincare method on $\mathbb{R}^n$

Theorem 2. *Let $\Omega \equiv \mathbb{R}^n$, $m \in \mathbb{N}_0$ and*

I. *a) Conditions (A_1) and (A_6) hold;*

b) The coefficients $\eta_{\alpha,\beta}(x, \varepsilon)$ $(\alpha, \beta \in \mathscr{N})$ of the operator L_ε bounded with its derivatives of x_n up to order $m+1$ on $\overline{\Omega} \times [0, \overline{\varepsilon}]$;

II. *a) Problem $\mathfrak{D}_0$ is solvable;*

b) The solution w_0 of Problem $\mathfrak{D}_0$ is smooth, i.e. $w_0 \in \mathbb{W}_2^\infty(\mathbb{R}^n)$;

III. *Problem $\mathfrak{D}_\varepsilon$ is uniformly solvable;*

Then the solution u_ε of the problem $\mathfrak{D}_\varepsilon$ admits the following asymptotic representation:

$$u_\varepsilon = w_0 + \sum_{i=1}^{m} \varepsilon^i w_i + z_m, \tag{7}$$

where w_0 is the solution of the problem $\mathfrak{D}_0$, w_i $(i = 1, \ldots, m)$ is the solution of the $\mathfrak{D}_0$ type problem, and for the remainder z_m holds the following estimate:

$$\|z_m\|_\varepsilon = O\left(\varepsilon^{m+1}\right) \tag{8}$$

($\|\cdot\|_\varepsilon$ is a norm from Condition III, see Definition 2).

Proof. Let $N \in \mathbb{N}_0$. Since Condition (A_1) the coefficients $\eta_{\alpha,\beta}$ can be represented as a finite power series with respect to ε with the remainder term of $(N+1)$-th order:

$$\eta_{\alpha,\beta}(x,\varepsilon) = \eta^{(0)}_{\alpha,\beta}(x) + \sum_{i=1}^{N} \varepsilon^i \eta^{(i)}_{\alpha,\beta}(x) + \varepsilon^{N+1} \bar{\eta}^{(N+1)}_{\alpha,\beta}(x,\varepsilon)\,(\alpha,\beta \in \mathscr{N}) \tag{9}$$

$(\eta^{(0)}_{\alpha,\beta}(x) \equiv \eta_{\alpha,\beta}(x,0))$, then by Condition (A_6)

$$L_\varepsilon = \sum_{s=0}^{N} \varepsilon^s L^{(s)} + \varepsilon^{N+1} L^{(N+1)}, \tag{10}$$

where

$$L^{(0)} \equiv L_0, \quad L^{(s)} \equiv L^{(s)}(D,x) \equiv \sum_{\substack{\alpha,\beta\in\mathscr{N} \\ 0\leqslant j\leqslant N \\ \psi(\alpha,\beta)+j=s}} D^\alpha \eta^{(j)}_{\alpha,\beta}(x)\, D^\beta \quad (s = 1,\ldots,N), \tag{11}$$

$$L^{(N+1)} \equiv L^{(N+1)}(D,x,\varepsilon) \equiv \sum_{\substack{\alpha,\beta\in\mathscr{N} \\ 0\leqslant s\leqslant N \\ \psi(\alpha,\beta)+s\geqslant N+1}} D^\alpha \eta^{(s)}_{\alpha,\beta}(x)\, D^\beta + \sum_{\alpha,\beta\in\mathscr{N}} \bar{\eta}^{(N+1)}_{\alpha,\beta}(x,\varepsilon). \tag{12}$$

Let $N = m$ and let w_0 is the solution of Problem $\mathfrak{D}_0$, and let $w_i \in \mathring{\mathbb{H}}_{\mathscr{N}_0}(\mathbb{R}^n)$ $(i = 1,\ldots,m)$ is the solution of the equation

$$L_0 w_i = -\sum_{s=1}^{i} L^{(s)} w_{i-s} \tag{13}$$

It is obvious that by Condition III

$$w_i \in \mathbb{W}^\infty_2(\mathbb{R}^n) \quad i = 1,\ldots,m. \tag{14}$$

Denote

$$u^{(m)} \equiv w_0 + \sum_{i=1}^{m} \varepsilon^i w_i.$$

Thus

$$L_\varepsilon u^{(m)} = L_0 w_0 + \sum_{i=1}^{m} \varepsilon^i \left(L_0 w_i + \sum_{s=1}^{i} L^{(s)} w_{i-s} \right) + \varepsilon^{N+1} \sum_{i=0}^{m} \sum_{r=0}^{i} \varepsilon^{i-r} L^{(N+1-r)} w_i. \tag{15}$$

It is not hard to see (using forms 11 and 12, by Conditions I, II and 14) that exists number $M > 0$ such that

$$\left\| L^{(N+1-r)} w_i \right\| \leqslant M \quad (r = 0, \ldots, i; i = 0, \ldots, m),$$

hence from (15) by (13) follows that exists number $K > 0$ such that

$$\left\| L_\varepsilon u^{(m)} - h \right\| \leqslant K \varepsilon^{m+1} \quad \forall x \in \mathbb{R}^n.$$

Let u_ε is the solution of Problem $\mathfrak{D}_\varepsilon$, and let $z_m = u_\varepsilon - u^{(m)}$ (it is easy to see that $z_m \in \mathring{\mathbb{H}}_{\mathscr{N}}(\mathbb{R}^n)$). Then by Condition III

$$\|z_m\|_\varepsilon \leqslant (L_\varepsilon z_m, z_m) = (L_\varepsilon u_\varepsilon, z_m) - \left(L_\varepsilon u^{(m)}, z_m \right) = -(L_\varepsilon u^{(m)} - h, z_m),$$

so by Cauchy type inequality

$$\|z_m\|_\varepsilon \leqslant \frac{1}{2} \left(\omega \left\| L_\varepsilon u^{(m)} - h \right\| + \frac{1}{\omega} \|z_m\| \right),$$

therefore

$$\|z_m\|_\varepsilon = O\left(\varepsilon^{m+1}\right).$$

Remark 3. Under Conditions $(\mathrm{A}_1) - (\mathrm{A}_6)$ the solution u_ε admits an asymptotic expansion (7) where w_0 is the solution of Problem $\mathfrak{D}_0$, and $w_i \in \mathring{\mathbb{W}}_2^{\mathscr{N}_0}(\mathbb{R}^n)$ $(i = 1, \ldots, m)$ is the solution of the equation (13) and for the remainder z_m holds the estimate (8).

5 Terms of regular degeneration

Denote

$$k_n \equiv \max_{\alpha \in \mathscr{N}_0} \alpha_n, \quad l_n \equiv \max_{\alpha \in \mathscr{N}} \alpha_n - k_n,$$
$$e^n \equiv (0, \ldots, 0, 1), \quad q_n \equiv \psi\left((k_n + l_n) e^n, (k_n + l_n) e^n\right).$$

We impose the following additional restriction on coefficients of operator L_ε
(A_7) for every $\alpha, \beta \in \mathscr{N} + \mathscr{N}$

$$\psi(\alpha, \beta) \geqslant \frac{(\alpha_n + \beta_n - 2k_n) q_n}{2l_n} \quad \text{with } \alpha + \beta = (\alpha_n + \beta_n) e^n,$$

$$\psi(\alpha,\beta) > \frac{(\alpha_n+\beta_n-2k_n)\,q_n}{2l_n} \quad \text{with } \alpha+\beta \neq (\alpha_n+\beta_n)\,e^n.$$

Let $\Omega \equiv \mathbb{R}^n_+ \equiv \{x \in \mathbb{R}^n : x_n > 0\}$, $\varkappa \in \mathbb{N}$, $N \in \mathbb{N}_0$ and $t = x_n \varepsilon^{-\varkappa}$. Then, under the Condition (A_1) the coefficients $\eta_{\alpha,\beta}$ can be represented as in formula (9), and in addition the functions $\eta^{(i)}_{\alpha,\beta}$ can be represented as a finite power series with respect to x_n:

$$\eta^{(i)}_{\alpha,\beta}(x) = \eta^{(i,0)}_{\alpha,\beta}\left(x^{(n)}\right) + \sum_{j=1}^{N} x_n^j \eta^{(i,j)}_{\alpha,\beta}\left(x^{(n)}\right) + \varepsilon^{N+1}\bar{\eta}^{(i,N+1)}_{\alpha,\beta}(x,\varepsilon)$$
$$(\alpha,\beta \in \mathscr{N};\, i = 0,1,\ldots,N),$$

where $\eta^{(i,0)}_{\alpha,\beta}\left(x^{(n)}\right) \equiv \eta^{(i)}_{\alpha,\beta}\left(x^{(n)},0\right)$.

Since
$$\frac{\partial^s}{\partial x_n^s} = \varepsilon^{-s\varkappa}\frac{\partial^s}{\partial t^s} \ (s \geqslant 1),$$
then
$$D_x^\alpha \left(x_n^j \eta^{(i,j)}_{\alpha,\beta}\left(x^{(n)}\right)\right) D_x^\beta = \varepsilon^{-\varkappa(\alpha_n+\beta_n)+\varkappa j} D_y^\alpha \left(t^j \eta^{(i,j)}_{\alpha,\beta}\left(x^{(n)}\right)\right) D_y^\beta \tag{16}$$

$$(\alpha,\beta \in \mathscr{N};\, i = 0,1,\ldots,N;\, j = 0,1,\ldots,N),$$

where $y \equiv \left(x^{(n)}, t\right)$.

Using (16), the operator L_ε can be represented as follows:

$$L_\varepsilon = \sum_{\alpha,\beta\in\mathscr{N}} \sum_{i=0}^{N}\sum_{j=0}^{N} \varepsilon^{i-\varkappa(\alpha_n+\beta_n)+\varkappa j+\psi(\alpha,\beta)} D_y^\alpha \left(t^j \eta^{(i,j)}_{\alpha,\beta}\left(x^{(n)}\right)\right) D_y^\beta. \tag{17}$$

Denote
$$\gamma \equiv \max_{\alpha,\beta\in\mathscr{N}} \left(\psi(\alpha,\beta) - \varkappa(\alpha_n+\beta_n)\right).$$

From (17), combining terms with equal powers of ε, we get:

$$L_\varepsilon = \varepsilon^\gamma \left\{ M_0 + \sum_{s=1}^{N} \varepsilon^s R_s + \varepsilon^{N+1} R_{N+1} \right\}, \tag{18}$$

where

$$M_0 \equiv \sum_{\substack{\alpha,\beta\in\mathscr{N} \\ \psi(\alpha,\beta)-\varkappa(\alpha_n+\beta_n)=\gamma}} D_y^\alpha \eta_{\alpha,\beta}^{(0,0)}\left(x^{(n)}\right) D_y^\beta = \sum_{\substack{\alpha,\beta\in\mathscr{N} \\ \psi(\alpha,\beta)-\varkappa(\alpha_n+\beta_n)=\gamma}} D_y^\alpha \eta_{\alpha,\beta}\left(x^{(n)},0,0\right) D_y^\beta, \tag{19}$$

$$R_s \equiv \sum_{\substack{\alpha,\beta\in\mathscr{N} \\ 0\leqslant i\leqslant N, 0\leqslant j\leqslant N \\ \psi(\alpha,\beta)-\varkappa(\alpha_n+\beta_n)+i+\varkappa j=\gamma+s}} D_y^\alpha\left(t^j \eta_{\alpha,\beta}^{(i,j)}\left(x^{(n)}\right)\right) D_y^\beta \quad (s=1,\ldots,N),$$

$$R_{N+1} \equiv$$

$$\equiv \sum_{\substack{\alpha,\beta\in\mathscr{N} \\ 0\leqslant i\leqslant N, 0\leqslant j\leqslant N \\ \psi(\alpha,\beta)-\varkappa(\alpha_n+\beta_n)+i+\varkappa j>\gamma+N}} \varepsilon^{i-\varkappa(\alpha_n+\beta_n)+\varkappa j+\psi(\alpha,\beta)-N-1} D_y^\alpha\left(t^j \eta_{\alpha,\beta}^{(i,j)}\left(x^{(n)}\right)\right) D_y^\beta +$$

$$+\sum_{\substack{\alpha,\beta\in\mathscr{N} \\ 0\leqslant i\leqslant N}} \bar{\eta}_{\alpha,\beta}^{(i,N+1)}(x,\varepsilon).$$

Proposition 1. To M_0 was an ordinary differential operator of order $2\left(k_n+l_n\right)$ with a minor member of the order $2k_n$, it is necessary and sufficient to

1^0) $\gamma=-\frac{k_n q_n}{l_n}$;

2^0) $\varkappa=\frac{q_n}{2l_n}$ is a natural number;

3^0) $\psi(\alpha,\beta)\geqslant\frac{(\alpha_n+\beta_n-2k_n)q_n}{2l_n}$ for $\alpha+\beta=(\alpha_n+\beta_n)e^n$,

$\psi(\alpha,\beta)>\frac{(\alpha_n+\beta_n-2k_n)q_n}{2l_n}$ for $\alpha+\beta\neq(\alpha_n+\beta_n)e^n$.

Remark 4. Note that under the Condition (A_6) we can assume that $\varkappa=\frac{q_n}{2l_n}$ is a natural number (otherwise we can reach this by replacement variable).

Remark 5. Under the Conditions (A_1), (A_6) and (A_7) (in respect that 4) if $\varkappa=\frac{q_n}{2l_n}$ then the operator M_0 is an ordinary differential operator.

Let M_0 (introduced in (19)) is a ordinary differential operator satisfies the conditions of the proposition 1. We introduce the following equation (which is the characteristic equation of the operator M_0):

$$\lambda^{2k_n} Q(\lambda) \equiv \lambda^{2k_n} \sum_{\substack{\alpha_n e^n, \beta_n e^n \in \mathscr{N} \\ \psi(\alpha_n e^n, \beta_n e^n) - \varkappa(\alpha_n + \beta_n) = \gamma}} \eta_{\alpha_n e^n, \beta_n e^n} \left(x^{(n)}, 0, 0 \right) \lambda^{\alpha_n + \beta_n - 2k_n} = 0. \tag{20}$$

Definition 3. The degeneration of the Problem $\mathfrak{D}_\varepsilon$ into Problem $\mathfrak{D}_0$ is called regular if the Conditions (A_1), (A_6) and (A_7) hold and characteristic polynomial $Q(\lambda)$ has exactly l_n pairwise different roots with negative real parts.

For the complete symbol of the operator L_ε, we introduce the notation

$$L_\varepsilon(x, i\xi) \equiv \sum_{\alpha, \beta \in \mathscr{N}} \varepsilon^{\psi(\alpha,\beta)} \eta_{\alpha,\beta}(x, \varepsilon) (i\xi)^{\alpha+\beta}.$$

Theorem 3. *Let the Conditions (A_1), (A_6) and (A_7) hold and there exists a number C such that for all $\xi_n \in \mathbb{R}$ and $\varepsilon \in (0, \varepsilon_0]$*

$$\mathrm{Re} L_\varepsilon(x, 0, \ldots, 0, i\xi_n) \geqslant C \sum_{s=0}^{l_n} \varepsilon^{\varphi^{opt}_{\mathscr{N}+\mathscr{N}}(2(k_n+s)e^n)} |\xi_n|^{2(k_n+s)}. \tag{21}$$

Then the degeneration of the Problem $\mathfrak{D}_\varepsilon$ into Problem $\mathfrak{D}_0$ is regular.

6 Boundary layer method on $\mathbb{R}^n_+$

Definition 4. (see [12], p. 7). Let $v_\varepsilon(x) = v_\varepsilon(x_1, \ldots, x_n)$ be an s ($s \in \mathbb{N}$) times differentiable function in a domain $Q \subset \mathbb{R}^n$. Then v_ε is called boundary layer type function of order k ($k < s$), if

1. for every closed subset $\overline{K}$ of the domain Q ($\overline{K} \subset Q$), which does not intersect the boundary ∂Q of the domain Q ($\overline{K} \cap \partial Q = \varnothing$) and for every $\delta > 0$ there exists positive number ε_0 such that

$$|D^\alpha v_\varepsilon(x)| \leqslant \delta \quad \forall \varepsilon \in (0, \varepsilon_0], \forall x \in \overline{K}, |\alpha| \leqslant s;$$

2. there exist positive numbers M and ε_0 such that

$$|D^\alpha v_\varepsilon(x)| \leqslant M \quad \forall \varepsilon \in (0, \varepsilon_0], \forall x \in Q, |\alpha| = k;$$

3. for every $\delta > 0$ there exists positive number ε_0 such that

$$|D^\alpha v_\varepsilon(x)| \leqslant \delta \quad \forall \varepsilon \in (0, \varepsilon_0], \forall x \in \overline{Q}, |\alpha| < k;$$

Example 1. The typical examples of boundary layer type functions of order k on the positive semiaxis are

$$\varepsilon^k e^{-\frac{\lambda t}{\varepsilon}} \text{ and } \varepsilon^k P\left(\frac{t}{\varepsilon}\right) e^{-\frac{\lambda t}{\varepsilon}},$$

where $\lambda > 0$ and P is a polynomial.

Suppose $\tau \in (0, \infty)$, and $\phi(y)$ is an infinitely differentiable function of one variable, that equals to 1 when $y \leqslant \frac{\tau}{2}$ and vanishes when $y \geqslant \tau$.

It is possible to prove the following result.

Theorem 4. *Let $\Omega \equiv \mathbb{R}^n_+$, $m \in \mathbb{N}_0$ and*

I. *a) Conditions (A_1) and (A_6) hold;*

b) The coefficients $\eta_{\alpha,\beta}(x, \varepsilon)$ $(\alpha, \beta \in \mathscr{N})$ of the operator L_ε bounded with its derivatives of x_n up to order $m + k_n + 1$ on $\overline{\Omega} \times [0, \overline{\varepsilon}]$;

II. *a) Problem $\mathfrak{D}_0$ is solvable;*

b) The solution w_0 of Problem $\mathfrak{D}_0$ is smooth, i.e. $w_0 \in \mathbb{W}^\infty_2\left(\mathbb{R}^n_+\right)$;

III. *Problem $\mathfrak{D}_\varepsilon$ is uniformly solvable;*

IV. *The degeneration of Problem $\mathfrak{D}_\varepsilon$ into Problem $\mathfrak{D}_0$ is regular.*

Then the solution u_ε of Problem $\mathfrak{D}_\varepsilon$ admits the following asymptotic representation:

$$u_\varepsilon = w_0 + \sum_{i=1}^{m} \varepsilon^i w_i + \sum_{i=0}^{m+k_n} \varepsilon^i \left(v_i + \varepsilon \phi(x_n) \alpha_i\right) + z_m,$$

where w_0 is the solution of Problem $\mathfrak{D}_0$, w_i $(i = 1, \ldots, m)$ is the solution of the $\mathfrak{D}_0$ type problem, $v_i = \varepsilon^{k_n} \overline{v}_i$ $(i = 0, \ldots, m + k_n)$ is a boundary layer type function of order k_n, α_i $(i = 0, \ldots, m + k_n)$ is a polynomial of degree $k_n - 1$ with respect to x_n, and for the remainder z_m holds the following estimate:

$$\|z_m\|_\varepsilon = O\left(\varepsilon^{m+1}\right)$$

($\|.\|_\varepsilon$ is a norm from Condition III, see Definition 2).

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MODIFIED METHOD OF ENERGY INEQUALITIES AND AVERAGING OPERATORS WITH VARIABLE STEP IN THEORY OF LINEAR BOUNDARY VALUE PROBLEMS

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Key words: method of energy inequalities and averaging operators with variable step, strong solution, partial differential equation of the fourth order, partial differential equation of composite type

AMS Mathematics Subject Classification: 35D35

Abstract. The method of energy inequalities and averaging operators with variable step was applied in the article to prove the existence and uniqueness of strong solution of boundary value problem for the fourth-order partial differential equation of composite type.

1 Introduction

In the 1930s, studying boundary value problems for partial differential equations [1–3], J. Hadamard introduced the concept of well-posed problem. This and studying of Cauchy problem by I. G. Petrovskii, whose results were published in 1937 [4, 5], were a key factor of modern theory of partial differential equations creation.

The theory of solvability of various problems of differential equations and their systems receives further development using different methods of functional analysis. One of such tools is energy inequality

$$\|u\|_B \leqslant \|Lu\|_H, \tag{1}$$

where the operator L generated by considered problem acts from Banach space B into Hilbert space H, elements u from the domain $\mathcal{D}(L)$ of the operator L, a constant $c > 0$ is independent of u, $\|\cdot\|_B$ and $\|\cdot\|_H$ are norms in the spaces B and H.

The uniqueness in the space B of the solution of equation

$$Lu = F, \quad u \in \mathcal{D}(L), \tag{2}$$

for the the original problem generated operator equation (2) follows from inequality (1) immediately (if it exists).

The elliptic theory, integral transforms and other methods are used along with inequality (1) or various its modifications when proving theorems of existence [6–9].

But, for example, when studying many mixed and other problems for nonstationary equations given in noncylindrical domains, i. e. in domains changing with time, the mentioned mathods of proving solvability can not be migrate to this case automatically. In the same time the considering of such problems is dictated by real physical problems when modeling of specific phenomena. In the book [10, chapter 3, problems 11.9–11.11] J. L. Lions notes such problems for equations of evolutionary type in noncylindrical domains as problems.

2 Strong solution

We assume that the considered linear problem can be written in the form of linear operator equation (2), where an operator $L\colon B \to H$ with dense in the space B domain $\mathcal{D}(L)$. We denote the range of the operator L by $\mathcal{R}(L)$.

To prove the existence of solution of equation (2) for all $F \in H$ we need to show $\mathcal{R}(L)$ coincidence with H. As a rule, the equality $\mathcal{R}(L) = H$ doesn't hold, and even more so if the domain $\mathcal{D}(L)$ of the operator L represents a set of sufficiently smooth functions. In this connection the extension of the operator L is done.

We consider the extension of L in the strong topology. We assume that the operator $L\colon B \to H$ admits closure $\overline{L}$.

Definition 1. A solution of operator equation

$$\overline{L}u = F,\ u \in \mathcal{D}(\overline{L}), \tag{3}$$

is called a *strong* solution of equation (2) or differential problem which equation (2) describes.

If energy inequality (1) holds for the operator $L\colon B \to H$ and it admits closure $\overline{L}\colon B \to H$, then equality

$$\overline{\mathcal{R}(L)} = \mathcal{R}(\overline{L}) \tag{4}$$

is proved, where $\overline{R(L)}$ is the closure in H of the set $\mathcal{R}(L)$, $\mathcal{R}(\overline{L})$ is the range of the operator $\overline{L}$.

It follows from equality (4) that it suffices to prove energy inequality (1), the density of the range $\mathcal{R}(L)$ of the operator L in the space H and that the operator L admits closure $\overline{L}$ within the spaces B and H for proving the existence of the solution of equation (2) (the strong solution of equation (2)) for all $F \in H$. One can use averaging operators to achieve the formulated goals.

We note that the introduction of the averaging operators is related in a certain sense to approximation of defined functions by infinitely differentiable or other smoother functions. In their research S. L. Sobolev [11, 12] and K. Friedricks [13] offer integral operators with infinitely differentiable kernel as averaging operators. The structure of averaging operators with variable step based on the structures these integral operators and the partition of unity was offered in the papers by Deny and Lions [14] and Burenkov [15, 16], which allows to take into account the boundary conditions (see also [17]). It is known that the averaging operators allow to build the sequence of smooth functions, are used for the partition of unity, for the integral representation, for the theory of the continuation of functions. In the same time, the averaging operatirs with variable step can be used in the proof of the solvability of many boundary value problems for partial differential equations. More exactly: in the proof of the density of the set $\mathcal{R}(L)$ of the corresponding to the considered problem operator L in H.

Many boundary value problems for partial differential equations were consudered by the just described scheme of proved energy inequalities and averaging operators of variable step [19–34]. We note that we had to evaluate the commutator of the differential operators and the averaging operators here. If the differential expression is of the high order, then some problems arise when evaluating such commutators. In this connection we consider a slightly modified method of energy inequalities and averaging operatoes with variable step for a boundary value problem for the fourth-order differential equation of composite type (see also [35]).

3 Statement of problem for differential equation

Let $Q = (0,T) \times \Omega$, $\Omega \subset \mathbb{R}^n$, be a domain in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ of independent variables $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$, $x_0 \in (0,T)$, $\boldsymbol{x}' = (x_1, \ldots, x_n) \in \Omega$. For a function $u : \ Q \ni \boldsymbol{x} \to u(\boldsymbol{x}) \in \mathbb{R}$ we consider the fourth-order linear differential equation

$$\mathcal{L}u \equiv \mathcal{L}^{(0)}u + A^{(2)}u = f \tag{5}$$

with an operator of composite type in the leading part. Here $\mathcal{L}^{(0)} = \mathcal{L}^{(1)}\mathcal{L}^{(2)}$, $\mathcal{L}^{(1)} = \dfrac{\partial^2}{\partial x_0^2} - a^2\Delta$, $\mathcal{L}^{(2)} = \dfrac{\partial^2}{\partial x_0^2} + b^2\Delta$, a and b are some real constants, $a^2 < b^2$, $\Delta = \sum\limits_{i=1}^{n} \partial^2/\partial x_i^2$ is the Laplace operator, $A^{(2)}u = \sum\limits_{|\boldsymbol{\alpha}|\leqslant 2} a^{(\boldsymbol{\alpha})}(\boldsymbol{x})\boldsymbol{D}^{\boldsymbol{\alpha}}u$, $a^{(\boldsymbol{\alpha})} \in C^2(\overline{Q})$,

$\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_n)$, $|\boldsymbol{\alpha}| = \sum\limits_{i=0}^{n} \alpha_i$, α_i, $i = 0, \dots, n$, are nonnegative integers, $\boldsymbol{D}^{\boldsymbol{\alpha}} u = \dfrac{\partial^{|\boldsymbol{\alpha}|} u}{\partial x_0^{\alpha_0} \cdots \partial x_n^{\alpha_n}}$, $f:\ Q \ni \boldsymbol{x} \to f(\boldsymbol{x}) \in \mathbb{R}$ is a given square integrable function.

The boundary ∂Q of the domain Q consists of the bottom face $\Omega^{(0)} = \{\boldsymbol{x} \in \partial Q |\ x_0 = 0\}$, the top face $\Omega^{(T)} = \{\boldsymbol{x} \in \partial Q |\ x_0 = T\}$ and the lateral surface $\Gamma = \{\boldsymbol{x} \in \partial Q |\ 0 < x_0 < T\}$.

We consider equation (5) with the boundary conditions

$$u|_{\Omega^{(0)}} = \left.\frac{\partial u}{\partial x_0}\right|_{\Omega^{(0)}} = 0,\ lu \equiv \left.\frac{\partial^3 u}{\partial x_0^3}\right|_{\Omega^{(0)}} = \varphi(\boldsymbol{x}'), \left.\frac{\partial u}{\partial x_0}\right|_{\Omega^{(T)}} = 0 \tag{6}$$

on $\Omega^{(0)}$ and $\Omega^{(T)}$ and the boundary conditions

$$u|_{\Gamma} = \left.\frac{\partial^2 u}{\partial \boldsymbol{\nu}^2}\right|_{\Gamma} = 0 \tag{7}$$

on the lateral surface Γ, where $\varphi:\ \Omega \ni \boldsymbol{x}' \to \varphi(\boldsymbol{x}') \in \mathbb{R}$ is a given square integrable function, $\boldsymbol{\nu} = (\nu_0, \dots, \nu_n)$ is the unit normal to Γ outward with respect to Q. One can give conditions

$$\left.\frac{\partial u}{\partial x_0}\right|_{\Gamma} = \left.\frac{\partial^3 u}{\partial \boldsymbol{\nu}^3}\right|_{\Gamma} = 0 \tag{8}$$

instead of conditions (7) on the lateral area Γ.

The existence and uniqueness of weak solution of problem (5)–(7) are proved in [18].

We rewrite problem (5)–(7) in the operator form

$$\boldsymbol{L} u = \boldsymbol{F} \tag{9}$$

with operator $\boldsymbol{L} = (\mathcal{L}, l)$ and right-hand side $\boldsymbol{F} = (f, \varphi)$. For the domain $\mathcal{D}(\boldsymbol{L})$ of $\boldsymbol{L}$, we take the set of functions that are four times continuously differentiable in the closure $\overline{Q}$ of Q and satisfy the homogeneous boundary conditions in (6) and (7); i. e.,

$$\mathcal{D}(\boldsymbol{L}) = \left\{ u \in C^4(\overline{Q}) |\ u|_{\Omega^{(0)}} = \left.\frac{\partial u}{\partial x_0}\right|_{\Omega^{(0)}} = \left.\frac{\partial u}{\partial x_0}\right|_{\Omega^{(T)}} = u|_{\Gamma} = \left.\frac{\partial^2 u}{\partial \boldsymbol{\nu}^2}\right|_{\Gamma} = 0 \right\}.$$

By B we denote the Banach space obtained as the closure of $\mathcal{D}(\boldsymbol{L})$ in the norm

$$\|u\|_B = \sum_{|\boldsymbol{\alpha}|=3} \|\boldsymbol{D}^{\boldsymbol{\alpha}} u\|_{L_2(Q)} + \sup_{0<\tau<T} \sum_{|\boldsymbol{\alpha}|\leqslant 2} \|\boldsymbol{D}^{\boldsymbol{\alpha}} u\|_{L_2(\Omega)}(\tau),$$

and by $\boldsymbol{H}$ we denote the Hilbert space of right-hand sides of the operator equation (9); that is, $\boldsymbol{H} = L_2(Q) \times L_2(\Omega)$, $\|\boldsymbol{F}\|_{\boldsymbol{H}} = \|f\|_{L_2(Q)} + \|\varphi\|_{L_2(\Omega)}$.

In what follows, we consider equation (9) only in the spaces B and $\boldsymbol{H}$ thus introduced; $\boldsymbol{L}: \ B \ni u \to \boldsymbol{L}u \in \boldsymbol{H}$.

4 Energy inequality

Theorem 1. *The energy inequality*

$$q\|u\|_B \leqslant c\|\boldsymbol{L}u\|_{\boldsymbol{H}} \tag{10}$$

$\forall u \in \mathcal{D}(\boldsymbol{L})$ holds for the operator $\boldsymbol{L}$ in the operator equation (9), where c is some constant independent of u.

Proof. Consider the expression $2\mathcal{L}u\mathcal{M}u$, where $\mathcal{M}u = (T-x_0)\,\partial^3 u/\partial x_0^3 + (b^2 - a^2)(T-x_0)\partial/\partial x_0 \Delta u$, and represent it in the divergence form

$$\begin{aligned}
2\mathcal{L}u\mathcal{M}u &= \frac{\partial}{\partial x_0}\left((T-x_0)\left(\frac{\partial^3 u}{\partial x_0^3}\right)^2\right) + \left(\frac{\partial^3 u}{\partial x_0^3}\right)^2 + \\
&+ 2(b^2-a^2)\sum_{i=1}^n \frac{\partial}{\partial x_0}\left(\frac{\partial^3 u}{\partial x_0^3}(T-x_0)\frac{\partial^3 u}{\partial x_0 \partial x_i^2}\right) + 2(b^2-a^2)\sum_{i=1}^n \frac{\partial}{\partial x_0}\left(\frac{\partial^2 u}{\partial x_0^2}\frac{\partial^3 u}{\partial x_0 \partial x_i^2}\right) - \\
&- 2(b^2-a^2)\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial^2 u}{\partial x_0^2}\frac{\partial^3 u}{\partial x_0^2 \partial x_i}\right) + 2(b^2-a^2)\sum_{i=1}^n \left(\frac{\partial^3 u}{\partial x_0^2 \partial x_i}\right)^2 + \\
&+ 2(b^2-a^2)^2 \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial^3 u}{\partial x_0^2 \partial x_i}(T-x_0)\frac{\partial^3 u}{\partial x_0 \partial x_j^2}\right) - \\
&- 2(b^2-a^2)^2 \sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(\frac{\partial^3 u}{\partial x_0^2 \partial x_i}(T-x_0)\frac{\partial^3 u}{\partial x_0 \partial x_i \partial x_j}\right) + \\
&+ (b^2-a^2)^2 \sum_{i,j=1}^n \frac{\partial}{\partial x_0}\left((T-x_0)\left(\frac{\partial^3 u}{\partial x_0 \partial x_i \partial x_j}\right)^2\right) + \\
&+ ((b^2-a^2)^2 + 3a^2b^2)\sum_{i,j=1}^n \left(\frac{\partial^3 u}{\partial x_0 \partial x_i \partial x_j}\right)^2 - 2a^2b^2 \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial^3 u}{\partial x_i \partial x_j^2}(T-x_0)\frac{\partial^3 u}{\partial x_0^3}\right) +
\end{aligned}$$

$$+ 2a^2b^2 \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}(T-x_0)\frac{\partial^4 u}{\partial x_0^3 \partial x_i}\right) -$$

$$- 2a^2b^2 \sum_{i,j=1}^{n} \frac{\partial}{\partial x_0}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}(T-x_0)\frac{\partial^4 u}{\partial x_0^2 \partial x_i \partial x_j}\right) +$$

$$+a^2b^2 \sum_{i,j=1}^{n} \frac{\partial}{\partial x_0}\left((T-x_0)\left(\frac{\partial^3 u}{\partial x_0 \partial x_i \partial x_j}\right)^2\right) - 2a^2b^2 \sum_{i,j=1}^{n} \frac{\partial}{\partial x_0}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial^3 u}{\partial x_0 \partial x_i \partial x_j}\right) -$$

$$- 2a^2b^2(b^2-a^2) \sum_{i,j,k=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial^3 u}{\partial x_i \partial x_j^2}(T-x_0)\frac{\partial^3 u}{\partial x_0 \partial x_k^2}\right) +$$

$$+ 2a^2b^2(b^2-a^2) \sum_{i,j,k=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}(T-x_0)\frac{\partial^4 u}{\partial x_0 \partial x_i \partial x_k^2}\right) -$$

$$- 2a^2b^2(b^2-a^2) \sum_{i,j,k=1}^{n} \frac{\partial}{\partial x_k}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}(T-x_0)\frac{\partial^4 u}{\partial x_0 \partial x_i \partial x_j \partial x_k}\right) +$$

$$+ a^2b^2(b^2-a^2) \sum_{i,j,k=1}^{n} \frac{\partial}{\partial x_0}\left((T-x_0)\left(\frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}\right)^2\right) +$$

$$+ a^2b^2(b^2-a^2) \sum_{i,j,k=1}^{n} \left(\frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}\right)^2 + A^{(2)}u\mathcal{M}u.$$

Then we integrate the resulting relation over the domain Q,

$$2\int_Q \mathcal{L}u\mathcal{M}u\,d\boldsymbol{x} = \int_Q \left(\frac{\partial^3 u}{\partial x_0^3}\right)^2 d\boldsymbol{x} + 2(b^2-a^2)\sum_{i=1}^{n}\int_Q \left(\frac{\partial^3 u}{\partial x_0^2 \partial x_i}\right)^2 d\boldsymbol{x} +$$

$$+ ((b^2-a^2)^2 + 3a^2b^2)\sum_{i,j=1}^{n}\int_Q \left(\frac{\partial^3 u}{\partial x_0 \partial x_i \partial x_j}\right)^2 d\boldsymbol{x} +$$

$$+, a^2b^2(b^2-a^2)\sum_{i,j,k=1}^{n}\left(\frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}\right)^2 d\boldsymbol{x} - T\int_\Omega \varphi^2(\boldsymbol{x}')d\boldsymbol{x}' + 2\int_Q A^{(2)}u\mathcal{M}u\,d\boldsymbol{x}. \tag{11}$$

Using the inequality

$$2|ab| \leqslant \varepsilon a^2 + \frac{1}{\varepsilon}b^2 \quad \forall \varepsilon > 0, \tag{12}$$

we estimate the left-hand side of relation (11) as

$$2\int\limits_Q \mathcal{L}u\mathcal{M}u\,d\boldsymbol{x} \leqslant \frac{1}{\varepsilon_1}\int\limits_Q (\mathcal{L}u)^2\,d\boldsymbol{x} + \varepsilon_1 T^2\int\limits_Q \left(\frac{\partial^3 u}{\partial x_0^3}\right)^2 d\boldsymbol{x} + \\ + \varepsilon_1(b^2-a^2)^2T^2\sum_{i=1}^n\int\limits_Q\left(\frac{\partial^3 u}{\partial x_0\partial x_i^2}\right)^2 d\boldsymbol{x}.$$

By using this estimate, from (11), we obtain the inequality

$$\int\limits_Q\left(\frac{\partial^3 u}{\partial x_0^3}\right)^2 d\boldsymbol{x} + 2(b^2-a^2)\sum_{i=1}^n\int\limits_Q\left(\frac{\partial^3 u}{\partial x_0^2\partial x_i}\right)^2 d\boldsymbol{x}+ \\ + ((b^2-a^2)^2+3a^2b^2)\sum_{i,j=1}^n\int\limits_Q\left(\frac{\partial^3 u}{\partial x_0\partial x_i\partial x_j}\right)^2 d\boldsymbol{x}+ \\ + a^2b^2(b^2-a^2)\sum_{i,j,k=1}^n\int\limits_Q\left(\frac{\partial^3 u}{\partial x_i\partial x_j\partial x_k}\right)^2 d\boldsymbol{x} + 2\int\limits_Q A^{(2)}u\mathcal{M}u\,d\boldsymbol{x} \leqslant \\ \leqslant \frac{1}{\varepsilon_1}\int\limits_Q(\mathcal{L}u)^2\,d\boldsymbol{x} + T\int\limits_\Omega \varphi^2(\boldsymbol{x}')\,d\boldsymbol{x}' + \varepsilon_1 T^2\int\limits_Q\left(\frac{\partial^3 u}{\partial x_0^3}\right)^2 d\boldsymbol{x} + \\ + \varepsilon_1(b^2-a^2)^2T^2\sum_{i=1}^n\int\limits_Q\left(\frac{\partial^3 u}{\partial x_0\partial x_i^2}\right)^2 d\boldsymbol{x},$$

where we pass to the norm of the space $L_2(Q)$,

$$(1-\varepsilon_1T^2)\left\|\frac{\partial^3 u}{\partial x_0^3}\right\|^2_{L_2(Q)} + 2(b^2-a^2)\sum_{i=1}^n\left\|\frac{\partial^3 u}{\partial x_0^2\partial x_i}\right\|^2_{L_2(Q)} + \\ + ((b^2-a^2)^2+3a^2b^2-\varepsilon_1(b^2-a^2)^2T^2\delta_{ij})\sum_{i,j=1}^n\left\|\frac{\partial^3 u}{\partial x_0\partial x_i\partial x_j}\right\|^2_{L_2(Q)} + \\ + a^2b^2(b^2-a^2)\sum_{i,j,k=1}^n\left\|\frac{\partial^3 u}{\partial x_i\partial x_j\partial x_k}\right\|^2_{L_2(Q)} + 2\int\limits_Q A^{(2)}u\mathcal{M}u\,d\boldsymbol{x} \leqslant \\ \leqslant \frac{1}{\varepsilon_1}\|\mathcal{L}u\|^2_{L_2(Q)} + T\|\varphi\|^2_{L_2(\Omega)}, \quad (13)$$

where δ_{ij} is the Kronecker delta.

Take an ε_1 such that $1-\varepsilon_1 T^2 > 0$ and $(b^2-a^2)^2+3a^2b^2-\varepsilon_1(b^2-a^2)^2T^2 > 0$.

To obtain an expression for the norm of the space B, on the left-hand side in inequality (13), we add the missing terms $\sup\limits_{0<\tau<T}\|\boldsymbol{D}^{\boldsymbol{\alpha}}u\|^2_{L_2(\Omega)}(\tau)$, $|\boldsymbol{\alpha}|\leqslant 2$. To this end, we integrate the relation

$$c_1\frac{\partial}{\partial x_0}\left(\frac{\partial^2 u}{\partial x_0^2}\right)^2 = 2c_1\frac{\partial^2 u}{\partial x_0^2}\,\frac{\partial^3 u}{\partial x_0^3}$$

over the domain $Q^{(\tau)}=(0,\tau)\times\Omega$, $\tau\in(0,T)$, where c_1 is some sufficiently large positive constant,

$$c_1\int\limits_{\Omega}\left(\frac{\partial^2 u}{\partial x_0^2}\right)^2(\tau,\boldsymbol{x}')d\boldsymbol{x}' = 2c_1\int\limits_{Q^{(\tau)}}\frac{\partial^2 u}{\partial x_0^2}\,\frac{\partial^3 u}{\partial x_0^3}\,d\boldsymbol{x}\leqslant\frac{c_1}{\varepsilon_2}\int\limits_{Q^{(\tau)}}\left(\frac{\partial^2 u}{\partial x_0^2}\right)^2 d\boldsymbol{x}+$$
$$+\varepsilon_2 c_1\int\limits_{Q^{(\tau)}}\left(\frac{\partial^3 u}{\partial x_0^3}\right)^2 d\boldsymbol{x}\leqslant\frac{c_1}{\varepsilon_2}\int\limits_{Q^{(\tau)}}\left(\frac{\partial^2 u}{\partial x_0^2}\right)^2 d\boldsymbol{x}+\varepsilon_2 c_1\left\|\frac{\partial^3 u}{\partial x_0^3}\right\|^2_{L_2(Q)},\qquad(14)$$

$\varepsilon_2>0$.

Set

$$v(\tau)=c_1\int\limits_{\Omega}\left(\frac{\partial^2 u}{\partial x_0^2}\right)^2(\tau,\boldsymbol{x}')d\boldsymbol{x}'.$$

Then

$$c_1\int\limits_{Q^{(\tau)}}\left(\frac{\partial^2 u}{\partial x_0^2}\right)^2 d\boldsymbol{x}=\int\limits_0^{\tau}v(t)\,dt$$

and inequality (14) acquires the form

$$v(\tau)\leqslant\frac{1}{\varepsilon_2}\int\limits_0^{\tau}v(t)\,dt+\varepsilon_2 c_1\left\|\frac{\partial^3 u}{\partial x_0^3}\right\|^2_{L_2(Q)}.$$

By adding it to inequality (13) and by using the Gronwall inequality, we obtain the relations

$$(1-\varepsilon_1 T^2-\varepsilon_2 c_1)\left\|\frac{\partial^3 u}{\partial x_0^3}\right\|^2_{L_2(Q)}+2(b^2-a^2)\sum_{i=1}^{n}\left\|\frac{\partial^3 u}{\partial x_0^2\partial x_i}\right\|^2_{L_2(Q)}+$$

$$+\left((b^2-a^2)^2+3a^2b^2-\varepsilon_1(b^2-a^2)^2T^2\delta_{ij}\right)\sum_{i,j=1}^{n}\left\|\frac{\partial^3 u}{\partial x_0\partial x_i\partial x_j}\right\|^2_{L_2(Q)}+$$
$$+a^2b^2(b^2-a^2)\sum_{i,j,k=1}^{n}\left\|\frac{\partial^3 u}{\partial x_i\partial x_j\partial x_k}\right\|^2_{L_2(Q)}+2\int\limits_Q A^{(2)}u\mathcal{M}u\,d\boldsymbol{x}+v(\tau)\leqslant$$
$$\leqslant e^{\tau/\varepsilon_2}\left(\frac{1}{\varepsilon_1}\|\mathcal{L}u\|^2_{L_2(Q)}+T\|\varphi\|^2_{L_2(\Omega)}\right)\leqslant e^{T/\varepsilon_2}\left(\frac{1}{\varepsilon_1}\|\mathcal{L}u\|^2_{L_2(Q)}+T\|\varphi\|^2_{L_2(\Omega)}\right). \quad (15)$$

Take an ε_2 such that $1-\varepsilon_1T^2-\varepsilon_2c_1>0$. The right-hand side of inequality (15) is independent of τ; therefore, one can pass to the least upper bound with respect to τ on the left-hand side. As a result, inequality (15) acquires the form

$$(1-\varepsilon_1T^2-\varepsilon_2c_1)\left\|\frac{\partial^3 u}{\partial x_0^3}\right\|^2_{L_2(Q)}+2(b^2-a^2)\sum_{i=1}^{n}\left\|\frac{\partial^3 u}{\partial x_0^2\partial x_i}\right\|^2_{L_2(Q)}+$$
$$+\left((b^2-a^2)^2+3a^2b^2-\varepsilon_1(b^2-a^2)^2T^2\delta_{ij}\right)\sum_{i,j=1}^{n}\left\|\frac{\partial^3 u}{\partial x_0\partial x_i\partial x_j}\right\|^2_{L_2(Q)}+$$
$$+a^2b^2(b^2-a^2)\sum_{i,j,k=1}^{n}\left\|\frac{\partial^3 u}{\partial x_i\partial x_j\partial x_k}\right\|^2_{L_2(Q)}+c_2\sup_{0<\tau<T}\left\|\frac{\partial^2 u}{\partial x_0^2}\right\|^2_{L_2(\Omega)}(\tau)+$$
$$+2\int\limits_Q A^{(2)}u\mathcal{M}u\,d\boldsymbol{x}\leqslant e^{T/\varepsilon_2}\left(\frac{1}{\varepsilon_1}\|\mathcal{L}u\|^2_{L_2(Q)}+T\|\varphi\|^2_{L_2(\Omega)}\right).$$

In a similar way, by adding the terms $\sup_{0<\tau<T}\left\|\frac{\partial^2 u}{\partial x_0\partial x_i}\right\|^2_{L_2(\Omega)}(\tau)$, $i=1,\dots,n$, $\sup_{0<\tau<T}\left\|\frac{\partial^2 u}{\partial x_i\partial x_j}\right\|^2_{L_2(\Omega)}(\tau)$, $i,j=1,\dots,n$, and $\sup_{0<\tau<T}\|u\|_{L_2(\Omega)}(\tau)$, we arrive at the inequality

$$\sum_{|\boldsymbol{\alpha}|=3}\|\boldsymbol{D}^{\boldsymbol{\alpha}}u\|_{L_2(Q)}+c_2\sup_{0<\tau<T}\sum_{|\boldsymbol{\alpha}|\leqslant 2}\|\boldsymbol{D}^{\boldsymbol{\alpha}}u\|_{L_2(\Omega)}(\tau)+2\int\limits_Q A^{(2)}u\mathcal{M}u\,d\boldsymbol{x}\leqslant$$
$$\leqslant c\left(\|\mathcal{L}u\|^2_{L_2(Q)}+\|\varphi\|^2_{L_2(\Omega)}\right),$$

where the constant c_2 is sufficiently large.

By estimating the integral $\int_Q A^{(2)}u\mathcal{M}u\,d\boldsymbol{x}$ from above with the use of inequality (12), we obtain an energy inequality for the operator $\boldsymbol{L}$. The proof of the theorem is complete. $\square$

5 Existence of strong solution

Then we prove that operator $\boldsymbol{L}$ admits closure with the help of

Definition 2. An operator $L\colon B \to H$ admits closure if and only if it follows from the condition $u^{(m)} \to 0$ in B and $Lu^{(m)} \to F$ in H as $m \to \infty$ that $F = 0$ in the norm of the space H, where $u^{(m)} \in \mathcal{D}(L)$ [36].

Theorem 2. *The range* $\mathcal{R}(J\boldsymbol{L}) = \bigcup_{k=1}^{\infty} \mathcal{R}(J_{(k)}\boldsymbol{L})$ *of the operators* $J_{(k)}\boldsymbol{L}$ *is dense in* $\boldsymbol{H}$.

Proof. Let an element $\boldsymbol{v} = (v(\boldsymbol{x}), v^{(0)}(\boldsymbol{x}')) \in \boldsymbol{H}$ be orthogonal to the set $\mathcal{R}(J_{(k)}\boldsymbol{L})$. It means that the relation

$$(J_{(k)}\mathcal{L}u, v)_{L_2(Q)} + (lu, v^{(0)})_{L_2(\Omega)} = 0 \tag{16}$$

holds for any $u \in \mathcal{D}(J_{(k)}\boldsymbol{L})$. Taking in (16), in particular, u equal to any element from $\mathcal{D}^{(0)}(J_{(k)}\boldsymbol{L})$, where $\mathcal{D}^{(0)}(J_{(k)}\boldsymbol{L}) = \{u \in \mathcal{D}(J_{(k)}\boldsymbol{L}) \mid lu = 0\}$, we obtain from (16) the relation

$$(J_{(k)}\mathcal{L}u, v)_{L_2(Q)} = 0 \tag{17}$$

for all $u \in \mathcal{D}^{(0)}(J_{(k)}\boldsymbol{L})$. Show that it follows from relation (17) that it holds only for $v = 0$ in $L_2(Q)$.

Transform relation (17) in the following way:

$$(J_{(k)}\mathcal{L}u, v)_{L_2(Q)} = (\mathcal{L}u, J^\star_{(k)}v)_{L_2(Q)} = (u, \mathcal{L}' J^\star_{(k)}v)_{L_2(Q)} + \mathcal{M}(u, J^\star_{(k)}v; \partial Q), \tag{18}$$

where

$$\mathcal{L}' J^\star_{(k)}v = \mathcal{L}^{(0)} J^\star_{(k)}v + A^{(2)\prime} J^\star_{(k)}v,\ A^{(2)\prime} J^\star_{(k)}v = \sum_{|\boldsymbol{\alpha}|\leqslant 2} (-1)^{|\boldsymbol{\alpha}|} \boldsymbol{D}^{\boldsymbol{\alpha}}(a^{(\boldsymbol{\alpha})} J^\star_{(k)}v),$$

$$\begin{aligned}
(u, J^\star_{(k)}v; \partial Q) = \int\limits_{\partial Q} \biggl(&\frac{\partial}{\partial x_0}\left(\frac{\partial^3 u}{\partial x_0^3} J^\star_{(k)}v\right) + (b^2 - a^2)\sum_{i=1}^{n} \frac{\partial}{\partial x_0}\left(\frac{\partial^3 u}{\partial x_0 \partial x_i^2} J^\star_{(k)}v\right) - \\
&- a^2 b^2 \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial^3 u}{\partial x_i \partial x_j^2} J^\star_{(k)}v\right) - \frac{\partial}{\partial x_0}\left(\frac{\partial^2 u}{\partial x_0^2} \frac{\partial}{\partial x_0} J^\star_{(k)}v\right) - \\
&- (b^2 - a^2)\sum_{i=1}^{n} \frac{\partial}{\partial x_0}\left(\frac{\partial^2 u}{\partial x_i^2} \frac{\partial}{\partial x_0} J^\star_{(k)}v\right) + a^2 b^2 \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial}{\partial x_i} J^\star_{(k)}v\right) + \\
&+ \frac{\partial}{\partial x_0}\left(\frac{\partial u}{\partial x_0} \frac{\partial^2}{\partial x_0^2} J^\star_{(k)}v\right) + (b^2 - a^2)\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i} \frac{\partial^2}{\partial x_0^2} J^\star_{(k)}v\right) -
\end{aligned}$$

$$- a^2b^2 \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial x_j} \frac{\partial^2}{\partial x_i \partial x_j} J^{\star}_{(k)} v \right) - \frac{\partial}{\partial x_0} \left(u \frac{\partial^3}{\partial x_0^3} J^{\star}_{(k)} v \right) -$$

$$- (b^2 - a^2) \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(u \frac{\partial^3}{\partial x_0^2 \partial x_i} J^{\star}_{(k)} v \right) + a^2b^2 \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(u \frac{\partial^3}{\partial x_i^2 \partial x_j} J^{\star}_{(k)} v \right) \Big) d\boldsymbol{x}.$$

From the homogeneous conditions (6), (7) ($u \in \mathcal{D}^{(0)}(J_{(k)}\boldsymbol{L})$) we have $\mathcal{M}(u, J^{\star}_{(k)}v; \partial Q) = \mathcal{M}(u, J^{\star}_{(k)}v; \Omega^{(0)}) + \mathcal{M}(u, J^{\star}_{(k)}v; \Omega^{(T)}) + \mathcal{M}(u, J^{\star}_{(k)}v; \Gamma)$, where

$$\mathcal{M}(u, J^{\star}_{(k)}v; \Omega^{(0)}) = - \int\limits_{\Omega^{(0)}} \frac{\partial^2 u}{\partial x_0^2} \frac{\partial}{\partial x_0} J^{\star}_{(k)} v \, d\boldsymbol{x}',$$

$$\mathcal{M}(u, J^{\star}_{(k)}v; \Omega^{(T)}) =$$
$$= \int\limits_{\Omega^{(T)}} \left(\frac{\partial^3 u}{\partial x_0^3} J^{\star}_{(k)} v - \frac{\partial^2 u}{\partial x_0^2} \frac{\partial}{\partial x_0} J^{\star}_{(k)} v - (b^2 - a^2) \frac{\partial^2 u}{\partial x_i^2} \frac{\partial}{\partial x_0} J^{\star}_{(k)} v - - u \frac{\partial^3}{\partial x_0^3} J^{\star}_{(k)} v \right) d\boldsymbol{x}',$$

$$\mathcal{M}(u, J^{\star}_{(k)}v; \Gamma) = \int\limits_{\Gamma} \Bigg(-a^2b^2 \sum_{i,j=1}^{n} \frac{\partial^3 u}{\partial x_i \partial x_j^2} J^{\star}_{(k)} v \, \nu_i +$$

$$+ (b^2 - a^2) \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial^2}{\partial x_0^2} J^{\star}_{(k)} v \, \nu_i - a^2b^2 \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial^2}{\partial x_i \partial x_j} J^{\star}_{(k)} v \, \nu_i \Bigg) ds.$$

Varying in relation (18) the function u in the bound of the set $\mathcal{D}^{(0)}(J_{(k)}\boldsymbol{L})$, it is possible to show that it holds if and only if the relations

$$(u, \mathcal{L}' J^{\star}_{(k)} v)_{L_2(Q)} = 0, \tag{19}$$

$$\mathcal{M}(u, J^{\star}_{(k)}v; \partial Q) = 0 \tag{20}$$

hold, since in (19) and (20) the domains of integrating are different.

Relation (20) generates the boundary conditions

$$\frac{\partial}{\partial x_0} J^{\star}_{(k)} v \bigg|_{\Omega^{(0)}} = J^{\star}_{(k)} v|_{\Omega^{(T)}} = \frac{\partial}{\partial x_0} J^{\star}_{(k)} v \bigg|_{\Omega^{(T)}} =$$

$$= \frac{\partial^3}{\partial x_0^3} J^\star_{(k)} v \bigg|_{\Omega^{(T)}} = J^\star_{(k)} v|_\Gamma = \frac{\partial^2}{\partial \boldsymbol{\nu}^2} J^\star_{(k)} v \bigg|_\Gamma = 0. \quad (21)$$

As far as the set $\mathcal{D}^{(0)}(J_{(k)}\boldsymbol{L})$ is dense in $L_2(Q)$, relation (19) can be extended with the help of passaging to the limit to all functions $u \in L_2(Q)$. We suppose $u = (T - x_0)\dfrac{\partial^3}{\partial x_0^3} J^\star_{(k)} v + (b^2 - a^2)(T - x_0)\dfrac{\partial}{\partial x_0}\Delta J^\star_{(k)} v$ in relation (19) and, by repeating the proof of Theorem 1, obtain that $\|J^\star_{(k)} v\|_{H^3(Q)} = 0$ for all $k = 1, 2, \dots$. As long as $\{J^\star_{(k)} v\}$ converges to v as $k \to \infty$, it follows from here that $\|v\|_{L_2(Q)} = 0$. Here $H^l(Q)$ is the Hilbert space of functions $u \in L_2(Q)$ and have generalized derivatives $\boldsymbol{D}^{\boldsymbol{\alpha}} u$, $|\boldsymbol{\alpha}| \leqslant l$, also belonging to $L_2(Q)$. On $H^l(Q)$ the scalar product $H^l(Q) \times H^l(Q) \ni u, v \to (u, v)_{H^l(Q)} = \sum\limits_{|\boldsymbol{\alpha}| \leqslant l} (\boldsymbol{D}^{\boldsymbol{\alpha}} u, \boldsymbol{D}^{\boldsymbol{\alpha}} v)_{L_2(Q)}$ is introduced and the norm $H^l(Q) \ni u \to \|u\|_{H^l(Q)} = (u, u)^{1/2}_{H^l(Q)}$.

Returning again to (16), taking into account (17) we have the relation of orthogonality

$$(lu, v^{(0)})_{L_2(\Omega)} = 0 \quad (22)$$

for all $u \in \mathcal{D}(J_{(k)}\boldsymbol{L})$. If the function $u \in \mathcal{D}(J_{(k)}\boldsymbol{L})$, then $lu = \dfrac{\partial^3 u(0, \boldsymbol{x}')}{\partial x_0^3} \in C^1(\overline{\Omega})$. It is evidently that for any function $u^{(0)} \in C^1(\overline{\Omega})$ $u \in C^1(\overline{Q})$ exists such that $\dfrac{\partial^3 u(0, \boldsymbol{x}')}{\partial x_0^3} = u^{(0)}(\boldsymbol{x}')$. The set $C^1(\Omega)$ is dense in $L_2(\Omega)$. Since the set $\{lu\}$ is dense in $L_2(\Omega)$, if u runs through the whole set $\mathcal{D}(J_{(k)}\boldsymbol{L})$, then relation (22) holds if and only if $v^{(0)} = 0$ in $L_2(\Omega)$. Theorem 2 is proved. □

So, Theorem 3 is proved.

Theorem 3. *For arbitrary functions $f \in L_2(Q)$, $\varphi \in L_2(\Omega)$ there exists a unique strong solution $u \in B$ of (5)–(7), and the estimate*

$$\|u\|_B \leqslant c(\|f\|_{L_2(Q)} + \|\varphi\|_{L_2(\Omega)}),$$

holds, where c is the same constant as in inequality (10).

The proof of the theorem analogous to Theorem 3 for problem (5), (6), (8) is a verbatim repetition of the above proof.

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SINGULAR INTEGRAL EQUATION WITH NORMALIZED NUCLEI

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Abstract. Singular inergral equations with a regular part of a specific kind is viewed and the theorems about their decidability for them are obtained.

On the smooth closed path $L = \cup L_j$, consisting of the finite number of closed curves which are mutually disjoint and non-passing through the infinite point singular integral equation is regarded

$$K\varphi(t) \equiv a(t)\varphi(t) + b(t)(S\varphi(t) - H_{m-1}(t, S\varphi, \Delta)) = f(t). \tag{1}$$

Further we'll be using the generally adopted abbreviation s.i.e.

Here the singular integration operator is the integral Cauchy type along the contour L, $H_{m-1}(t, g, \Delta)$ one of the interpolational polynominals by Lagrange, Taylor and Hermite [3], built according to the g function meaning and its derivation in the interpolation points $\Delta = \{t_1^{n_1}, t_2^{n_2}, ..., t_q^{n_q}\}$, $n_1 + ... + n_q = m - 1$.

The equation coefficient $a(t)$, $b(t)$ and its right $f(t)$ belong to the class $C_\alpha^{(m-1)}(L)$ and here $a^2(t) - b^2(t) \equiv 1$. Suggested investigations adjoin to the author's works [5]- [8].

S.i.e. (1) is a complete singular integral equation [1]. The peculiarity of this equation lies in the presence of the Hermite polynominal, which can be regarded as a quasi-regular part of the operator K.

According to the definition [1] the integral $\int\limits_L k(t, \tau)\varphi(\tau)\, d\tau$ is part of the singular integral operator, the nucleus $k(t, \tau)$ of which can be presented in the form of the fraction $k(t, \tau) = k_1(t, \tau)/|t - \tau|^\alpha$, $k_1(t, \tau)$ is a piecewise Hölder function and α — is such a real number as $0 \leq \alpha < 1$. In the equation (1) under discussion the Hermite polynominal can be registered in the shape of integral, but the nucleus of this integral representation can have peculiarities of the first order terminal type in the nodes of interpolation.

The singular integral equation can be solved with the help of Carleman-Veku regularization method [1], which consists of s.i.e. reduction (1) to the characteristic equation $K^0\varphi = F$ with the right side in the form of $F(t) = f(t) - H_{m-1}(t, S\varphi, \Delta)$. The general solution and the picture of solvability in the characteristic equation is well-known by the present time. Indefinite constants are a part of them which depend on the required s.i.e. solution (1). If we want to calculate these constants we'll have to compose a system of linear algebraic equations which is investigated by the higher algebra methods through the rank of this system.

At present we suggest drawing a direct analytical continuation of s.i.e. into the surface of a complex variable z. Such an approach allows not only to construct a general solution of the s.i.e. (1), but to investigate effectively the conditions of this equation solvability. This method works according to one and the same scheme for all kinds of the Hermite polynominal. That's why we'll describe this method in detail for the case when the Hermite polynominal is part of the Taylor one while for other situations we'll show only the peculiarities.

1°. Let L be a smooth closed composite contour and $t_0 \in L$ — as a fixed point. Let us regard on L a singular integral equation of the type (1)

$$a(t)\psi(t) + b(t)\left\{(S\psi)(t) - \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!}(S\psi)^{(k)}(t_0)\right\} = f(t). \tag{2}$$

We can notice that from the qualities of the function $a(t)$ and $b(t)$ it follows that if $a(t_0) = a^{(1)}(t_0) = ... = a^{(p-1)}(t_0) = 0$, we'll $f(t_0) = f'(t_0) = ... = f^{(p-1)}(t_0) = 0$. We'll consider that these conditions are fulfilled.

1 An auxiliary problem about a jump

In the surface of a complex variable z to find the piecewise-analytical function of $\Psi(z)$ which on L meets the boundary condition

$$\begin{aligned} &\Phi^+(t) - \Phi^-(t) = g(t), \quad g \in C_\alpha^{n-1}(L), \\ &[\Phi^+(t) + \Phi^-(t)]^{(k)}\Big|_{t=t_0} = 0, \quad k = 0, .., n-1, \end{aligned} \tag{3}$$

while in the infinite point it can have an order terminal not higher than $n-1$.

The boundary problem (3) differs from a usual Riemann problem [2] only by additional conditions (3). Taking into account them, let us write down a general solution of the problem (3) as follows:

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_L \frac{g(\tau)\,d\tau}{\tau - z} + \sum_{k=0}^{n-1} C_k(z-t_0)^k.$$

Let's make it dependent on the condition (3). We have

$$\Phi^+(t) + \Phi^-(t) = (Sg)(t) + 2\sum_{k=0}^{n-1} C_k(t-t_0)^k. \tag{4}$$

Then from the condition (3) has only one solution which is done according to the formula (4).

2 Obtaining the equation (2) to the Riemann boundary problem

Let us introduce an auxiliary function

$$\Psi(z) = \frac{1}{2\pi i}\int\limits_L \frac{\psi(\tau)\,d\tau}{\tau - z} - \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{k!}(z-t_0)^k(S\psi)^{(k)}(t_0). \tag{5}$$

In the infinite point this function has an order terminal not higher than $n-1$ while its limiting meanings on L are connected by Yu.V. Sokhotsky formulas

$$\begin{aligned} &\Psi^+(t) - \Psi^-(t) = \psi(t), \\ &\Psi^+(t) + \Psi^-(t) = (S\psi)(t) - \sum_{k=0}^{n-1}\frac{1}{k!}(t-t_0)^k(S\psi)^{(k)}(t_0). \end{aligned} \tag{6}$$

And by point wise conditions

$$(\Psi^+ + \Psi^-)^{(k)}(t_0) = 0, \quad k = 0, ..., n-1. \tag{7}$$

Using the formulas (6) we come from the equation (2) to the boundary problem. "Find the piecewise-analytical function of $\Psi(z)$ which meets the L piecewise condition

$$\Psi^+(t) = \left(\frac{b-a}{a+b}\right)(t)\Psi^-(t) + \left(\frac{f}{a+b}\right)(t) \tag{8}$$

and the correlations (7) and in (8) the infinitely distant point having a point order not more than $n-1$".

The boundary problem (7)–(8) is equivalent to the equation (2). It is proved by a usual scheme [1] with the use of the results p.1. From the proved equation (2) equivalence and the boundary problems (7)–(8) it follows that first it's necessary to solve the boundary problem and then to get the equation (2) solution using the formula (6).

3 Boundary problem solution (7)–(8)

Let us mark through $\chi(z)$ the canonical function of the homogeneous problem (1), corresponding to the problem (7)–(8) and through κ — its order in the infinitely distant point. The general solution of the boundary problem (7)–(8) can be put this way:

$$\Psi(z) = \chi(z)\left[\frac{1}{2\pi i}\int\limits_L \frac{f(\tau)}{Z(\tau)}\frac{d\tau}{\tau - z} + \sum_{k=0}^{n+\kappa-1} C_k(z-t_0)^k\right], \tag{9}$$

where $Z(t) = \chi^+(t)(a(t)+b(t))$.

Here $n+\kappa \le 0$ should be regarded all the $C_k = 0$ and $n+\kappa < 0$ in equations the solution should be carried out

$$\int\limits_L f(\tau)Z^{-1}(\tau)\tau^k d\tau = 0, \quad k = 0,..,-n-\kappa-1, \tag{10}$$

which present the integral conditions of solvability.

Let us now meet the conditions (7). For this we calculate the sum of the limit meanings in the function (9)

$$\Psi^+(t) + \Psi^-(t) = -(Qf)(t) + \sum_{k=0}^{n+\kappa-1} C_k\psi_k(t),$$

where

$$(QF)(t) = -b(t)f(t) + a(t)Z(t)\left(S\frac{f}{Z}\right)(t), \quad \psi_k(t) = 2a(t)Z(t)(t-t_0)^k.$$

Then the conditions (7) are rechanged into:

$$\sum_{k=0}^{n+\kappa-1} C_k\psi_k^{(l)}(t_0) = (Qf)^{(l)}(t_0), \quad l = 0,...,n-1. \tag{11}$$

Thus, a general solution of the problem (7)–(8) has a shape (9) when the conditions (10),(11) are fulfilled.

4 Equation (2) solution and the picture of its solvability

Now according to the first one from the formulas (6) and to the formula (9) we get a general equation (2) solution

$$\psi(t) = Rf(t) + \sum_{k=1}^{n-\kappa} C_k \psi_k(t), \tag{12}$$

where

$$(Rf)(t) = (af)(t) - b(t)Z(t)\left(S\frac{f}{Z}\right)(t), \tag{13}$$

while constants C_k are connected with condition (11).

Because of the function $f(t)$ the general solution belongs to class $C_\alpha^{n-1}(L)$. It must be also noted that the differentiability of coefficients s.i.e (1) and its right part guarantee the existence and calculation derived from the integral of the Cauche type involved in the s.i.e. (2) solution.

Let us now describe the picture of the equation (2) solvability. A direct analysis of the condition (11) shows that the rank of the system (11) depends on the number of the derived coefficient $a(t)$ turning into a zero at the point $t = t_0$. Using simple arguments the next theorem is proved.

Theorem 1. *Let r be a number of the derived coefficient $a(t)$ of the integral operator K which turns into zero at point t_0, including into this number meaning of the function proper. Then, if $n+\kappa-r \geq 0$, the homogeneous s.i.e. (1) has $n+\kappa-r$ linear independent solutions. If $n+\kappa-r < 0$, then a inhomogeneous s.i.e. (1) is solvable while carrying out $r-n-\kappa$ integral and pointwise conditions.*

2°. Let L be a differentiable close composite contour and $\Delta = \{t_1, t_2, ..., t_n\}$ — the divisor from arbitory fixed points let us regard on L singular integral equation of type (1).

$$a(t)\psi(t) + b(t)\{(S\psi)(t) - L_{n-1}(t, S\psi, \Delta)\} = f(t), \tag{14}$$

where

$$L_{n-1}(t, \varphi, \Delta) \equiv \sum_{j=1}^{n} \varphi(t_j) \cdot \omega_j(t), \quad \omega_j(t) \equiv \prod_{\substack{k=1,\\ k\neq j}}^{n} \frac{t-t_k}{t_j-t_k}. \tag{15}$$

In this case s.i.e. (15) leads to the equivalent of the Riemann boundary problem with the help of auxiliary piecewise analytical function

$$\Phi(z) \equiv \frac{1}{2}\left(\frac{1}{\pi i}\int\limits_L \frac{\varphi(\tau)\,d\tau}{\tau - z} - L_{n-1}(z, S\varphi, \Delta)\right). \tag{16}$$

The boundary condition of this problem has a shape

$$\begin{cases} \Phi^+(t) = \dfrac{a(t) - b(t)}{a(t) + b(t)}\Phi^-(t) + \dfrac{f(t)}{a(t) + b(t)}, \\ \Phi^+(t_k) + \Phi^-(t_k) = 0, \quad k = 1, 2, ..., n \end{cases} \tag{17}$$

and its solution is looked for in the class of functions, having at the point $z = \infty$ an order terminal $n = 1$.

A general solution of s.i.e. in this case looks as follows

$$\varphi(t) = Rf(t) + b(t)Z(t)P_{n+\kappa-1}(t), \tag{18}$$

where R is a linear integral operators which allows a partial solution of the inhomogeneous equation (15), $Z(t)$ is quite a definite function which is part of the operator R, and

$$P_{n+\kappa-1}(t) = \begin{cases} \sum\limits_{k=0}^{n+\kappa-1} C_k t^k, \quad n + \kappa - 1 \geq 0, \\ 0, \quad n + \kappa - 1 < 0. \end{cases} \tag{19}$$

If $n + \kappa - 1 \leq -1$, then for the s.i.e (15) solvability it is necessary to fulfill $1 - n - \kappa$ integral conditions

$$\int\limits_L \frac{f(t)}{Z(t)} t^{l-1} dt = 0, \quad l = 1, 2, ..., 1 - n - \kappa. \tag{20}$$

Here $Z(t) = (a(t) + b(t))X^+(t)$, $X(z)$ is a canonic function of the Riemann homogeneous problem corresponding to the problem (17), l — is the index of this problem. Besides, pointwise conditions (17) give birth to pointwise of the s.i.e. solvability (1)

$$a(t_k)Z(t_k)P_{n+\kappa-1}(t_k) = -Qf(t_k), \quad k = 1, 2, .., n, \tag{21}$$

Q is a linear integral operator of the shape

$$Qf(t) = -b(t)f(t) + a(t)Z(t) \cdot S\left(\frac{f}{Z}\right)(t). \tag{22}$$

The conditions (20) and (21) are necessary and sufficient ones fro the s.i.e. solvability (1).

It is more convenient to analyze the picture of the s.i.e. (15) solvability if to mark polynominal $P_{n+\kappa-1}(t)$ in the form of the linear combination in the elementary Langrage $\omega_j(t)$ polynominals which are the basis polynominals degree of the linear space not more than $n-1$.

The main result which is obtained here is expressed by the following theorem.

Theorem 2. *Let r be the point number from the set group Δ in which the coefficient $a(t)$ of the operator K turns into zero. If $n+\kappa-r \geq 0$, then a homogeneous s.i.e. (15) has $n+\kappa-r$ a linear independent solutions. If $n+\kappa-r < 0$, then a inhomogeneous s.i.e. (15) can be solved while fulfilling $r-n-\kappa$ integral and pointwise conditions.*

3°. The previous situations are by no means summarized if we take the quasi-reular part of the integral operator K in the form of $k\varphi(t) = -b(t) \cdot H_{m-1}(t, S\varphi, \Delta)$, where $H_{m-1}(t, S\varphi, \Delta)$ is an interpolational of the Hermite polynominal (3), drawn by the divisor $\Delta = (t_1^{\lambda_1}, t_2^{\lambda_2}, ..., t_n^{\lambda_n})$, $m = \lambda_1 + ... + \lambda_n$ for the singular integral operator S, studied in the polynominals of functions, differentiated by $m-1$ times on L contour. The coefficients $a(t)$, $b(t)$ and the right part of s.i.e. (1) are taken from the same class $C_\alpha^{n-1}(L)$.

The method of the s.i.e. (1) solution in the case of 3 is a reduction to the Riemann equivalent problem with pointwise conditions in the form of

$$(\Phi^+)^{(j)}(t_k) + (\Phi^-)^{(j)}(t_k) = 0, \quad j = 0, ..., \lambda_k - 1, \quad k = 1, ..., n. \tag{23}$$

The analogy prompts and the calculations show that a general s.i.e. (1) solution in case 3 is given by the same formulas (13), (18). The conditions (23) give birth to pointwise conditions of s.i.e. (1) solvability which are analogous the mentioned conditions (11), (21) in the cases of 1 and 2.

The final theorem of case 3 about the s.i.e. (1) solvability coincide with the content of the previous theorems 1 and 2, if by the quantity r the sum of highest orders derived from coefficient $a(t)$ of the integral operator K, turning into zero at divisor Δ points is understood.

Supplement 1. *The s.i.e. (1) of case 2 appears while solving the s.i.e.* $K^0\varphi = f$ *in the class of non-summarized functions in the form*

$$\varphi(t) = \frac{\psi(t)}{\Omega(t)}, \quad \Omega(t) = (t - t_1)...(t - t_n), \quad \psi(t) \in H(L), \tag{24}$$

where all the points t_k *are inner points of the* L *curve and are different in pairs.*

If in this equation the required function is changed according to the formula (24), then the mentioned characteristic equation twins into the equation (15).

Supplement 2. *The s.i.e. (1) in all the three considered above cases appears in the s.i.e. solution as*

$$b(t)(\varphi(t) - M(t, \varphi, \Delta)) + a(t)S\varphi(t) = f(t),$$

where $M(t, \varphi, \Delta)$ *is one of the interpolational of the Langrage, Taylor or Hermite polynominals for the function* $\varphi(t)$ *on the set group* Δ*. For this it is enough to change the desired functions as* $S\varphi = \psi$*.*

And here the integration order permutation formula is summarized in case of the composite closed contour for a repeated singular integral.

Supplement 3. *The singular integral equation connected with the s.i.e. (1) in case 2, it is natural to solve it in class (24) which is discovered in a concrete formation the adjoint operator.*

Supplement 4. *The s.i.e. (1) appears if it is solved in the N.I. Mousheshvili [2] class and it is changed (24). Here the s.i.e. (15) appears while appearing in this case the Riemann problem has an additional condition on* L *contour.*

$$\Phi^+(t_k) \pm \Phi^-(t_k) = 0,$$

where points t_k *coincide with the groups which define the N.I. Moushelishvili function class.*

Supplement 5. *The s.i.e. in cases 1–3 appears if as S.G. Samko suggests bilding the composition of integral operators with power and logarithmic nuclei and that of the integral operator* K *(1). And here the integral equations of Fredholm of the first kind, solving a closed form are received.*

Supplement 6. *The s.i.e. with the Hilbert nucleous, interpolated by a periodic analog of the interpolational polynominals by Langrange, Taylor or Hermite are solved by the method of reduction to the boundary Riemann problem with the symmetry (4) condition and with additional pointwise conditions.*

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CRITERION OF FUNCTION CLASSES IMBEDDING $E^{(N)}_{P\vartheta}(\lambda;\rho_N)$ IN SPACE OF LORENTZ WITH HERMITE WEIGHT BY THE STRONG PARAMETER

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Abstract. In present work based on the elements of Lorentz's spaces with Hermite weight$f \in L_{p,\vartheta}(\mathbb{R}_n;\rho_n)$, $1 \leqslant p < +\infty$, $0 < \vartheta \leqslant +\infty$ was defined class $E^{(n)}_{p,\vartheta}(\lambda;\rho_n)$ and established criterions of embeddings $E^{(n)}_{p\vartheta}(\lambda;\rho_n) \subset L_{q,\tau}(\mathbb{R}_n;\rho_n)$, $1 \leqslant p < q < +\infty$, $0 < \vartheta \leqslant +\infty$, $0 < \tau < +\infty$ $E^{(n)}_{p\vartheta}(\lambda;\rho_n) \subset L_{q,\infty}(\mathbb{R}_n;\rho_n)$, $1 \leqslant p < q < +\infty$, $0 < \vartheta \leqslant +\infty$.

1 Definitions and auxiliary theorems

Integral properties of functions in terms of the rate of decrease to zero the best approximations of periodic functions by trigonometric polynomials were investigated in [1,2]. Afterwards this theme had got rapid development in works of many mathematicians, including [3]– [7].

In present work based on the elements of Lorentz space with Hermite weight is defining class $E^{(n)}_{p,\vartheta}(\lambda;\rho_n)$ and establishing criterion of its embedding in spaces $L_{q,\tau}(\mathbb{R}_n;\rho_n)$, $L_{q,\infty}(\mathbb{R}_n;\rho_n)$, where $1 \leqslant p < q < +\infty$, $0 < \vartheta \leqslant +\infty$, $0 < \tau < +\infty$.

Let $1 \leqslant p \leqslant +\infty$, $1 \leqslant \vartheta \leqslant +\infty$ and $f(\bar{x})$- measurable in the sense of Lebesgue on $\mathbb{R}_n$ function; let $\rho_n(\bar{x}) = e^{-\frac{|\bar{x}|^2}{2}}$, $\bar{x} \in \mathbb{R}_n$; $|\bar{x}| = \left(\sum\limits_{k=1}^{n} x_k^2\right)^{\frac{1}{2}}$.

By $F(|f\rho_n|;t)$ — denote nonincreasing rearrangement of functions $|f(\bar{x})\rho_n(\bar{x})|$ on $\mathbb{R}_n$, $t \in [0;+\infty)$. Let's say, that $f \in L_{p,\vartheta}(\mathbb{R}_n;\rho_n)$, [8] if finite value:

$$\|f\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} = \left\{ \frac{\vartheta}{p} \int\limits_0^{+\infty} t^{\frac{\vartheta}{p}-1} (F(|f\rho_n|;t))^{\vartheta} dt \right\}^{\frac{1}{\vartheta}}, \quad \text{by} \quad 0 < \vartheta < +\infty,$$

$$\|f\|_{L_{p\infty}(\mathbb{R}_n;\rho_n)} = \sup_{t>0} \left\{ t^{\frac{1}{p}} F(|f\rho_n|;t) \right\}, \quad \text{by} \quad \vartheta = +\infty.$$

Space $L_{p\infty}(\mathbb{R}_n;\rho_n)$ — is also called Marcinkiewicz space.

By $\mathcal{P}_{m_1,\ldots,m_n}(\mathbb{R}_n)$ let's define set of all possible algebraic polynomials of m_k order, by variable x_k, $k=1,\ldots,n$;

$$E_{m_1,\ldots,m_n}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} = \\ = \inf\left\{\|f-P_{m_1,\ldots,m_n}\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} : P_{m_1,\ldots,m_n}(\bar{x}) \in \mathcal{P}_{m_1,\ldots,m_n}(\mathbb{R}_n)\right\} -$$

complete best approximation of functions f in the metric of space $L_{p,\vartheta}(\mathbb{R}_n;\rho_n)$, $1\leqslant p<+\infty$, $0<\vartheta\leqslant+\infty$ by means of algebraic polynomials.

Definition 1. let $\{\lambda_m\}_{m=1}^{+\infty}$ — decreasing to zero given sequence of numbers. Then, for $E_{p,\vartheta}^{(n)}(\lambda;\rho_n)$, $1\leqslant p<+\infty$, $0<\vartheta\leqslant+\infty$ let's define class of all functions $f\in L_{p,\vartheta}(\mathbb{R}_n;\rho_n)$, for which $E_{m,\ldots,m}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}\leqslant\lambda_m, \forall m\in\mathbb{N}$.

Lemma 1. *Let* $0<\alpha<+\infty$, $0<\gamma<+\infty$, $\{\lambda_m\}_{m=1}^{+\infty}$ *is given sequence of numbers.*

a) If series $\sum\limits_{m=1}^{+\infty} m^{\gamma\alpha-1}\lambda_m^{\gamma}$ *is diverging than exists the sequence of natural numbers* $\{m_k\}_{k=1}^{+\infty}$ *and sequence of positive numbers* $\{\mu_k\}_{k=1}^{+\infty}$ *possessing such qualities: 1)* $\mu_k\leqslant\lambda_\nu$, *for* $m_k\leqslant\nu<m_{k+1}$; $2m_k\leqslant m_{k+1}$; $\lambda_{m_{k+1}}\leqslant\mu_k$, $\forall k\in\mathbb{N}$; *2)* $\mu_{k+1}\leqslant\frac{1}{2}\mu_k$, $\forall k\in\mathbb{N}$; *3)* $\sum\limits_{k=1}^{+\infty} m_{k+1}^{\gamma\alpha}\mu_k^{\lambda}=+\infty$.

b) Let $1\leqslant p<q<+\infty$. *If* $\sup\limits_{m\in\mathbb{N}}\left(m^{-\frac{n}{2q}}\sum\limits_{k=1}^{m}k^{\frac{n}{2p}-1}\lambda_k\right)=+\infty$, *then exist the sequence of natural numbers* $\{l_k\}_{k=1}^{+\infty}$ *and sequence of positive numbers* $\{\mu_k\}_{k=1}^{+\infty}$ *such, that satisfy the conditions 1 and 2 of point a) of given lemma, for which the assertion 4) is true* $\sup\limits_{\nu\in\mathbb{N}}\left(l_{\nu+1}^{-\frac{n}{2q}}\sum\limits_{s=1}^{\nu}l_{s+1}^{\frac{n}{2p}}\mu_s\right)=+\infty$.

The proof of point a) of this lemma is in [5], and point b) is establishing with the help of the same technique by verbatim repetition of proof of point a).

Lemma 2 (see [9]). *Let* $0<p<+\infty$, $1\leqslant\vartheta\leqslant+\infty$. *There is exists sequence of negative algebraic polynomials* $\{P_m*(x)\}_{m=1}^{+\infty}$, $x\in\mathbb{R}_1$ *of degree not higher* $(m-1)$ *such, that*

$$P_m^*(0)=1 \ \text{ and } \ c_p'm^{-\frac{1}{2p}}\leqslant\|P_m^*\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}\leqslant c_p''m^{-\frac{1}{2p}}, m\in\mathbb{N}.$$

Here $\rho(x)=e^{-\frac{x^2}{2}}$, $x\in\mathbb{R}$ *and multipliers* $c_p'>0$, $c_p''>0$ *depend only on mentioned parameters.*

Lemma 3. *Let $L_{p,\vartheta}(\mathbb{R}_n;\rho_n)$, $1 \leqslant p < +\infty$, $0 < \vartheta \leqslant +\infty$. Then, for nonincreasing rearrangements $F(t)$ of functions $|f(\bar{x})\rho_n(\bar{x})|$ the inequality is valid:*

$$F\left(2^{-\frac{n(m+1)}{2}}\right) \leqslant A_{p\vartheta n}\left\{\|f\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} + \sum_{k=0}^{m} 2^{\frac{(k+1)n}{2p}} E_{2^k,\dots,2^k}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}\right\}.$$

The proof of this approval is carried by analogy with proof of similar theorem 1 from [4].

2 Theorems of embedding in Lorentz spaces $L_{q,\tau}(\mathbb{R}_n;\rho_n)$, $1 \leqslant p < q < +\infty$, $0 < \tau < +\infty$.

Theorem 1 (see [10]). *Let $1 \leqslant p < +\infty$, $0 < \vartheta \leqslant +\infty$ and $\{l_k\}_{k=0}^{+\infty} \subset \mathbb{Z}^+$ is such that $l_0 = 1$, $1 < a \leqslant l_{k+1} \cdot l_k^{-1}$, $\forall k \in \mathbb{Z}^+$. Let $f \in L_{p,\vartheta}(\mathbb{R}_n;\rho_n)$ and sequence of algebraic polynomials $\{P_{l_k,\dots,l_k}(\bar{x})\}_{k=0}^{+\infty}$ is such, that in metric of space $L_{p,\vartheta}(\mathbb{R}_n;\rho_n)$ valid representation is*

$$f(\bar{x}) = P_{1,\dots,1}(\bar{x}) + \sum_{k=1}^{+\infty}\left(P_{l_k,\dots,l_k}(\bar{x}) - P_{l_{k-1},\dots,l_{k-1}}(\bar{x})\right) = \sum_{k=0}^{+\infty} \Delta_{l_k,\dots,l_k}(f;\bar{x}).$$

If for some q and τ: $p < q < +\infty$, $0 < \tau < +\infty$ series $\sum\limits_{k=0}^{+\infty} l_k^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)} \|\Delta_{l_k,\dots,l_k}(f)\|^{\tau}_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}$ converges, then $f \in L_{q,\tau}(\mathbb{R}_n;\rho_n)$ and with it the following inequality is valid:

$$\|f\|_{L_{q,\tau}(\mathbb{R}_n;\rho_n)} \leqslant C_{pq\vartheta\tau n}\left[\sum_{k=0}^{+\infty} l_k^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)} \|\Delta_{l_k,\dots,l_k}(f)\|^{\tau}_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}\right]^{\frac{1}{\tau}}.$$

Theorem 2 (see [10]). *Let $1 \leqslant p < q < +\infty$, $1 \leqslant \vartheta \leqslant +\infty$, $1 \leqslant \tau < +\infty$, sequence $\{l_k\}_{k=0}^{+\infty} \subset \mathbb{Z}^+$ is such, that $l_0 = 1$, $l_{k+1} \cdot l_k^{-1} \geqslant a_0 > 1$. Let $f \in L_{p\vartheta}(\mathbb{R}_n;\rho_n)$ and in metric of space $L_{p\vartheta}(\mathbb{R}_n;\rho_n)$ the following representation is valid:*

$$f(\bar{x}) = P_{1,\dots,1}(\bar{x}) + \sum_{k=1}^{+\infty}\left(P_{l_k,\dots,l_k}(\bar{x}) - P_{l_{k-1},\dots,l_{k-1}}(\bar{x})\right) = \sum_{k=0}^{+\infty} \Delta_{l_k,\dots,l_k}(\bar{x}),$$

where $P_{m,\dots,m}(\bar{x}) \in \mathcal{P}_{m,\dots,m}$, $m \in \mathbb{N}$ are algebraic polynomials.

Then the following inequality is valid:

$$\|f\|_{L_{p\vartheta}(\mathbb{R}_n;\rho_n)} \geqslant A_{pq\vartheta\tau n}\left\{\sum_{k=0}^{+\infty} l_k^{\vartheta\left(\frac{n}{2q}-\frac{n}{2p}\right)} \|\Delta_{l_k,\dots,l_k}\|^{\vartheta}_{L_{q\tau}(\mathbb{R}_n;\rho_n)}\right\}^{\frac{1}{\vartheta}}.$$

Here $A_{pq\vartheta\tau n} > 0$ depends on mentioned parameters.

Theorem 3. *Let $f \in L_{p\vartheta}(\mathbb{R}_n;\rho_n)$, $1 \leqslant p < +\infty$, $0 < \vartheta \leqslant +\infty$ and sequence of integers $\{l_k\}_{k=0}^{+\infty}$ is such, that $l_0 = 1$, $l_{k+1}\cdot l_k^{-1} \geqslant a_0 > 1$, $\forall k \in \mathbb{Z}^+$.*

If for some numbers q and τ: $p < q < +\infty$, $0 < \tau < +\infty$ series $\sum\limits_{m=0}^{+\infty} l_{m+1}^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)} E^{\tau}_{l_m,\dots,l_m}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}$ converges, then $f \in L_{q,\tau}(\mathbb{R}_n;\rho_n)$. While there is inequality:

$$\|f\|_{L_{q,\tau}(\mathbb{R}_n;\rho_n)} \leqslant$$
$$\leqslant C_{pq\vartheta\tau n}\left\{\|f\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} + \left[\sum_{m=1}^{+\infty} l_{m+1}^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)} E^{\tau}_{l_m,\dots,l_m}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}\right]^{\frac{1}{\tau}}\right\},$$

here $C_{pq\vartheta\tau n} > 0$ depends only on mentioned parameters.

Proof. Let $f \in L_{p\vartheta}(\mathbb{R}_n;\rho_n)$, $1 \leqslant p < +\infty$, $0 < \vartheta \leqslant +\infty$ and $\{P_{l_k,\dots,l_k}(\bar{x})\}_{k=0}^{+\infty}$-sequence of algebraic polynomial of best approximation of this function by space metric $L_{p,\vartheta}(\mathbb{R}_n;\rho_n)$: $E_{m,\dots,m}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} = \|f - P_{m,\dots m}\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}, \quad \forall m \in \mathbb{N}$.

That is why in sense of space $L_{p\vartheta}(\mathbb{R}_n;\rho_n)$ the following inequality is valid

$$f(\bar{x}) = P_{l_0,\dots,l_0}(\bar{x}) + \sum_{k=1}^{+\infty}\left(P_{l_k,\dots,l_k}(\bar{x}) - P_{l_{k-1},\dots,l_{k-1}}(\bar{x})\right),$$

where $\{l_k\}_{k=0}^{+\infty}$ is some integer sequence which satisfy conditions $l_0 = 1$, $l_{k+1}\cdot l_k^{-1} \geqslant a_0 > 1$, $\forall k \in \mathbb{Z}^+$.

Let's consider the series

$$\sum_{m=0}^{+\infty} l_m^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)} \|\Delta_{l_m,\dots,l_m}\|^{\tau}_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} \leqslant$$
$$\leqslant 2^{\tau+1}\left\{\|f\|^{\tau}_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} + \sum_{m=1}^{+\infty} l_{m+1}^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)} E_{l_m,\dots,l_m}(f)^{\tau}_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}\right\}.$$

According to the condition of theorem the series on the right side converges, so according to the theorem 1 function $f \in L_{q\tau}(\mathbb{R}_n;\rho_n)$ and takes place inequality mentioned in statement of theorem. $\square$

If in theorem 3 put $l_k = 2^k$, $\forall k \in \mathbb{Z}^+$, then we get variant of theorem equivalent to approval:

Theorem 4. *Let* $f \in L_{p\vartheta}(\mathbb{R}_n;\rho_n)$, $1 \leqslant p < +\infty$, $0 < \vartheta \leqslant +\infty$. *If for some numbers* q *and* τ: $p < q < +\infty$, $0 < \tau < +\infty$ *series* $\sum\limits_{k=1}^{+\infty} k^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)-1} E^{\tau}_{k,\dots,k}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}$ *converges, then* $f \in L_{q,\tau}(\mathbb{R}_n;\rho_n)$ *and while there is inequality:*

$$\|f\|_{L_{q,\tau}(\mathbb{R}_n;\rho_n)} \leqslant$$
$$\leqslant D_{pq\vartheta\tau n}\left\{\|f\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} + \left[\sum_{k=1}^{+\infty} k^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)-1} E^{\tau}_{k,\dots,k}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}\right]^{\frac{1}{\tau}}\right\},$$

where multiplier $D_{pq\vartheta\tau n} > 0$ *depends only on mentioned parameters.*

Theorem 5. *Let* $1 \leqslant p < q < +\infty$, $0 < \vartheta \leqslant +\infty$, $0 < \tau < +\infty$. *For occurring of embedding* $E^{(n)}_{p\vartheta}(\lambda;\rho_n) \subset L_{q,\tau}(\mathbb{R}_n;\rho_n)$ *it is necessary and sufficient that the series converges* $\sum\limits_{k=1}^{+\infty} k^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)-1}\lambda_k^{\tau}$.

Proof. Sufficiency of theorem follows from theorem 4 and definition of class $E^{(n)}_{p\vartheta}(\lambda;\rho_n)$. Now let's prove the necessity of theorem's condition. For it let's suppose that $E^{(n)}_{p\vartheta}(\lambda;\rho_n) \subset L_{q,\tau}(\mathbb{R}_n;\rho_n)$, $1 \leqslant p < q < +\infty$, $0 < \vartheta \leqslant +\infty$, $0 < \tau < +\infty$ but, in this case the series $\sum\limits_{k=1}^{+\infty} k^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)-1}\lambda_k^{\tau}$ diverges. Then according to lemma 1 for $\gamma = \tau$, $\alpha = \frac{n}{2p} - \frac{n}{2q} > 0$ exists the sequence of natural numbers $\{m_k\}_{k=1}^{+\infty}$ and sequence of positive numbers $\{\mu_k\}_{k=1}^{+\infty}$ with mentioned in lemma 1 qualities, in particular $\sum\limits_{k=1}^{+\infty} m_{k+1}^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)}\mu_k^{\tau} = +\infty$.

Let's consider the functional series $f_0(\bar{x}) = \frac{1}{2(c''_p)^n}\sum\limits_{k=1}^{+\infty} m_{k+1}^{\frac{n}{2p}}\mu_k \prod\limits_{\nu=1}^{n} P^*_{m_{k+1}}(x_\nu)$, where $c''_p > 0$ is defined by lemma 2. By the direct estimation accountinf lemma 2, we can show that function f_0 belongs to $L_{p,\vartheta}(\mathbb{R}_n;\rho_n)$ and $E^{(n)}_{p\vartheta}(\lambda;\rho_n)$. Now let's show that as consequence of our assumption there is that $f_0 \notin L_{q,\tau}(\mathbb{R}_n;\rho_n)$.

If to introduce the notation $T_{m_l,\dots,m_l}(\bar{x}) = \sum\limits_{k=0}^{l-1} m_{k+1}^{\frac{n}{2p}} \mu_k \prod_{\nu=1}^{n} P_{m_{k+1}}^{*}(x_\nu)$, then $\Delta_{m_l,\dots,m_l}(f_0;\bar{x}) = m_l^{\frac{n}{2p}} \mu_{l-1} \prod_{\nu=1}^{n} P_{m_l}^{*}(x_\nu)$, $l \in \mathbb{Z}^+$.

Further by theorem 2 and lemma 2 there appears the following set of inequalities:

$$\|f_0\|_{L_{q,\tau}(\mathbb{R}_n;\rho_n)}^{\tau} \geqslant \frac{A_{pq\vartheta\tau n}^{\tau}}{2^{\tau}(c_p'')^{\tau n}} \sum_{k=2}^{+\infty} m_k^{\tau\left(\frac{n}{4q}-\frac{n}{2q}\right)} \|T_{m_k,\dots,m_k} - T_{m_{k-1},\dots,m_{k-1}}\|_{L_{2q,\vartheta}}^{\tau} =$$

$$= \frac{A_{pq\vartheta\tau n}^{\tau}}{2^{\tau}(c_p'')^{\tau n}} \sum_{k=2}^{+\infty} m_k^{-\tau\frac{n}{4q}} \cdot m_k^{\tau\frac{n}{2p}} \mu_{k-1}^{\tau} \left(\|P_{m_k}^{*}\|_{L_{2q,\vartheta}(\mathbb{R};\rho)}\right)^{n\tau} \geqslant$$

$$\geqslant 2^{-\tau} A_{pq\vartheta\tau n}^{\tau} \sum_{k=1}^{+\infty} m_{k+1}^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)} \cdot \mu_k^{\tau} = +\infty, \text{ i.e. } f_0 \notin L_{q,\tau}(\mathbb{R}_n;\rho_n).$$

Thus if there is imbedding $E_{p\vartheta}^{(n)}(\lambda;\rho_n) \subset L_{q,\tau}(\mathbb{R}_n;\rho_n)$, $1 \leqslant p < q < +\infty$, $0 < \vartheta \leqslant +\infty$, $0 < \tau < +\infty$, then series $\sum\limits_{k=1}^{+\infty} k^{\tau\left(\frac{n}{2p}-\frac{n}{2q}\right)-1} \lambda_k^{\tau}$ must converge. By this the necessity of theorem condition is established. □

3 Theorem about the embedding in Marcinkiewicz space $L_{q,\infty}(\mathbb{R}_n;\rho_n)$

Theorem 6. *Let $f \in L_{p\vartheta}(\mathbb{R}_n;\rho_n)$, $1 \leqslant p < +\infty$, $0 < \vartheta \leqslant +\infty$. If for some q: $p < q < +\infty$ final value $\sup\limits_{m\in\mathbb{Z}^+} 2^{-\frac{nm}{2q}} \sum\limits_{k=0}^{m} 2^{\frac{nk}{2p}} E_{2^k,\dots,2^k}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}$, then $f \in L_{q,\infty}(\mathbb{R}_n;\rho_n)$ and the following inequality is valid:*

$$\|f\|_{L_{q,\infty}(\mathbb{R}_n;\rho_n)} \leqslant$$

$$\leqslant C_{pq\vartheta\tau n} \left\{ \|f\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} + \sup_{m\in\mathbb{Z}^+} 2^{-\frac{nm}{2q}} \sum_{k=0}^{m} 2^{\frac{nk}{2p}} E_{2^k,\dots,2^k}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} \right\}.$$

Here multiplier $C_{pq\vartheta\tau n} > 0$ depends only on mentioned parameters.

Proof. Let $f \in L_{p\vartheta}(\mathbb{R}_n;\rho_n)$, $1 \leqslant p < q < +\infty$, $0 < \vartheta \leqslant +\infty$. According to the definition $\|f\|_{L_{q,\infty}(\mathbb{R}_n;\rho_n)} \leqslant \sup\limits_{0\leqslant t\leqslant 1} \left\{ t^{\frac{1}{q}} F(|f\rho_n|;t) \right\} + \sup\limits_{t\geqslant 1} \left\{ t^{\frac{1}{q}} F(|f\rho_n|;t) \right\} = J_1 + J_2$.

Firstly let's estimate J_2:

$$J_2 \leqslant \sup_{t\geqslant 1}\left\{t^{\frac{1}{p}}F(|f\rho_n|;t)\cdot t^{-\left(\frac{1}{p}-\frac{1}{q}\right)}\right\} \leqslant \sup_{t\geqslant 1}\left\{t^{\frac{1}{p}}F(|f\rho_n|;t)\right\}. \qquad (1)$$

By property of the monotonicity of nonincreasing rearrangement:

$$t^{\frac{1}{p}}F(|f\rho_n|;t) = F(|f\rho_n|;t)\cdot\left\{\frac{\vartheta}{p}\int_0^t y^{\frac{\vartheta}{p}-1}dy\right\}^{\frac{1}{\vartheta}} \leqslant$$

$$\leqslant \left\{\frac{\vartheta}{p}\int_0^t y^{\frac{\vartheta}{p}-1}(F(|f\rho_n|;t))^{\vartheta}dy\right\}^{\frac{1}{\vartheta}} \leqslant \|f\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}. \qquad (2)$$

Thus from (1), (2) we have, that $J_2 \leqslant \|f\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}$.
Further

$$J_1 \leqslant \sup_{\nu\in\mathbb{Z}^+}\ \sup_{2^{-\frac{(\nu+1)n}{2}}\leqslant t\leqslant 2^{-\frac{\nu n}{2}}}\left\{t^{\frac{1}{q}}F(|f\rho_n|;t)\right\} \leqslant$$

$$\leqslant \sup_{\nu\in\mathbb{Z}^+}\left\{2^{-\frac{\nu n}{2q}}F\left(|f\rho_n|;2^{-\frac{(\nu+1)n}{2}}\right)\right\} \leqslant \text{(lemma3)} \leqslant$$

$$\leqslant A_{p\vartheta n}\sup_{\nu\in\mathbb{Z}^+}\left\{2^{-\frac{\nu n}{2q}}\left[\|f\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} + \sum_{k=0}^{\nu}2^{\frac{n(k+1)}{2p}}E_{2^k,\ldots,2^k}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}\right]\right\}.$$

Thus

$$\|f\|_{L_{q,\infty}(\mathbb{R}_n;\rho_n)} \leqslant A'_{p\vartheta n}\left\{\|f\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} + \sup_{\nu\in\mathbb{Z}^+}2^{-\frac{n\nu}{2q}}\sum_{k=0}^{\nu}2^{\frac{nk}{2p}}E_{2^k,\ldots,2^k}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}\right\}.$$

Theorem 7. *Let $f \in L_{p\vartheta}(\mathbb{R}_n;\rho_n)$, $1 \leqslant p < +\infty$, $0 < \vartheta \leqslant +\infty$. If for some number q: $p < q < +\infty$ final value is $\sup_{m\in\mathbb{N}} m^{-\frac{n}{2q}}\sum_{k=1}^{m}k^{\frac{n}{2p}-1}E_{k,\ldots,k}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}$, so $f \in L_{q,\infty}(\mathbb{R}_n;\rho_n)$ and the following inequality is valid:*

$$\|f\|_{L_{q,\infty}(\mathbb{R}_n;\rho_n)} \leqslant$$

$$\leqslant C_{pq\vartheta n}\left\{\|f\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} + \sup_{m\in\mathbb{N}} m^{-\frac{n}{2q}}\sum_{k=1}^{m}k^{\frac{n}{2p}-1}E_{k,\ldots,k}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}\right\}.$$

Proof. By property of monotonicity of sequence of complete best approximation
$\left\{E_{k,\dots,k}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}\right\}_{k=1}^{+\infty}$ there is inequality:

$$2^{\frac{n\nu}{2p}}E_{2^\nu,\dots,2^\nu}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} \leqslant C_{pn}\sum_{k=2^{\nu-1}+1}^{2^\nu-1}k^{\frac{n}{2p}-1}E_{k,\dots,k}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}, \forall\nu\in\mathbb{N}.$$

Hence we obtain

$$\sum_{k=0}^{\nu}2^{\frac{nk}{2p}}E_{2^k,\dots,2^k}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)} \leqslant$$
$$\leqslant C'_{pn}\left\{\|f\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}+\sum_{l=2}^{2^\nu}l^{\frac{n}{2p}-1}E_{l,\dots,l}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}\right\}.$$

Let now $2^\nu\leqslant m<2^{\nu+1}$, $\nu\in\mathbb{Z}^+$. Then

$$\sup_{\nu\in\mathbb{Z}^+}2^{-\frac{n\nu}{2q}}\sum_{k=0}^{\nu}2^{\frac{nk}{2p}}E_{2^k,\dots,2^k}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}\leqslant$$
$$\leqslant C'_{pn}\left\{\|f\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}+2^{\frac{n}{2q}}\sup_{m\in\mathbb{N}}m^{-\frac{n}{2p}}\sum_{l=1}^{m}l^{\frac{n}{2p}-1}E_{l,\dots,l}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}\right\}.$$

Since according to the condition of theorem right side of inequality is final then left side is also final. So according to the theorem 6 function $f\in L_{q,\infty}(\mathbb{R}_n;\rho_n)$ and also inequality

$$\|f\|_{L_{q,\infty}(\mathbb{R}_n;\rho_n)}\leqslant$$
$$\leqslant A_{pq\vartheta n}\left\{\|f\|_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}+\sup_{m\in\mathbb{N}}m^{-\frac{n}{2q}}\sum_{l=1}^{m}l^{\frac{n}{2p}-1}E_{l,\dots,l}(f)_{L_{p,\vartheta}(\mathbb{R}_n;\rho_n)}\right\}.$$

Theorem 8. *Let $1<p<q<+\infty$, $1\leqslant\vartheta\leqslant+\infty$. For occurring the embedding $E_{p\vartheta}^{(n)}(\lambda;\rho_n)\subset L_{q,\infty}(\mathbb{R}_n;\rho_n)$ it is necessary and sufficient that value*

$$\sup_{m\in\mathbb{N}}m^{-\frac{n}{2q}}\sum_{k=1}^{m}k^{\frac{n}{2p}-1}\lambda_k$$

was final.

Proof. Sufficiency of theorem condition follows from the definitions of class $E^{(n)}_{p\vartheta}(\lambda;\rho_n)$ and theorem 7. Let's prove the necessity of theorem condition. For it let's suppose that there is embedding $E^{(n)}_{p\vartheta}(\lambda;\rho_n)\subset L_{q,\infty}(\mathbb{R}_n;\rho_n)$, $1<p<q<+\infty$, $1\leqslant\vartheta\leqslant+\infty$, but $\sup\limits_{m\in\mathbb{N}} m^{-\frac{n}{2q}}\sum\limits_{k=1}^{m} k^{\frac{n}{2p}-1}\lambda_k=+\infty$.

Let's consider functional series $f_0(\bar{x})=\frac{1}{2(c''_p)^n}\sum\limits_{k=1}^{+\infty} l_{k+1}^{\frac{n}{2p}}\mu_k\prod\limits_{\nu=1}^{n} P^*_{l_{k+1}}(x_\nu)$, where $c''_p>0$ is defined by lemma 2.

By estimation with the help of lemma 2, we can show that this series is converges in sense of space $L_{p,\vartheta}(\mathbb{R}_n;\rho_n)$ and its amount $f_0\in E^{(n)}_{p\vartheta}(\lambda;\rho_n)$, $1<p<+\infty$, $1\leqslant\vartheta\leqslant+\infty$.

$$\|f_0\|_{L_{q,\infty}(\mathbb{R}_n;\rho_n)}\geqslant C_q\sup_{t\geqslant 0}\left\{t^{\frac{1}{q}-1}\int\limits_0^t F(|f\rho_n|;t)dt\right\}\geqslant$$

$$\geqslant C_q\sup_{t\in\left[l_{s+1}^{-n/2},l_s^{-n/2}\right]}\left\{t^{\frac{1}{q}-1}\sup_{|E|=t}\int\limits_E |f_0(\bar{x})\rho_n(\bar{x})|d\bar{x}\right\}\geqslant$$

$$\geqslant C_q l_{s+1}^{-\frac{n}{2q}}\cdot l_s^{\frac{n}{2}}\int\limits_0^{l_s^{-\frac{1}{2}}}\dots\int\limits_0^{l_s^{-\frac{1}{2}}}|f_0(\bar{x})\rho_n(\bar{x})|d\bar{x}\geqslant$$

$$\geqslant C_q l_{s+1}^{-\frac{n}{2q}}\cdot l_s^{\frac{n}{2}}\left(\sum_{k=1}^{s} l_{k+1}^{\frac{n}{2p}}\mu_k\prod_{k=1}^{n}P^*_k(0)\rho_n(\bar{x})\right)l_s^{-\frac{n}{2}}=C_q l_{s+1}^{-\frac{n}{2q}}\sum_{k=1}^{s} l_{k+1}^{\frac{n}{2p}}\mu_k,\forall s\in\mathbb{N}.$$

Since $\sup\limits_{s\in\mathbb{N}} l_{s+1}^{-\frac{n}{2q}}\sum_{k=1}^{s} l_{k+1}^{\frac{n}{2p}}\mu_k=+\infty$, then $\|f_0\|_{L_{q,\infty}(\mathbb{R}_n;\rho_n)}=+\infty$.

This fact contradict that $E^{(n)}_{p\vartheta}(\lambda;\rho_n)\subset L_{q,\infty}(\mathbb{R}_n;\rho_n)$, $1<p<q<+\infty$, $1\leqslant\vartheta\leqslant+\infty$. Consequently if before this embedding existed then it must be

$$\sup_{m\in\mathbb{N}} m^{-\frac{n}{2q}}\sum_{k=1}^{m} k^{\frac{n}{2p}-1}\lambda_k<+\infty.$$

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NEW RESULTS IN THE STUDY OF LYAPUNOV STABILITY

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Key words: abstract parabolic differential equation, stability by Lyapunov, asymptotic stability

AMS Mathematics Subject Classification: 4 7J35, 377K45

Abstract. In Banach space semi-linear differential equation with the singular linear operator is considered. The stability questions of its trivial solution on the base of the generalized Lyapunov–Schmidt method are investigated. On this way the different extensions of Lyapunov theorem about stability on the base of linear approximation for the DE are obtained.

1 Regular case of stability

Let X be real or complex Banach space. The initial problem for differential equation

$$\dot{x} = Ax + R(t,x), \quad x(0) = a \tag{1}$$

is considered under the fulfillment of the following conditions.

I. A is closed linear operator mapping its dense in X domain $D(A)$ and its range $R(A)$ is closed in X. Furthermore let A be infinitesimal operator of continuous semi group $T(t)$.

II. Nonlinear operator $R(t,x)$ is continuous at $t \in R^+$, $x: \ ||x|| \leqslant \rho$, $R(0,t) = 0$ for all $t \in R^+$.

Introduce the concepts of classical and generalized solutions of initial problem (1).

$x = x(t)$ is classical solution of (1) if and only if $x(t)D(A)$ continuously differentiable and satisfies (1) on Re^+.

The continuous on Re^+ solution $x(t)$ to the integral equation

$$x(t) = U(t)a + \int_0^t U(t-s)R(s,u(s))]ds \tag{2}$$

will be called generalized solution of Cauchy problem (1).

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We interesting in stability properties of trivial solution of DE from (1). Consider the corresponding Cauchy linear problem

$$\dot{x} = Ax, \quad x(0) = a \tag{3}$$

Introduce the following restriction to operator A.

III. There exist constants $M > 0$ and $\alpha > 0$ such that for all $t \in \mathrm{Re}^+$ the inequality $||U(t)|| \leqslant M \exp(-\alpha t)$ is fulfilled.

Let III is satisfied then solution of (3)$x(t) = U(t)a$ and $||x(t)|| \leqslant M||a|| \exp(-\alpha t)$. This means that the trivial solution of the linearized DE is asymptotic stable.

Introduce the restriction to nonlinear part of DE in (1) which guaranties the asymptotic stability of trivial solution to nonlinear equation from (1).

IV. There exist constants $C > 0$ and $\beta > 0$ such that for all $t \in \mathrm{Re}^+, \quad x_1, x_2 \in S$ the inequality

$$||R(x_1,t) - R(x_2,t)|| \leqslant C \overset{\beta}{\max}(||x_1||, ||x_2||)||x_1 - x_2||$$

is fulfilled.

Remark 1. From condition IV it follows that for all $x \in S$ and any $t \in \mathrm{Re}^+$ the inequality $||R(x,t)|| \leqslant C||x||^{1+\beta}$ is true.

Following to [4, 5] introduce a convenient family of Banach spaces of abstract functions.

Definition 1. Let $\gamma > 0$. The space $C_\gamma(X)$ will be called a set of all abstract functions $u(t)$, defined and continuous on semi axis Re^+ with natural operations of addition and multiplication on scalars taking values in X, for which the norm $|||u|||_\gamma = sup_{\mathrm{Re}^+}||u(t)||exp(\gamma t)$ is finite.

Note, that C_γ is Banach space. For $\gamma = 0$ we have the limit case - the space $C_0(X)$ of bounded on Re^+ continuous abstract functions.

In the articles [4, 5] it was established that by the realization of the restrictions I-IV condition take place the asymptotic stability of the trivial solution to nonlinear D Eq (1).

Theorem 1. *Let the condition I-IV be realized. Then there exist the numbers* $r_* > 0, \quad \rho_* > 0$ *such that for any* $a : ||a|| \leqslant \rho_*$ *the initial problem (1) has in the ball* $||x|| \leqslant r_*$ *the unique general on* Re^+ *solution* $x = x(t,a) \in C_\gamma$. *This solution is continuous on* a *in the ball* $||x_0|| \leqslant \rho_*$ *and* $x(0) = a$.

In the work [5] we show that if the Holder condition to $R(t, x)$ is fulfilled then the generalized solution is classic one.

In the work [6] we assume that the result of theorem be reserved if some growth $R(x, t)$ by t goes to plus infinity damps by semi group $T(t)$ decreasing.

2 Singular case of stability

Under the fulfillment of the restrictions I,III the operator A is continuously invertible (see [2]), that is its range is coincided with all space$X$ and the inverse to A operator is bounded.

Here it is stated the problem to stability study of principally another case when the operator A is non-invertible. Apparently, here more typical is the case of Lyapunov stability of trivial solution to D Eq, but not the case of its asymptotic stability. It is supposed further that for the operator A the following very general conditions are realized. More precisely we suppose the following restriction be realized.

V. The set $V = N(A)$ of zeroes of the operator A and the range $U = R(A)$ of the operator A are nontrivial and closed in X. Let the space X can be represented in the form of direct sum of U and V $X = U + V$.

Note that the finite-dimensionality of the subspace V is not assumed. Thus in that case the question is about the generalization of Fredholm operator notion.

Introduce two projectors. Let P and Q are the projectors of X to U and V respectively. Now the semi group $U(t)$ does not satisfies the condition of the exponentially decrease because of $U(t)v = v$ for all v from V. This condition we replace to the following one.

IV. Let the restriction of $T(t)$ to subspace U be the exponentially decreasing semi group, i.e. there exist constants $M > 0$ and $\alpha > 0$ such that for all $t \in \mathrm{Re}^+$ and for all $u \in U$ the inequality $||T(t)u|| \leqslant M \exp(-\alpha t)||u||$ is fulfilled.

Consider the linear problem (3) where we take $x = u + v$, $u \in U$, $\quad v \in V$.

Project obtained identity on U and than on V. Now we can rewrite (3) as the system of initial problems

$$\begin{aligned} \dot{u} &= Au, \quad u(0) = Pa, \\ \dot{v} &= 0, \quad v(0) = Qa. \end{aligned} \tag{4}$$

The solution of this system has the form $u(t) = T(t)Pa, \quad v(t) = Qa$.

Hence the solution of (3) $x(t) = U(t)Pa + Qa$ is bounded on Re^+ because of restrictions I,III,IV. This means that the trivial solution of linear problem is stable by Lyapunov (not asymptotic).

Introduce now two following special local Lipschitz restrictions for nonlinear part of DE. They guarantee the Lyapunov stability of trivial solution to nonlinear problem (1).

V.Let $\forall x_1, x_2 : \quad ||x_1|| \leqslant \rho, \quad ||x_2|| \leqslant \rho \ (\beta_k > 0, \quad k = 1, 2)$:

$$||P(R(x_1) - R(x_2))|| \leqslant C_1(t) \overset{\beta_1}{\max}(||x_1||, ||x_2||)||x_1 - x_2||$$

$$||Q(R(x_1) - R(x_2))|| \leqslant C_2(t) \overset{\beta_2}{\max}(||x_1||, ||x_2||)||x_1 - x_2||$$

are fulfilled.

Theorem 2. *Suppose conditions I-II, IV-V are fulfilled. If on* $R^+, \quad C_1(t) \leqslant$ *const*, $\quad \int_0^{+\infty} C_2(t)dt$ *leconst then there exist the numbers* $r_* > 0, \quad \rho_* > 0$ *such that for any* $a : ||a|| \leqslant \rho_*$ *the initial problem (1) has in the ball* $||x|| \leqslant r_*$ *the unique bounded on* R^+ *generalized solution* $x = x(t, a) \in C_0$ *for all initial values* a *such that* $||a|| < \rho_*$. *This solution is continuous on* a *in the ball* $||x_0|| \leqslant \rho_*$ *and* $x(0) = a$.

Give the plan of theorem 2 proof. In (1) take $x = u + v$, where $u \in U, \quad v \in V$, project obtained identity on U and than on V. Now we can rewrite (1) as the system of initial problems

$$\begin{gathered} \dot{u} = Au + PR(t, u + v), \quad u(0) = Pa \\ \dot{v} = QR(t, u + v), \quad v(0) = Qa \end{gathered} \tag{5}$$

Replace the system (3)-(4) by the equivalent integral equations system

$$\begin{gathered} u(t) = U(t)Pa + \int_0^t U(t-s)PR(s, u(s) + v(s))]ds, \\ v(t) = Qa + \int_0^t QR(s, u(s) + v(s))]ds. \end{gathered} \tag{6}$$

The two integral equations may be written as the nonlinear operator equation in the space of couple $x(t) = (u(t), v(t))$ of continuous bounded on $R+$ functions. In other words write the system of integral equations (6)–(7) in the form of operator equation with unknown $x \in C_0(X)$ and with small parameter $a \in X$ and with corresponding operators D and F

$$x = Da + F(x). \tag{7}$$

Here $Da(t)$ is continuous and bounded on Re^+. Then we establish that the operator $F(x)$ maps the ball of sufficiently small radius into this ball and satisfies in this ball the Lipschitz condition with the constant tending to zero together with radius. Applying to this this equation the variant of implicit operator theorem from [2] we obtain the theorem 2 statement.

As the consequences of the theorem 2 one has the following previously established propositions.

Theorem 3 (V.A. Trenogin, A-V. Ion [11]). *Suppose the conditions I-II, IV-V are fulfilled. If*

$$\int_0^{+\infty} C_k(t) \leqslant const_k, \quad k = 1, 2$$

then theorem 2 conclusion are fulfilled.

This result was represented in the joint work of author and professor Anca Veronica Ion (Romania) carried out in the process of International grant Russia–Romania realization. In this work one can find also different useful additions and generalizations.

Theorem 4 (V.A. Trenogin). *Suppose the conditions I–II, IV–V are fulfilled. If $R = R(x)$ and $C_2 = 0$ or $QR(x) = 0$ then theorem 2 conclusions are fulfilled.*

The result of theorem 4 is published in the work [10] for the more general case of presence nontrivial Jordan structure of operator A.

Corollary. By additional Holder condition to nonlinear operator $R(t, x)$ with respect to t one can proof as in the work [5] that the generalized solution indicated in the theorems 2-4 will be classic one.

These theorems are new variants of the known Lyapunov theorem about the asymptotic stability of trivial solution of DE but now the question is about Lyapunov stability and not asymptotic stability.

In conclusion we want to note that the more general situation of problem (1) must also investigated. Let now the following restrictions to operator A are fulfilled. $V = N(A)$ and $W = R(A)$ are nontrivial, closed in X and have in X the nontrivial direct complements U and Z respectively so that$X = U + V$ and $X = W + Z$. Now the existence of nontrivial Jordan structure for elements from V is not except as our preceding text where $W = U$ and $Z = V$. In the work [10] and another our last publications the analogue of theorem 4 are illuminate. In consequence we plan extend this ideas to theorem 4 analog.

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II.4. Fixed Point Theory and Applications

(Sessions organizers: E. Karapinar)

REMARKS ON «FIXED AND PERIODIC RESULTS IN TVS-CONE METRIC SPACES»

Thabet Abdeljawad, Erdal Karapinar

Key words: cone metric space, TVS-cone metric space, fixed point theorems

AMS Mathematics Subject Classification: 46N40, 47H10, 54H25, 46T99

Abstract. Recently, M. Abbas and B.E. Rhoades proved some fixed point theorems on the class of complete cone metric space with normal cone. In 2010, Wei-Shih Du introduces TVS-cone metric space. In this paper, the results of M. Abbas and B.E. Rhoades are extended to the class of complete TVS-cone metric spaces without using any normality assumption.

1 Introduction

Many authors attempted to generalize the notion of the metric space. In 2007, Huang and Zhang [8] announced the notion of cone metric spaces (CMS) by using the same idea, namely, by replacing real numbers with an ordering real Banach space. In that paper, they also discussed some properties of convergence of sequences and proved the fixed point theorems of contractive mapping for cone metric spaces: Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leqslant k < 1$, the inequality $d(Tx, Ty) \leqslant kd(x, y)$, for all $x, y \in X$, has a unique fixed point. Lately, many results on fixed point theorems have been extended to cone metric spaces (see e.g. [4, 8, 14–16], [5]– [13], [2, 3]).

Recently, Du [7] gave the definition of generalized cone metric space, namely topological vector space-cone metric space (TVS-CMS), and proved some fixed point theorem on that class. The author show also that Banach contraction principles in usual metric spaces and in TVS-CMS are equivalent.

In this manuscript, the results of [1] are generalized to TVS-cone metric spaces and without any normality assumptions.

Throughout this paper, E stands for real topological vector space (t.v.s.) with zero vector and $\mathbb{Z}_+$ represent the set of all positive integer, as usual. A non-empty subset P of E is called a cone if $P+P \subseteq P$, $\lambda P \subseteq P$ for $\lambda \geqslant 0$ and $P \cap (-P) = \{0\}$. For a given cone P, one can define a partial ordering (denoted by $\leqslant$: or $\leqslant P$) with respect to P by $x \leqslant y$ if and only if $y - x \in P$. The notation $x < y$ indicate that $x \leqslant y$ and $x \neq y$ while $x \ll y$ will show $y - x \in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P.

Throughout this manuscript, E is locally convex Hausdorff t.v.s. with its zero vector, P is a proper, closed and pointed convex cone in E with $int(P) \neq \emptyset$, $c \in int(P)$ and $\leqslant$ is a partial ordering with respect to P.

Definition 1 (see [5–7]). For $c \in IntP$, the nonlinear scalarization function $\varphi_c : E \to \mathbb{R}$ is defined by $\varphi_c(y) = \inf\{t \in \mathbb{R} : y \in tc - P\}$, for all $y \in E$.

Lemma 1 (see [5–7]). *For each $t \in \mathbb{R}$ and $y \in E$, the following are satisfied:*

1. $\varphi_c(y) \leqslant t \Leftrightarrow y \in tc - P$,
2. $\varphi_c(y) > t \Leftrightarrow y \notin tc - P$,
3. $\varphi_c(y) \geqslant t \Leftrightarrow y \notin tc - int(P)$,
4. $\varphi_c(y) < t \Leftrightarrow y \in tc - int(P)$,
5. $\varphi_c(y)$ *is positively homogeneous and continuous on* E,
6. *if* $y_1 \in y_2 + K$, *then* $\varphi_c(y_2) \leqslant \varphi_c(y_1)$,
7. $\varphi_c(y_1 + y_2) \leqslant \varphi_c(y_1) + \varphi_c(y_2)$, *for all* $y_1, y_2 \in E$.

Definition 2. Let X be non-empty set. Suppose a vector-valued function $p : X \times X \to Y$ satisfies:

1. $0 \leqslant p(x, y)$ for all $x, y \in X$,
2. $p(x, y) = 0$ if and only if $x = y$,
3. $p(x, y) = p(y, x)$ for all $x, y \in X$
4. $p(x, y) \leqslant p(x, z) + p(z, y)$, for all $x, y, z \in X$.

Then, p is called TVS-cone metric on X, and the pair (X, p) is called a TVS-cone metric space (in short, TVS-CMS).

Note that in [8], the authors considered E as a real Banach space in the definition of TVS-CMS. Thus, a cone metric space (in short, CMS) in the sense of Huang and Zhang [8] is a special case of TVS-CMS.

The cone P of a real Banach space E is called *normal* if there is a number $K \geqslant 1$ such that for all $x, y \in E$: $0 \leqslant x \leqslant y \Rightarrow \|x\| \leqslant K\|y\|$. The least positive integer K, satisfying this inequality, is called the normal constant of P. Also, P is said to be *regular* if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geqslant 1}$ is a sequence such that $x_1 \leqslant x_2 \leqslant \cdots \leqslant y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. For the definition of normality of cones in locally convex topological vector spaces we may refer the reader to [17].

Lemma 2 (see [7]). *Let (X, p) be a TVS-CMS. Then, $d_p : X \times X \to [0, \infty)$ defined by $d_p = \varphi_c(y) \circ p$ is a metric.*

Remark 1. Since a cone metric space (X, d) in the sense of Huang and Zhang [8], is a special case of TVS-CMS, then $d_p : X \times X \to [0, \infty)$ defined by $d_p = \varphi_c(y) \circ d$ is also a metric.

Definition 3 (see [7]). Let (X, p) be a TVS-CMS, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X.

(*i*) $\{x_n\}_{n=1}^{\infty}$ TVS-cone converges to $x \in X$ whenever for every $0 \ll c \in E$, there is a natural number M such that $p(x_n, x) \ll c$ for all $n \geqslant M$ and denoted by $cone - \lim_{n\to\infty} x_n = x$ (or $x_n \overset{cone}{\to} x$ as $n \to \infty$),

(*ii*) $\{x_n\}_{n=1}^{\infty}$ TVS-cone Cauchy sequence in (X, p) whenever for every $0 \ll c \in E$, there is a natural number M such that $p(x_n, x_m) \ll c$ for all $n, m \geqslant M$,

(*iii*) (X, p) is TVS-cone complete if every sequence TVS-cone Cauchy sequence in X is a TVS-cone convergent.

Lemma 3 (see [7]). *Let (X, p) be a TVS-CMS, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X. Set $d_p = \varphi_c(y) \circ p$. Then the following statements hold:*

(i) If $\{x_n\}_{n=1}^{\infty}$ converges to x in TVS-CMS (X, p), then $d_p(x_n, x) \to 0$ as $n \to \infty$,

(ii) If $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in TVS-CMS (X, p), then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence (in usual sense) in (X, d_p),

(iii) If (X, p) is complete TVS-CMS, then (X, d_p) is a complete metric space.

Proposition 1 (see [7]). Let (X, p) is complete TVS-CMS and $T : X \to X$ satisfy the contractive condition

$$p(Tx, Ty) \leqslant kp(x, y) \tag{1}$$

for all $x, y \in X$ and $0 \leqslant k < 1$. Then, T has a unique fixed point in X. Moreover, for each $x \in X$, the iterative sequence $\{T^n x\}_{n=1}^{\infty}$ converges to fixed point.

2 Common Fixed Point Theorems

Theorem 1. *Let (X, p) be a complete TVS-CMS and $f, g : X \to X$ self-mappings. Suppose the following condition is satisfied:*

$$p(fx, gy) \leqslant \alpha p(x, y) + \beta[p(x, fx) + p(y, gy)] + \gamma[p(x, gy) + p(y, fx)] \tag{2}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geqslant 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then, f and g have a unique fixed point in X. Moreover, any fixed point of f is a fixed point of g, and conversely.

Proof. By Lemma 2, $d_p = \varphi_c(y) \circ p$ is a (usual) metric. Due to Lemma 3, (X, d_p) is a complete metric space. Taking Lemma 1 into account, (2) turns into

$$d_p(fx, gy) \leqslant \alpha d_p(x, y) + \beta[d_p(x, fx) + d_p(y, gy)] + \gamma[d_p(x, gy) + d_p(y, fx)] \quad (3)$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geqslant 0$ and $\alpha + 2\beta + 2\gamma < 1$.

The rest of the proof is standard. Indeed, take an arbitrary $x_0 \in X$ and let $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = fx_{2n+1}$ for all $n \in \{0, 1, 2, \cdots\}$. By (3), routine calculation implies that $d_p(x_{2n+1}, x_{2n+2}) \leqslant \frac{\alpha+\beta+\gamma}{1-(\alpha+\beta)} d_p(x_{2n}, x_{2n+1})$. Analogously, $d_p(x_{2n+3}, x_{2n+2}) \leqslant \frac{\alpha+\beta+\gamma}{1-(\alpha+\beta)} d_p(x_{2n+2}, x_{2n+1})$. Thus, one can get

$$d_p(x_{n+1}, x_{n+2}) \leqslant t d_p(x_n, x_{n+1}) \leqslant \cdots \leqslant t^{n+1} d_p(x_0, x_1)$$

for any n, where $t = \frac{\alpha+\beta+\gamma}{1-(\alpha+\beta)} < 1$. Hence, for $m > n$

$$\begin{aligned} d_p(x_m, x_n) & \leqslant d_p(x_n, x_{n+1}) + d_p(x_{n+1}, x_{n+2}) + \cdots + d_p(x_{m-1}, x_m) \\ & (t^n + t^{n+1} + \cdots + t^{m-1}) d_p(x_0, x_1) \leqslant \tfrac{t^n}{1-t} \end{aligned}$$

Thus, $d_p(x_m, x_n) \to 0$ as $n, m \to \infty$ and $\{x_n\}$ is a Cauchy sequence. Since (X, d_p) is complete metric space, $\{x_n\}$ converges to some point in X, say z. To show z is a fixed point, Taking account into (3), routine calculation yields that

$$d_p(z, gz) \leqslant d_p(z, x_{2n+1}) + d_p(x_{2n+1}, gz)$$

$$\begin{aligned} d_p(z, gz) \leqslant \frac{1}{1-(\alpha+\beta)} [& d_p(z, x_{2n+1}) + \alpha d_p(z, x_{2n}) + \\ & + \beta d_p(x_{2n}, x_{2n+1}) + \gamma(d_p(x_{2n}, z) + d_p(z, x_{2n+1}))]. \quad (4) \end{aligned}$$

The right hand side of (4) converges to zero as $n \to \infty$. Thus, $gz = z$. Regarding (3) and $gz = z$, we have

$$\begin{aligned} d_p(fz, z) = d_p(fz, gz) \leqslant [& \alpha d_p(z, z) + \\ & + \beta(d_p(z, fz) + d_p(z, gz))\gamma(d_p(z, gz) + d_p(z, fz))] = (\beta + \gamma) d_p(z, fz) \end{aligned}$$

which is possible only if $d_p(fz, z) = 0$, that is, $fz = z$.

For the uniqueness follows by the method of reductio ad absurdum. Suppose w is another common fixed point of f and g.

$$d_p(z,z) = d_p(fz,gz) \leqslant [\alpha d_p(z,w)+ \\ + \beta(d_p(z,fz)+d_p(w,gw))\gamma(d_p(z,gw)+d_p(w,fz))] = (\beta+2\gamma)d_p(z,w)$$

which is possible only if $d_p(z,w)=0$, that is, $w=z$. □

Remark 2. Theorem 1 generalizes Theorem 2.1 in [1] and the consequent results (see Corollaries 2.2–2.8 in [1]) for proper choice of $f, g, \alpha, \beta, \gamma$. Notice also that we remove all additional conditions on the cone P.

Corollary 1. Let (X,p) be a complete TVS-CMS. A self-mapping $f : X \to X$ has a unique fixed point if one of the following conditions is satisfied (for all $x, y \in X$):

1. $p(f^p x, f^q y) \leqslant \alpha p(x,y) + \beta[p(x,f^p x)+p(y,f^q y)] + \gamma[p(x,f^q y)+p(y,f^p x)]$ where $\alpha,\beta,\gamma \geqslant 0$ and $\alpha+2\beta+2\gamma<1$ and $p,q\in\mathbb{N}$,
2. $p(fx,fy) \leqslant \alpha p(x,y) + \beta[p(x,fx)+p(y,fy)] + \gamma[p(x,fy)+p(y,fx)]$ where $\alpha,\beta,\gamma \geqslant 0$ and $\alpha+2\beta+2\gamma<1$,
3. $p(fx,fy) \leqslant \alpha_1 p(x,y)+\alpha_2 p(x,fx)+\alpha_3 p(y,fy)+\alpha_4 p(x,fy)+\alpha_5 p(y,fx)$ where $0\leqslant\alpha_i$ for $i=1,2,\cdots,5$, and $\sum_{i=1}^{5}\alpha_i<1$,
4. $p(fx,fy)\leqslant\alpha p(x,y)$ where $0\leqslant\alpha_i<1$,
5. $p(fx,fy)\leqslant\beta[p(x,fx)+p(y,fy)]$ where $\beta\geqslant 0$ and $2\beta<1$,
6. $p(fx,fy)\leqslant\alpha p(x,y)+\gamma[p(x,fy)+p(y,fx)]$ where $\gamma\geqslant 0$ and $2\gamma<1$,
7. $p(fx,fy)\leqslant\alpha p(x,y)+\beta[p(x,fx)+p(y,fy)]$ where $\alpha,\beta\geqslant 0$ and $\alpha+2\beta<1$.

3 Periodic Fixed Point Theorems

Let $z\in X$ be an fixed point of $f : X\to X$. Then $f^n(z) = f^{n-1}(f(z)) = f(f(...f(z))) = z$. Thus, z is also a fixed point of f^n for any $n\in\mathbb{N}$. In general, the converse is not true. For example, consider $X=\mathbb{R}$ and $f : X\to X$ in a way that $fx = 1-x$. Then $f^2x = x$ and thus $f^n x = x$ for all $n>1$. Thus, each $x\in\mathbb{R}$ is a fixed point of $f^n x = x$ for all $n>1$ but only fixed point of f is $\frac{1}{2}$. Let $Fix(f)$ denotes a set of all fixed point of f.

Theorem 2. *Let (X,p) be a complete TVS-CMS. A self-mapping $f : X\to X$ satisfies the following condition:*

(i) $p(fx,f^2x)\leqslant\alpha p(x,fx)$ for all $x\in X$, where $0\leqslant\alpha<1$

If $Fix(f)\neq\emptyset$ then $Fix(f)=Fix(f^n)$.

Proof. By Lemma 2, $d_p = \varphi_c(y)\circ p$ is a (usual) metric. Due to Lemma 3, (X,d_p) is a complete metric space. Taking Lemma 1. into account, the condition of the theorem turns into

(i)' $d_p(fx, f^2x) \leqslant \alpha d_p(x, fx)$ for all $x \in X$, where $0 \leqslant \alpha < 1$

The rest of the proof is obtained by standard methods. Indeed, suppose $z \in Fix(f^n)$ for $1 \leqslant n$ and (i) holds. Thus $(i)'$ holds and

$$d_p(z, fz) = d_p(ff^{n-1}z, f^2f^{n-1}z) \leqslant \alpha d_p(f^{n-1}z, f^n z) \leqslant \cdots \leqslant \alpha^n d_p(z, fz).$$

The right hand side of the inequality converges to zero as $n \to \infty$ which implies that $fz = z$ and so $Fix(f) = Fix(f^n)$. □

Remark 3. Theorem 2 generalizes Theorem 3.1 in [1]. Notice also that we remove all additional conditions on the cone P.

Theorem 3. *Suppose (X, p) is a complete TVS-CMS and a self-mapping $f : X \to X$ satisfies the condition (2) then $Fix(f) \cap Fix(g) = Fix(f^n) \cap Fix(g^n)$.*

Proof. Due to Theorem 1, the condition (3) holds and also $Fix(f) \cap Fix(g) \neq \emptyset$. Let $z \in Fix(f^n) \cap Fix(g^n)$. Then by (3) and triangle inequality

$$\begin{aligned} d_p(z, gz) &= d_p(ff^{n-1}z, gg^n z) \leqslant \\ &\leqslant \alpha d_p(f^{n-1}z, f^n z) + \beta(d_p(f^{n-1}z, f^n z) + d_p(f^n z, g^{n+1}z)) + \\ &\quad + \gamma(d_p(f^{n-1}z, g^{n+1}z) + d_p(g^n z, f^n z)) \leqslant \\ &\leqslant \alpha d_p(f^{n-1}z, z) + \beta(d_p(f^{n-1}z, z) + d_p(z, gz)) + \gamma(d_p(f^{n-1}z, gz) + d_p(z, z)) \leqslant \\ &\leqslant \alpha d_p(f^{n-1}z, z) + \beta(d_p(f^{n-1}z, z) + d_p(z, gz)) + \gamma(d_p(f^{n-1}z, z) + d_p(gz, z)). \end{aligned} \tag{5}$$

Thus, (5) implies $d_p(z, gz) \leqslant td_p(^{n-1}z, z)$ where $t = \frac{\alpha+\beta+\gamma}{1-(\alpha+\beta)} < 1$. Hence,

$$d_p(z, g^{n+1}z) = d_p(f^n z, gz) \leqslant td_p(^{n-1}z, z) \leqslant \cdots \leqslant t^n d(z, fz)$$

The right hand side of the inequality converges to zero as $n \to \infty$ which implies $z = gz$. By regarding Theorem 1 again, one can get $z = fz$. □

Theorem 4. *Suppose (X, p) is a complete TVS-CMS and a self-mapping $f : X \to X$ satisfies the condition 2 of Corollary 1 then $Fix(f) = Fix(f^n)$.*

Proof. It is clear that $Fix(f) \subset Fix(f^n)$. Due to Corollary 1, $Fix(f) \neq \emptyset$. One can repeat the process of proof Theorem 3, by replacing g with f and get $d_p(z, fz) \leqslant t^n d(z, fz)$ where $t = \frac{\alpha+\beta+\gamma}{1-(\alpha+\beta)} < 1$. Thus, $fz = z$ and hence $Fix(f) = Fix(f^n)$. □

Corollary 2. Suppose (X, p) is a complete TVS-CMS and a self-mapping $f : X \to X$ satisfies one of the condition 4-7 of Corollary 1 then $Fix(f) = Fix(f^n)$.

Remark 4. Normality can be removed also without making use of the nonlinear scalarization function φ_c. It can be removed by using the fact that for each $c \gg 0$ in (E, S) there exists $q \in S$ and $\delta > 0$ such that $q(b) < \delta,\ b \in E$ implies $b \ll c$. Where S is the system of the seminorms defining the locally convex topology of E.

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QUADRUPLE FIXED POINT THEOREMS FOR NONLINEAR CONTRACTIONS IN PARTIAL METRIC SPACES

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Abstract. The notion of tripled fixed point is introduced by Berinde and Borcut. Very recently Karapinar defined quadruple fixed point on partially ordered metric spaces. In this manuscript, we consider the existence and uniqueness of the quadruple fixed point on the class of partially ordered partial metric space.

1 Introduction

Berinde and Borcut [1] introduced the concept of tripled fixed point which is a generalization of coupled fixed point, defined by Bhaskar and Lakshmikantham [2]. Many authors focused on coupled and tripled fixed point and proved remarkable results (see e.g. [3, 4, 7–11]). Recently, Karapinar [5] introduced the concept of quadruple fixed point.

We consider the following partial order on the product space $X^4 = X \times X \times X \times X$:

$$(u, v, r, t) \leqslant (x, y, z, w) \text{ if and only if } x \geqslant u,\ y \leqslant v,\ z \geqslant r,\ t \leqslant w \tag{1}$$

where $(u, v, r, t), (x, y, z, w) \in X^4$. Regarding this partial order, we state the definition of the following mapping.

Definition 1 (see [5, 6]). Let $(X, \leqslant)$ be partially ordered set and $F : X^4 \to X$. We say that F has the mixed monotone property if $F(x, y, z, w)$ is monotone non-decreasing in x and z, and it is monotone non-increasing in y and w, that is, for any $x, y, z, w \in X$

$$\begin{aligned}
x_1, x_2 \in X,\ x_1 \leqslant x_2 &\Rightarrow\ F(x_1, y, z, w) \leqslant F(x_2, y, z, w),\\
y_1, y_2 \in X,\ y_1 \leqslant y_2 &\Rightarrow\ F(x, y_1, z, w) \geqslant F(x, y_2, z, w),\\
z_1, z_2 \in X,\ z_1 \leqslant z_2 &\Rightarrow\ F(x, y, z_1, w) \leqslant F(x, y, z_2, w),\\
w_1, w_2 \in X,\ w_1 \leqslant w_2 &\Rightarrow\ F(x, y, z, w_1) \geqslant F(x, y, z, w_2).
\end{aligned} \tag{2}$$

Definition 2 (see [5, 6]). An element $(x, y, z, w \in X^4$ is called a quadruple fixed point of $F : X^4 \to X$ if

$$F(x, y, z, w) = x \text{ and } F(x, w, z, y) = y \text{ and} \\ F(z, y, x, w) = z \text{ and } F(z, w, x, y) = w \tag{3}$$

For a metric space (X, d), the function $\rho : X^4 \to [0, \infty)$, given by,

$$\rho((x, y, z, w), (u, v, r, t)) := d(x, u) + d(y, v) + d(z, r) + d(w, t)$$

forms a metric space on X^4, that is, (X^4, ρ) is a metric induced by (X, d). For more details for quadruple fixed point theory, see [5, 6].

Metric spaces were introduced by Maurice René Fréchet in [12]. It is quite natural to attempt to get a generalization of the notion of metric: Pseudometric space, quasimetric space, semimetric spaces are well known examples of the generalizations of metric space. Here, we focus on another generalization of a metric space: Partial metric space. It was introduced by Matthews (see e.g. [13, 14]) for applying properly the concept of the metric space to apply to computer science [15]. Matthews suggested to non-zero self distance that was the basic idea of the construction of a partial metric space. In computer science this idea was appreciated (see e.g. [16, 17, 23, 24] and the references therein). In the last decade, on partial metric spaces a number of papers were reported (see e.g. [20]– [29] and the references therein)

We start our study by recalling some basic definitions and technical lemmas. A mapping $p : X \times X \to [0, \infty)$ is called a partial metric (see e.g. [13, 14]) on a nonempty set X if the following conditions are satisfied:

(PM1) $p(x, y) = p(y, x)$ (symmetry)
(PM2) If $0 \leqslant p(x, x) = p(x, y) = p(y, y)$ then $x = y$ (equality)
(PM3) $p(x, x) \leqslant p(x, y)$ (small self-distances)
(PM4) $p(x, z) + p(y, y) \leqslant p(x, y) + p(y, z)$ (triangularity)

Here, the pair (X, p) is called a partial metric space (PMS). Additionally, a triple $(X, p, \leqslant)$ is called a partially ordered partial metric space if (X, p) is a partial metric space and $(X, \leqslant)$ is a partially ordered set.

For a partial metric p on X, the mappings $d_p, d_m : X \times X \to \mathbb{R}^+$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{4}$$

$$d_m(x, y) = \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\} \tag{5}$$

are (usual) metrics on X. It is clear that d_p and d_m are equivalent. Notice also that each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Example 1 (see e.g. [13, 22, 23]). Consider $X = [0,\infty)$ with $p(x,y) = \max\{x,y\}$. Then (X,p) is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case $d_p(x,y) = |x-y|$ and $d_m(x,y) = \frac{1}{2}|x-y|$.

Example 2 (see [13]). Let $X = \{[a,b] : a,b,\in \mathbb{R},\ a \leqslant b\}$ and define $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (X,p) is a partial metric spaces.

Example 3 (see [13]). Let $X := [0,1] \cup [2,3]$ and define $p : X \times X \to [0,\infty)$ by

$$p(x,y) = \begin{cases} \max\{x,y\} \text{ if } \{x,y\} \cap [2,3] \neq \emptyset, \\ |x-y| \text{ if } \{x,y\} \subset [0,1]. \end{cases}$$

Then (X,p) is a complete partial metric space.

Throughout the paper, (X,p) will always denote a partial metric space and $(X,p,\leqslant)$ will denote a partially ordered complete partial metric space.

Definition 3 (see e.g. [13, 14]). (i) A sequence $\{x_n\}$ in (X,p) converges to $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$,

(ii) A sequence $\{x_n\}$ in (X,p) is called a Cauchy if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and finite),

(iii) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

(iv) Let $P = (x,y,z,w) \in X^4$ and $P_0 = (x_0,y_0,z_0,w_0)$ A mapping $F : X^4 \to X$ is said to be continuous at $(x_0,y_0,z_0,w_0) \in X^4$, if

$$F(x_0,y_0,z_0,w_0) = \lim_{P\to P_0} F(x,y,z,w) = F\left(\lim_{P\to P_0} x, \lim_{P\to P_0} y, \lim_{P\to P_0} z, \lim_{P\to P_0} w\right). \quad (6)$$

Lemma 1 (see e.g. [13, 14]). (A) *A sequence $\{x_n\}$ is Cauchy in (X,p) if and only if $\{x_n\}$ is Cauchy in the metric space (X,d_p),*

(B) *(X,p) is complete if and only if the metric space (X,d_p) is complete. Moreover,*

$$\lim_{n\to\infty} d_p(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m) \quad (7)$$

Lemma 2 (see e.g. [22, 25]). *Assume $x_n \to z$ as $n \to \infty$ in (X,p) such that $p(z,z) = 0$. Then $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y \in X$.*

Lemma 3 (see e.g. [13, 25]). *Let (X,p) be a PMS. If $p(x,y) = 0$ then $x = y$. Moreover if $x \neq y$, then $p(x,y) > 0$.*

Remark 1. Since d_p and d_m are equivalent, we can take d_m instead of d_p in the above Lemma.

2 Existence of Quadruple Fixed Point

The aim of this paper is to prove the existence and uniqueness of quadruple fixed point on the class of complete partially ordered partial metric space.

Theorem 1. *Let $(X, \leqslant)$ be partially ordered set and (X, p) be a complete partial metric space. Let $F : X^4 \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a constant $k \in [0, 1)$ such that*

$$p(F(x, y, z, w), F(u, v, r, t)) \leqslant \frac{k}{4}[p(x, u) + p(y, v) + p(z, r) + p(w, t)] \tag{8}$$

for all $x \geqslant u, y \leqslant v, z \geqslant r, w \leqslant t$. Suppose there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned} x_0 \leqslant F(x_0, y_0, z_0, w_0), \quad y_0 \geqslant F(x,w_0, z_0, y_0), \\ z_0 \leqslant F(z_0, y_0, x_0, w_0), \quad w_0 \geqslant F(z_0, w_0, x_0, y_0). \end{aligned}$$

Suppose either (a) F is continuous, or (b) X has the following property:

(i) if non-decreasing sequence $x_n \to x$ (respectively, $z_n \to z$), then $x_n \leqslant x$ (respectively, $z_n \leqslant z$) for all n,

(ii) if non-increasing sequence $y_n \to y$(respectively, $w_n \to w$), then $y_n \geqslant y$ (respectively, $w_n \geqslant w$) for all n,

then there exist $x, y, z, w \in X$ such that

$$\begin{aligned} F(x, y, z, w) = x, \quad F(x, w, z, y) = y, \\ F(z, y, x, w) = z, \quad F(z, w, x, y) = w. \end{aligned}$$

Proof. We construct a sequence $\{(x_n, y_n, z_n, w_n)\}$ in the following way: Set

$$\begin{aligned} x_1 = F(x_0, y_0, z_0, w_0) \geqslant x_0, \quad y_1 = F(x_0, w_0, z_0, y_0) \leqslant y_0, \\ z_1 = F(z_0, y_0, x_0, w_0) \geqslant z_0, \quad w_1 = F(z_0, w_0, x_0, y_0) \leqslant w_0, \end{aligned} \tag{9}$$

Since $x_0 \leqslant x_1$ (see (9)) then by the mixed monotone property of F we have $F(x_0, y_0, z_0, w_0) \leqslant F(x_1, y_0, z_0, w_0)$. Notice that $y_1 \leqslant y_0$ (see (9)). Thus, again by the mixed monotone property of F, we obtain that $F(x_1, y_0, z_0, w_0) \leqslant F(x_1, y_1, z_0, w_0)$. Since $z_0 \leqslant z_1$ (see (9)) then by the mixed monotone property of

F we get $F(x_1, y_1, z_0, w_0) \leqslant F(x_1, y_1, z_1, w_0)$. Regarding $w_1 \leqslant w_0$ (see (9)) and the mixed monotone property of F we observe that $F(x_1, y_1, z_1, w_0) \leqslant F(x_1, y_1, z_1, w_1)$. Combining all the inequalities above, we get the desired result, that is,

$$F(x_0, y_0, z_0, w_0) \leqslant F(x_1, y_1, z_1, w_1) \text{ or equivalently, } x_1 \leqslant x_2.$$

Analogously, one can gets $y_2 \leqslant y_1$, $z_1 \leqslant z_2$ and $w_2 \leqslant w_1$. Hence, for $n \geqslant 1$, inductively we get

$$\begin{aligned} x_n &= F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) \geqslant x_{n-1} \geqslant \cdots \geqslant x_0, \\ y_n &= F(x_{n-1}, w_{n-1}, z_{n-1}, y_{n-1}) \leqslant y_{n-1} \leqslant \cdots \leqslant y_0, \\ z_n &= F(z_{n-1}, y_{n-1}, x_{n-1}, w_{n-1}) \geqslant z_{n-1} \geqslant \cdots \geqslant z_0, \\ w_n &= F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}) \leqslant w_{n-1} \leqslant \cdots \leqslant w_0, \end{aligned} \tag{10}$$

Due to (8) and (10), we have

$$\begin{aligned} p(x_1, x_2) = p(F(x_0, y_0, z_0, w_0), F(x_1, y_1, z_1, w_1)) &\leqslant \\ \leqslant \frac{k}{4}[p(x_0, x_1) + p(y_0, y_1) + p(z_0, z_1) + p(w_0, w_1)] \end{aligned} \tag{11}$$

$$\begin{aligned} p(y_1, y_2) = p(F(x_0, w_0, z_0, y_0), F(x_1, w_1, z_1, y_1)) &\leqslant \\ \leqslant \frac{k}{4}[p(x_0, x_1) + p(w_0, w_1) + p(z_0, z_1) + p(y_0, y_1)] \end{aligned} \tag{12}$$

$$\begin{aligned} p(z_1, z_2) = p(F(z_0, y_0, x_0, w_0), F(z_1, y_1, x_1, w_1)) &\leqslant \\ \leqslant \frac{k}{4}[p(z_0, z_1) + p(y_0, y_1) + p(x_0, x_1) + p(w_0, w_1)] \end{aligned} \tag{13}$$

$$\begin{aligned} p(w_1, w_2) = p(F(z_0, w_0, x_0, y_0), F(z_1, w_1, x_1, y_1)) &\leqslant \\ \leqslant \frac{k}{4}[p(z_0, z_1) + p(w_0, w_1) + p(x_0, x_1) + p(y_0, y_1)] \end{aligned} \tag{14}$$

Regarding (8) together with (11), (12), (13) we have

$$\begin{aligned} p(x_2, x_3) = p(F(x_1, y_1, z_1, w_1), F(x_2, y_2, z_2, w_2)) &\leqslant \\ \leqslant \frac{k}{4}[p(x_1, x_2) + p(y_1, y_2) + p(z_1, z_2) + p(w_1, w_2)] \end{aligned} \tag{15}$$

$$p(y_2, y_3) = p(F(x_1, w_1, z_1, y_1), F(x_2, w_2, z_2, y_2)) \leqslant$$
$$\leqslant \frac{k}{4}[p(x_1, x_2) + p(w_1, w_2) + p(z_1, z_2) + p(y_1, y_2)] \quad (16)$$

$$p(z_2, z_3) = p(F(z_1, y_1, x_1, w_1), F(z_2, y_2, x_2, w_2)) \leqslant$$
$$\leqslant \frac{k}{4}[p(z_1, z_2) + p(y_1, y_2) + p(x_1, x_2) + p(w_1, w_2)] \quad (17)$$

$$p(w_2, w_3) = p(F(z_1, w_1, x_1, y_2), F(z_2, w_2, x_2, y_2)) \leqslant$$
$$\leqslant \frac{k}{4}[p(z_1, z_2) + p(w_1, w_2) + p(x_1, x_2) + p(y_1, y_2)] \quad (18)$$

Recursively we have

$$p(x_{n+1}, x_{n+2}) = p(F(x_n, y_n, z_n, w_n), F(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1})) \leqslant$$
$$\leqslant \frac{k}{4}[p(x_n, x_{n+1}) + p(y_n, y_{n+1}) + p(z_n, z_{n+1}) + p(w_n, w_{n+1})] \quad (19)$$

$$p(y_{n+1}, y_{n+2}) = p(F(x_n, w_n, z_n, y_n), F(x_{n+1}, w_{n+1}, z_{n+1}, y_{n+1})) \leqslant$$
$$\leqslant \frac{k}{4}[p(x_n, x_{n+1}) + p(w_n, w_{n+1}) + p(z_n, z_{n+1}) + p(y_n, y_{n+1})] \quad (20)$$

$$p(z_{n+1}, z_{n+2}) = p(F(z_n, y_n, x_n, w_n), F(z_{n+1}, y_{n+1}, x_{n+1}, w_{n+1})) \leqslant$$
$$\leqslant \frac{k}{4}[p(z_n, z_{n+1}) + p(y_n, y_{n+1}) + p(x_n, x_{n+1}) + p(w_n, w_{n+1})] \quad (21)$$

$$p(w_{n+1}, w_{n+2}) = p(F(z_n, w_n, x_n, y_n), F(z_{n+1}, w_{n+1}, x_{n+1}, y_{n+1})) \leqslant$$
$$\leqslant \frac{k}{4}[p(z_n, z_{n+1}) + p(w_n, w_{n+1}) + p(x_n, x_{n+1}) + p(y_n, y_{n+1})] \quad (22)$$

For simplicity, we can use the matrix notation as follow. Set

$$M = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \text{ and } D_n = \begin{pmatrix} p(x_{n+1}, x_n) \\ p(y_{n+1}, y_n) \\ p(z_{n+1}, z_n) \\ p(w_{n+1}, w_n) \end{pmatrix}, R = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

$$\text{Notice that } RM = R \text{ and } M^n = M \text{ for all } n \in \mathbb{N}. \tag{23}$$

So we have,

$$D_1 \leqslant kD_0, \tag{24}$$

$$D_2 \leqslant kMD_1 \leqslant k^2M^2D_0 = k^2MD_0, \text{and, inductively,} \tag{25}$$

$$D_n \leqslant kMD_{n-1} \leqslant k^nMD_0. \tag{26}$$

$$p(x_{n+1}, x_{n+2}) \leqslant kRD_n \begin{pmatrix} p(x_n, x_{n+1}) \\ p(y_n, y_{n+1}) \\ p(z_n, z_{n+1}) \\ p(w_n, w_{n+1}) \end{pmatrix}. \tag{27}$$

Hence, by (23), (8) and (10), we have

$$\begin{aligned} p(x_{n+1}, x_{n+2}) &= p(F(x_n, y_n, z_n, w_n), F(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1})) \leqslant \\ &\leqslant \frac{k}{4}[p(x_n, x_{n+1}) + p(y_n, y_{n+1}) + p(z_n, z_{n+1}) + p(w_n, w_{n+1})] \leqslant \\ &\leqslant kRD_n \leqslant k^{n+1}RMD_0 \leqslant k^{n+1}RD_0. \end{aligned} \tag{28}$$

We shall show the sequence $\{x_n\}$ is Cauchy easily by using (19)–(26). Without loss of generality, we may assume that $m > n$. By using (19)–(26) together with triangle inequality, we obtain that

$$\begin{aligned} p(x_m, x_n) &\leqslant p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + \cdots + p(x_{n+1}, x_n) \leqslant \\ &\leqslant k^{m-1}RD_0 + \cdots + k^nRD_0 \leqslant k^n(1 + \cdots + k^{m-n-1})RD_0 \leqslant \\ &\leqslant k^n\frac{1}{1-k}RD_0 \end{aligned} \tag{29}$$

Letting $n \to \infty$ in (29) and recalling that $k \in [0,1)$, $\lim_{n,m\to\infty} p(x_m, x_n) = 0$.

By definition, $d_p(x_n, x_m) = 2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m) \leqslant 2p(x_n, x_m)$. Thus, we have

$$\lim_{n\to\infty} d_p(x_n, x_m) = 0. \tag{30}$$

Since (X, p) is a complete partial metric spaces, then by Lemma 1,(X, d_p) is a complete metric space. Thus, $\{x_n\}$ converges in (X, d_p), say x. Again by Lemma 1, we have

$$p(x,x) = \lim_{n\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = 0. \tag{31}$$

Analogously, one can show that $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are also Cauchy sequences. Since (X, d_p) is complete metric space, there exists $y, z, w \in X$ such that

$$\begin{aligned} p(y,y) &= \lim_{n\to\infty} p(y_n, y_m) = \lim_{n\to\infty} p(y_n, y) = 0, \\ p(z,z) &= \lim_{n\to\infty} p(z_n, z_m) = \lim_{n\to\infty} p(z_n, z) = 0, \\ p(w,w) &= \lim_{n\to\infty} p(w_n, w_m) = \lim_{n\to\infty} p(w_n, w) = 0. \end{aligned} \tag{32}$$

Suppose now the assumption (a) holds. Then by (10) and (6), we have

$$\begin{aligned} x = \lim_{n\to\infty} x_n &= \lim_{n\to\infty} F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) = \\ &= F(\lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} z_{n-1}, \lim_{n\to\infty} w_{n-1}) = F(x,y,z,w) \end{aligned} \tag{33}$$

Analogously, we also observe that

$$\begin{aligned} y &= \lim_{n\to\infty} y_n = \lim_{n\to\infty} F(x_{n-1}, w_{n-1}, z_{n-1}, y_{n-1}) = F(x,w,z,y) \\ z &= \lim_{n\to\infty} z_n = \lim_{n\to\infty} F(z_{n-1}, y_{n-1}, x_{n-1}, w_{n-1}) = F(z,y,x,w) \\ w &= \lim_{n\to\infty} w_n = \lim_{n\to\infty} F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}) = F(z,w,x,y) \end{aligned} \tag{34}$$

Thus, we have

$$\begin{aligned} F(x,y,z,w) = x, \quad F(x,w,z,y) = y, \\ F(z,y,x,w) = z, \quad F(z,w,x,y) = w. \end{aligned}$$

Suppose now the assumption (b) holds. Since $\{x_n\}, \{z_n\}$ are non-decreasing and $x_n \to x,\ z_n \to z$ and also $\{y_n\}, \{w_n\}$ are non-increasing and $y_n \to y,\ w_n \to w$, then by assumption (b) we have

$$x_n \geqslant x, y_n \leqslant y, z_n \geqslant z, w_n \leqslant w$$

for all n. Regarding (8) and the triangle inequality, we have

$$\begin{aligned} p(x, F(x, y, z, w)) &\leqslant p(x, x_{n+1}) + p(x_{n+1}, F(x, y, z, w)) - p(x_{n+1}, x_{n+1}) = \\ &= p(x, x_{n+1}) + p(F(x_n, y_n, z_n, w_n), F(x, y, z, w)) \leqslant \\ &\leqslant p(x, x_{n+1}) + \frac{k}{4}[p(x_n, x) + p(y_n, y) + p(z_n, z) + p(w_n, w)] \end{aligned} \tag{35}$$

Taking $n \to \infty$ in (35) and using (31) and (32), we get that $p(x, F(x, y, z, w)) = 0$.

Again by (31) and (32), we have

$$p(F(x, y, z, w), F(x, y, z, w)) \leqslant \frac{k}{4}[p(x, x) + p(y, y) + p(z, z) + p(w, w)] = 0. \tag{36}$$

Hence, by (31),(36),(35) and definition

$$d_p(x, F(x, y, z, w)) = 2p(x, F(x, y, z, w)) - p(F(x, y, z, w), F(x, y, z, w)) - p(x, x) = 0. \tag{37}$$

Thus, we have $x = F(x, y, z, w)$. Analogously we get

$$F(y, z, w, x) = y, \quad F(z, w, x, y) = z, \quad F(w, x, y, z) = w.$$

Thus, we proved that F has a quadruple fixed point.

3 Uniqueness of Quadruple Fixed Point

In this section we shall prove the uniqueness of quadruple fixed point. For a product X^4 of a partial ordered set $(X, \leqslant)$ we define a partial ordering in the following way: For all $(x, y, z, t), (u, v, r, t) \in X \times X \times X \times X$

$$(x, y, z, w) \leqslant (u, v, r, t) \Leftrightarrow x \leqslant u, y \geqslant v, z \leqslant r, w \geqslant r. \tag{38}$$

We say that (x, y, z, w) is equal (u, v, r, t) if and only if $x = u,\ y = v,\ z = r$ and $w = t$.

Theorem 2. *In addition to hypothesis of Theorem 1., suppose that for all $(x, y, z, t), (u, v, r, t) \in X^4$, there exists $(a, b, c, d) \in X^4$ that is comparable to (x, y, z, t) and (u, v, r, t), then F has a unique quadruple fixed point.*

Proof. The set of quadruple fixed point of F is not empty due to Theorem 1. Assume, now, (x, y, z, t) and (u, v, r, t) are the quadruple fixed point of F, that is,

$$\begin{aligned} F(x, y, z, w) &= x, \quad F(u, v, r, t) = u, \\ F(x, w, z, y) &= y, \quad F(u, t, r, v) = v, \\ F(z, y, x, w) &= z, \quad F(r, v, u, t) = r, \\ F(z, w, x, y) &= w, \quad F(r, t, u, v) = t, \end{aligned}$$

We shall show that (x, y, z, w) and (u, v, r, t) are equal. By assumption, there exists $(a, b, c, d) \in X \times X \times X \times X$ that is comparable to (x, y, z, t) and (u, v, r, t). Define sequences $\{a_n\}, \{b_n\}, \{c_n\}$ and $\{d_n\}$ such that

$$a = a_0, \quad b = b_0, \quad c = c_0, \quad d = d_0 \quad \text{and}$$

$$\begin{aligned} a_n &= F(a_{n-1}, b_{n-1}, z_{n-1}, d_{n-1}), \\ b_n &= F(a_{n-1}, d_{n-1}, c_{n-1}, b_{n-1}), \\ c_n &= F(c_{n-1}, b_{n-1}, a_{n-1}, d_{n-1}), \\ d_n &= F(c_{n-1}, d_{n-1}, a_{n-1}, b_{n-1}). \end{aligned} \tag{39}$$

for all n. Since (x, y, z, w) is comparable with (a, b, c, d), we may assume that $(x, y, z, w) \geqslant (a, b, c, d) = (a_0, b_0, c_0, d_0)$. Recursively, we get that

$$(x, y, z, w) \geqslant (a_n, b_n, c_n, d_n) \text{ for all } n. \tag{40}$$

By (40) and (8), we have

$$\begin{aligned} p(x, a_{n+1}) = p(F(x, y, z, w), F(a_n, b_n, c_n, d_n)) \leqslant \\ \leqslant \frac{k}{4}[p(x, a_n) + p(y, b_n) + p(z, c_n) + p(w, d_n)] \end{aligned} \tag{41}$$

$$\begin{aligned} p(b_{n+1}, y) = p(F(a_n, d_n, c_n, b_n), F(x, w, z, y)) \leqslant \\ \leqslant \frac{k}{4}[p(a_n, x) + p(d_n, w) + p(c_n, z) + p(b_n, y)] \end{aligned} \tag{42}$$

$$\begin{aligned} p(z, c_{n+1}) = p(F(z, y, x, w), F(c_n, b_n, a_n, d_n)) \leqslant \\ \leqslant \frac{k}{4}[p(z, c_n) + p(y, b_n) + p(x, a_n) + p(w, d_n)] \end{aligned} \tag{43}$$

$$p(d_{n+1}, w) = p(F(c_n, d_n, a_n, b_n), F(z, w, x, y)) \leqslant$$
$$\leqslant \frac{k}{4}[p(c_n, z) + p(d_n, w) + p(a_n, x) + p(b_n, y)] \quad (44)$$

Set $\gamma_n = p(x, a_n) + p(y, b_n) + p(z, c_n) + p(w, d_n)$. Then, due to (44)–(44), we have

$$\gamma_{n+1} \leqslant k\gamma_n \leqslant k^n \gamma_0, \text{ for all } n. \quad (45)$$

Since $0 \leqslant k < 1$, the sequence $\{\gamma_n\}$ is decreasing and bounded below. Thus, there exists $\gamma \geqslant 0$ such that $\lim_{n\to\infty} \gamma_n = \gamma$. Now, we shall show that $\gamma = 0$. Letting $n \to \infty$ in (45), and having mind $0 \leqslant k < 1$, we obtain that $\gamma \leqslant 0$. Therefore, $\gamma = 0$. That is, $\lim_{n\to\infty} \gamma_n = 0$. Consequently, we have

$$\lim_{n\to\infty} p(x, a_n) = 0, \lim_{n\to\infty} p(y, b_n) = 0, \lim_{n\to\infty} p(z, c_n) = 0, \lim_{n\to\infty} p(w, d_n) = 0. \quad (46)$$

Analogously, we show that

$$\lim_{n\to\infty} p(u, a_n) = 0, \lim_{n\to\infty} p(v, b_n) = 0, \lim_{n\to\infty} p(r, c_n) = 0, \lim_{n\to\infty} p(s, d_n) = 0. \quad (47)$$

Combining (46) and (47) yield that (x, y, z, w) and (u, v, r, t) are equal. □

Example 4. Let $X = [0, 1]$ with the metric $p(x, y) = \max\{x, y\}$, for all $x, y \in X$ and the usual ordering. Let $F : X^4 \to X$ be given by

$$F(x, y, z, w) = \frac{x - y + z - w}{16}, \text{ for all } x, y, z, w \in X.$$

It is easy to check that all the conditions of Theorem 2 are satisfied and $(0, 0, 0, 0)$ is the unique quadruple fixed point of F.

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ON THE EXTREME POINTS OF THE CONVEX HULL OF GRAPH-DIRECTED SELF-SIMILAR SETS

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Key words: convex hull, Hausdorff dimension, self-similar set, extreme point, open set condition

AMS Mathematics Subject Classification: 28A80

Abstract. We define a convex attractor $H(\vec{K})$ for a graph-directed system $\mathcal{S}$ of similarities in $\mathbb{R}^d$. We show that if the system $\mathcal{S}$ acting in $\mathbb{R}^2$ satisfies the open convex set condition, then Hausdorff dimension of the set $\vec{F}$ of extreme points of it's convex attractor $H(\vec{K})$ is zero.

1 Introduction

The interplay between the concepts of self-similarity and convexity is a promising and still unexplored field in the theory of self-similar fractals. This sharply differs from the situation in the theory of Kleinian groups, where the study of convex hulls of the limit sets (which are self-conformal fractals) for a long time serves as one of the main research tools in the theory.

It was shown by the first author in [1] that for self-similar sets in $\mathbb{R}^2$ satisfying the open convex set condition, the set of the extreme points of their convex hull has zero Hausdorff dimension. Below we solve this problem for graph-directed systems of similarities in $\mathbb{R}^2$. The main result of the paper is the following

Theorem 1. *If a graph-directed system $\mathcal{S}$ of similarities in $\mathbb{R}^2$ satisfies OCSC, then Hausdorff dimension of the set $\vec{F}$ of extreme points of the convex attractor $H(\vec{K})$ of the system $\mathcal{S}$ is zero.*

1.1 Graph-directed IFS and it's attractor

Let $G = (V, E)$ be a directed multigraph, where V is a finite set of vertices and E is a finite set of edges. For each $u, v \in V$, $E_{uv} \subset E$ is the set of edges from u to v, so $E = \bigcup\limits_{u,v \in V} E_{uv}$. If $e \in E_{uv}$ then e has *initial* vertex $u = \alpha(e)$ and *final* vertex $v = \omega(e)$. We assume that every node $u \in V$ is the initial vertex for at least one edge. We call $\sigma = e_1 e_2 \ldots e_k$ a *path* in G if for $i = 1, \ldots, k-1$, $\alpha(e_{i+1}) = \omega(e_i)$. We define $\alpha(\sigma) = \alpha(e_1)$ and $\omega(\sigma) = \omega(e_k)$ as initial and final vertex of the path σ. We write $E_{uv}^{(k)}$ for the set of all paths $\sigma = e_1 e_2 \dots e_k$ of length k, with initial vertex u and final vertex v and set $E_{uv}^{(*)} = \bigcup\limits_{k=0}^{\infty} E_{uv}^{(k)}$, so that $E^{(*)} = \bigcup\limits_{u,v \in V} E_{uv}^{(*)}$. We say that $G = (V, E)$ is *strongly connected* if $E_{uv}^{(*)} \neq \emptyset$ for all $u, v \in V$.

Vector-sets. Let V be a finite set. An array $\{A_u, u \in V\}$ of sets A_u, will be called a *vector-set* $\vec{A}$ with components A_u; we call $\vec{A}$ *non-degenerate,* if neither of A_u is empty, and *open*, if all A_u are open sets, and *compact*, if all of A_u are compact sets. We say $\vec{A} \subset \vec{B}$ if $\vec{A} = \{A_u, u \in V\}$, $\vec{B} = \{B_u, u \in V\}$ and $A_u \subset B_u$ for each $u \in V$.

Graph-directed IFS. Let $\vec{X} = \{X_u, u \in V\}$ be a vector-set, with all of it's components being metric spaces $X_u = \mathbb{R}^d$ for certain d. Suppose for each $e \in E_{uv}$ we have a similarity $S_e : X_v \to X_u$, with contraction ratio $q_e : |S_e(x) - S_e(y)| = q_e|x - y|$ and (for the case $d = 2$) rotation angle ϑ_e. Assume $0 < q_e < 1$. Let $q_{\max} = \max\{q_e : e \in E\}$. For $\sigma = e_1 e_2 \dots e_k$ write $S_\sigma = S_{e_1} S_{e_2} \dots S_{e_k}$ and $q_\sigma = q_{e_1} q_{e_2} \dots q_{e_k}$. At the same time, $\vartheta_\sigma = \vartheta_{e_1} + \vartheta_{e_2} \dots + \vartheta_{e_k}$.

We call the family $\mathcal{S} = \{S_e, e \in E\}$ a *graph-directed iterated function system* or IFS. There is an unique non-degenerate compact vector-set $\vec{K} = \{K_u : u \in V\} \subset \vec{X}$ such that $K_u = \bigcup\limits_{v \in V} \bigcup\limits_{e \in E_{uv}} S_e(K_v)$ for all $u \in V$ [2, Theorem (4.3.5)]. The vector-set $\vec{K}$ is called *the invariant set*, or *the attractor*, of the system $\mathcal{S}$. We call the system $\mathcal{S}$ *regular* if the graph G is strongly connected.

1.2 The convex attractor of the system $\mathcal{S}$

Let A be a subset in $\mathbb{R}^d$. We denote by $H(A)$ or by $\tilde{A}$ the convex hull of the set A. Similarly, for a vector-set $\vec{A} \subset \vec{X} = \{X_u = \mathbb{R}^d, u \in V\}$ we define it's *convex hull* $H(\vec{A})$ or $\widetilde{\vec{A}}$ as a vector-set $H(\vec{A}) = \{\widetilde{A_u}, u \in V\}$.

Let $\mathcal{S}$ be a graph-directed system of contraction similarities $S_e : \mathbb{R}^d \to \mathbb{R}^d$ with the structure graph $G = (V, E)$ acting on $\vec{X} = \{X_u = \mathbb{R}^d, u \in V\}$ and let $\vec{K}$ be it's attractor. The convex hull $H(\vec{K}) = \{\widetilde{K_u}, u \in V\}$ of this attractor will be called *the convex attractor* of the system $\mathcal{S}$. It is easily verified that the components of $H(\vec{K})$ satisfy the equation $\widetilde{K_u} = H(\bigcup\limits_{v \in V} \bigcup\limits_{e \in E_{uv}} S_e(\widetilde{K_v}))$ for all $u \in V$.

Throughout this paper the system $\mathcal{S}$ is supposed to be regular, so for each $u \in V$, the dimension of the sets $\widetilde{K_u}$ is the same for all $u \in V$ and is supposed to be equal to d, so each $\widetilde{K_u}$ has nonempty interior $\dot{\widetilde{K_u}}$.

We denote by $\vec{F}$ the vector-set whose components $F_u, u \in V$ are the sets of extreme points of $\widetilde{K_u}$ and call it *the vector-set of extreme points* of $\vec{K}$.

2 Hausforff dimension for the set of extreme points

2.1 Open Convex Set Condition

Definition 1. We say, that a graph-directed system $\mathcal{S}$ satisfies the open convex set condition (OCSC), if there exists a non-degenerate open vector-set $\vec{\mathcal{O}} = \{\mathcal{O}_u, u \in V\}$ with convex components such that:
(i) for any $u, v \in V$, and $e \in E_{uv}$, $S_e(\mathcal{O}_v) \subset \mathcal{O}_u$;
(ii) for any $v_1, v_2 \in V$, $e_1 \in E_{uv_1}, e_2 \in E_{uv_2}$, $S_{e_1}(\mathcal{O}_{v_1}) \cap S_{e_2}(\mathcal{O}_{v_2}) = \emptyset$.

If OCSC is satisfied, we can take for $\vec{\mathcal{O}}$ the vector-set whose components are the interiors $\dot{\tilde{K}}_v$ of the convex hulls of the components K_v of the attractor $\vec{K}$ of the system $\mathcal{S}$:

Proposition 1. Let $U_v = \dot{\widetilde{K}_v}$. Then:
(i) for any $u, v \in V$, and $e \in E_{uv}$, $S_e(U_v) \subset U_u$;
(ii) for any $v1, v2 \in V$, $e_1 \in E_{uv_1}, e_2 \in E_{uv_2}$, $S_{e_1}(U_{v_1}) \cap S_{e_2}(U_{v_2}) = \emptyset$.

All further considerations in the paper will be performed for the systems of similarities in $\mathbb{R}^2$.

2.2 The number of boundary components $Q_{i,s}$

Lemma 1. *Let $U_1, \ldots, U_n$ be a collection of disjoint closed domains in $\mathbb{R}^2$. Let $\widetilde{U} = H(\bigcup_{i=1}^{n} U_i)$. For each $i = 1, \ldots, n$ consider the set Γ_i of such subarcs $\gamma \subset \partial\widetilde{U}$ with endpoints in U_i, that if $j \neq i$, then $\gamma \cap U_j = \emptyset$. Let $Q_{i,s}$ be the maximal subarcs inΓ_i . Let $m_i = \#\{Q_{i,s}\}$.*
Then each m_i is finite, and the sum of all m_i is less or equal to $2n - 2$.

Proof. We may suppose all U_i are simply-connected. If $n = 2$, then $m_1 = m_2 = 1$, so the statement is true. Take $n \geqslant 3$ and suppose that the statement of the Lemma is true for any collection of no more than $n - 1$ sets U_i. If all $m_i = 1$, the statement is true, so consider the case when for some $i, m_i \geqslant 2$. Denote those connected components of the set $\widetilde{U} \setminus U_i$, which are disjoint from any of $Q_{i,s}$, by $N_k, k = 1, \ldots, m_i$. Each of the sets $U_j, j \neq i$ is contained in the closure of one of the sets N_k; let n_k be the number of those $U_j, j \neq i$ which are contained in N_k. For each k, $1 \leqslant n_k \leqslant n - 2$. Therefore $m_i \leqslant n - 1$. Since $n_k + 1 \leqslant n - 1$, we apply the Lemma's statement to the subfamily $\{U_i\} \cup \{U_j | U_j \subset \bar{N}_k\}$ and to it's convex hull $\bar{N}_k \cup U_i$. As a result we have $\sum\limits_{U_j \subset \bar{N}_k} m_j + 1 \leqslant 2n_k$. Taking sum over all N_k, we get $m = \sum\limits_{j \neq i} m_j + m_i \leqslant 2 \sum n_k = 2(n - 1)$. $\square$

2.3 Boundary components for convex subcopies

Let $\mathcal{S}$ be a regular graph-directed system of similarities in $\mathbb{R}^2$ satisfying OCSC. Let $\widetilde{K}_v$ be the components of the convex attractor of the system $\mathcal{S}$. By $\tilde{K}_e$ we denote the subsets $S_e(\tilde{K}_{\omega(e)})$ of the sets $\tilde{K}_{\alpha(e)}$, and by $\tilde{K}_\sigma$ we denote the subsets $S_\sigma(\tilde{K}_{\omega(\sigma)})$ of the set $\tilde{K}_{\alpha(\sigma)}$. Since the sets $U_v = \dot{\tilde{K}}_v$ satisfy (i), (ii) in OCSC, acting the same way as in Lemma 1, we define the sets $Q_e = \tilde{K}_e \cap \partial\tilde{K}_{\alpha(e)}$, and denote their connected components by $Q_{e,s}$. Similarly, for a path σ in G we put $Q_\sigma = \tilde{K}_\sigma \cap \partial\tilde{K}_{\alpha(\sigma)}$, and denote connected components of the set Q_σ by $Q_{\sigma,s}$.

Proposition 2. 1) Each set $\partial\tilde{K}_u, u \in V$ contains a finite number m_e of subsets $Q_{e,s}$, the sum $\sum\limits_{e\in E_u} m_e$ being less or equal to $2n_u - 2$, where n_u is a number of elements of E_u.

2) If $n_u > 2$, then for any $e_1 \neq e_2$, $\#(Q_{e_1,s} \bigcap Q_{e_2,t}) \leqslant 1$.

3) For each two sets $Q_{\sigma,s}, Q_{\sigma',s'}$ either or one of these sets is a subset of the other whereas one of the two paths σ, σ' is the initial subpath of the other empty or one-point, or $\#(Q_{\sigma,s} \bigcap Q_{\sigma',s'}) \leqslant 1$.

Proof. Applying the argument of Lemma 1 to the sets $\tilde{K}_e, e \in E_u$ (instead of the sets U_i), we get that $\tilde{U} = \tilde{K}_u$, and connected components $Q_{e,s}$ of the sets $Q_e = \tilde{K}_e \cap \partial\tilde{K}_u$ are the maximal subarcs of $\partial\tilde{K}_u$ having their endpoints in $\tilde{K}_e$, whose interior does not intersect any of the sets $\tilde{K}_{e'}, e' \neq e$.

Indeed, if the interior of $Q_{e,s}$ has nonempty intersection with some $\tilde{K}_{e'}, e' \neq e$, then $\dot{\tilde{K}}_e \cap \dot{\tilde{K}}_{e'} \neq \emptyset$, and this contradicts OCSC.

Let $Q_{e,s}$, $Q_{e,t}$ be two connected components of the set $Q_e = \tilde{K}_e \cap \partial\tilde{K}_{\alpha(e)}$, and let γ be a subarc in $\partial\tilde{K}_u$, with one endpoint lying in $Q_{e,s}$, and the other in $Q_{e,t}$, whose interior is disjoint from Q_e. If γ is a straight line segment, then $\gamma \subset \tilde{K}_e$, which is impossible, because in this case the set $Q_{e,s} \cup \gamma \cup Q_{e,t}$ is connected and is contained in the set $Q_e = \tilde{K}_e \cap \partial\tilde{K}_{\alpha(e)}$. Therefore $\gamma \cap Q_{e'} \neq \emptyset$ for some $e' \neq e$, which ensures the maximality of the subarcs $Q_{e,s}$.

Therefore the statement 1) directly follows from Lemma 1. The statement 2) is obvious.

Since the inclusions $Q_{e_1} \supset Q_{e_1e_2} \ldots \supset Q_{e_1e_2\ldots e_p} \supset \ldots$ imply that the set $Q_{e_1e_2\ldots e_p} \cap Q_{e'_1e'_2\ldots e'_q}$ is contained in each of the sets $Q_{e_1e_2\ldots e_{p'}} \cap Q_{e'_1e'_2\ldots e'_{q'}}$ with $p > p', q > q'$, the statement 3) follows from 2). $\square$

2.4 The estimates for the number of boundary components

We will call two graph-directed systems $\mathcal{S}$ and $\mathcal{S}'$ with structure graphs $G = (V, E)$ and $G' = (V', E')$ *convex equivalent*, if $V = V'$ and the convex hulls $H(\vec{K})$ and $H(\vec{K}')$ of their invariant sets are the same.

Proposition 3. Let $\mathcal{S}$ be a regular graph directed system in $\mathbb{R}^2$ with the structure graph $\Gamma = (V, E)$, satisfying OCSC. There is such a system $\mathcal{S}'$ convex equivalent to $\mathcal{S}$ and satisfying OCSC , with a structure graph $\Gamma' = (V, E')$, that

$$\text{for each } \sigma \in {E'}^{(\omega)},\ Q_\sigma \text{ is connected,} \tag{1}$$

$$\text{for each } e \in E',\ \widetilde{K}_e \cap F_{\alpha(e)} \neq \emptyset, \tag{2}$$

$$\text{for each } e_1 \neq e_2, Q_{e_1} \not\subset Q_{e_2}. \tag{3}$$

Proof. Suppose for some $e \in E$, the set Q_e is not connected, and let $Q_{e,1} \dots Q_{e,s}$ be it's components. Denote the minimal distance between points belonging to different components of Q_e by δ_e. Let δ be the minimum of all such δ_e, $e \in E$.

For each component $Q_{e,k}$ denote by $N_{e,k}$ the set of all outer normal vectors at points $x \in Q_{e,k}$. For each line segment $l \subset \partial \widetilde{K}_u$, with both endpoints belonging to Q_e, it is contained in Q_e, therefore l lies in some component $Q_{e,k}$. Therefore the sets $N_{e,k}$ are disjoint closed subarcs of the unit circle C. Their complement $C \setminus \bigcup\limits_k N_{e,k}$ is a collection of s open circular arcs; let λ_e be the length of the smallest of those arcs and λ be the minimum of λ_e, $e \in E$.

By Lemma II in [5] there is such an integer p, that for each path $\sigma \in E^{(p)}$,

$$q_\sigma \leqslant \frac{\delta}{\max\{\operatorname{diam}(K_u), u \in V\}}.$$

For each path $\sigma = e_1 \dots e_p$ the set $S_\sigma(\widetilde{K}_{\omega(\sigma)})$ has the diameter smaller than δ, therefore the intersection $S_\sigma(\widetilde{K}_{\omega(\sigma)}) \cap \partial \widetilde{K}_{\alpha(\sigma)} = Q_\sigma$ is contained in one of the components of the set $S_{e_1}(\widetilde{K}_{\omega(e_1)}) \cap \partial \widetilde{K}_{\alpha(\sigma)} = Q_{e_1}$. Also, for each such path $\sigma_1 = \tau \cdot \sigma$ that $\sigma \in E^{(p)}$ the set $Q_{\sigma_1} = S_{\sigma_1}(\widetilde{K}_{\omega(\sigma_1)}) \cap \partial \widetilde{K}_{\alpha(\sigma_1)}$ is contained in one of the components of the set $Q_\tau = S_\tau(\widetilde{K}_{\omega(\tau)}) \cap \partial \widetilde{K}_{\alpha(\sigma_1)}$.

Consider the path $\sigma_1 = \tau \cdot \sigma$, $\sigma = e_1 \dots e_p$, $\tau = e'_1 \dots e'_m$. Suppose the set Q_{σ_1} is not connected. Let $Q_1^{(1)}$ and $Q_2^{(1)}$ be two it's adjacent components and $\xi^{(1)}$, $\eta^{(1)}$ are the endpoints of the interval $\Delta^{(1)}$ in the set $\partial \widetilde{K}_{\alpha(\sigma_1)}$, separating $Q_1^{(1)}$ from $Q_2^{(1)}$.

Each of the sets $S_{e'_k\dots e'_m e_1\dots e_p}(\widetilde{K}_{\omega(\sigma)}) \cap \partial\widetilde{K}_{\alpha(e'_k)}$ is also non-connected, and contains two such components $Q_1^{(k)}$, $Q_2^{(k)}$ that $S_{e'_1\dots e'_{k-1}}(Q^{(k)}) \cap \partial\widetilde{K}_{\alpha(\sigma_1)} = Q_i^{(1)}$. In this case the points $\xi^{(k)} = S^{-1}_{e'_1\dots e'_{k-1}}(\xi)$, $\eta^{(k)} = S^{-1}_{e'_1\dots e'_{k-1}}(\eta)$ are the endpoints of the interval $\Delta^{(k)}$, separating $Q_1^{(k)}$ from $Q_2^{(k)}$. Consider the sets of normal vectors $N(\Delta^{(k)})$. They are the equal open arcs. They cannot coincide when $k \neq l$, because all the distances $d(\xi^{(k)}, \eta^{(k)})$ are different. Due to OCSC their intersection is empty.

Therefore the number of such arcs is not larger than $\dfrac{2\pi}{\lambda} \cdot \#V$.

So, if $p_1 > \dfrac{2\pi}{\lambda} \cdot \#V + p$, then each of the sets $Q_{e_1\dots e_{p_1}} = S_{e_1\dots e_{p_1}}(\widetilde{K}_{\omega(e_{p_1})}) \cap \partial\widetilde{K}_{\alpha(e_1)}$ is connected.

To obtain the new system $\mathcal{S}'$ we take the set of all such $S_{e_1\dots e_{p_1}}$ that the set $Q_{e_1\dots e_{p_1}}$ is non-empty. The system $\mathcal{S}'$ satisfies the OCSC. So the condition (3) is proved.

To make sure that (1) (2) are fulfilled too, delete those similarities $S_{e'}$, which violate these conditions; the resulting system $\mathcal{S}$", is convex equivalent to $\mathcal{S}$. The reader may verify that the resulting system is also regular. □

A system $\mathcal{S}$, satisfying (1), (2), (3) and OCSC, is called a *proper* one.

Lemma 2. *If $\mathcal{S}$ is a proper system then for each natural p and each $u \in V$, the number of order p components $Q_{e_1\dots e_p} \subset \partial\widetilde{K_u}$, having nonempty interior in $\partial\widetilde{K}_u$, is less or equal to p^{n+1}, where n is the number $\#E$ of the edges of the graph Γ.*

Proof. For $p = 1$ the statement is obvious.

Suppose it holds for the components of order $p - 1$.

If the component $Q_{e_1\dots e_{p-1}}$ contains a component $Q_{e_1\dots e_{p-1}f}$ of order p which is not equal to $Q_{e_1\dots e_{p-1}}$, then one of the endpoints ξ_f, η_f of the component Q_f either $S_{e_1\dots e_{p-1}}(\xi_f)$, or $S_{e_1\dots e_{p-1}}(\eta_f)$ lies in the interior of the arc $Q_{e_1\dots e_{p-1}}$.

This allows us to evaluate the number of those components $Q_{e_1\dots e_p}$ of order p, whose interior is non-empty.

For each of these components consider a set of unit normal vectors $\dot{N}_{e_1\dots e_p} = \bigcup\limits_{x \in \dot{Q}_{e_1\dots e_p}} N_x$, where $\dot{Q}_{e_1\dots e_p}$ is the interior of the component $Q_{e_1\dots e_p}$ in $\partial\widetilde{K}_{\alpha(e_1)}$. If $\dot{Q}_{e_1\dots e_p}$ is non-empty, then $\dot{N}_{e_1\dots e_p}$ — is an open subarc of the unit circle, so that $\dot{N}_{e_1} \supset \dot{N}_{e_1 e_2} \supset \dots \supset \dot{N}_{e_1\dots e_p} \supset \dots$ and $\dot{N}_{e_1\dots e_p} = S_{e_1\dots e_p}(\dot{N}_{e_p}) \cap \dot{N}_{e_1\dots e_{p-1}}$.

The set $\dot{N}_{e_1\dots e_p}$ is different from $\dot{N}_{e_1\dots e_{p-1}}$ if at least one of the endpoints of the arc $\dot{N}_{e_1\dots e_p}$ lies in the interior of the arc $\dot{N}_{e_1\dots e_{p-1}}$.

Let $\beta^-_{e_p}$ and $\beta^+_{e_p}$ be the endpoints of the arc $\dot{N}_{e_p}$. Then one of the following inequalities is true:

$$\beta^-_{e_1\dots e_p} < \vartheta_{e_1} + \dots + \vartheta_{e_{p-1}} + \beta^-_{e_p} < \beta^+_{e_1\dots e_p}$$
$$\beta^-_{e_1\dots e_p} < \vartheta_{e_1} + \dots + \vartheta_{e_{p-1}} + \beta^+_{e_p} < \beta^+_{e_1\dots e_p}$$

Observe now, that first, the sets $\dot{N}_{e_1\dots e_{p-1}}$ are disjoint for each fixed $u = \alpha(e_1)$ and second, that the value of the sum $\vartheta_{e_1} + \dots + \vartheta_{e_{p-1}}$ remains the same for all permutations of the symbols $e_1 \dots e_{p-1}$. Therefore for each $u \in V$ denote by $\dot{N}^u_\gamma$ the union of all those sets $\dot{N}_{e_1\dots e_{p-1}}$, for which $u = \alpha(e_1)$ and $\vartheta_{e_1} + \dots + \vartheta_{e_{p-1}} = \gamma$; and denote by P^u_γ the union of all corresponding open components $\dot{Q}_{e_1\dots e_{p-1}}$, and let p^u_γ be the number of these components.

Then, if $\dot{Q}_{e_1\dots e_p} \subset P^u_\gamma$ and if $\dot{Q}_{e_1\dots e_p}$ is not equal to $\dot{Q}_{e_1\dots e_{p-1}}$, then at least one of the conditions $\gamma + \beta^-_{e_p} \in \dot{N}^u_\gamma$ and $\gamma + \beta^+_{e_p} \in \dot{N}^u_\gamma$ is satisfied.

When one passes from $p-1$ to p, the number of components increases, if some component $\dot{Q}_{e_1\dots e_{p-1}}$ contains at least two components $Q_{e_1\dots e_p}$, $Q_{e_1\dots e_{p-1}e'_p}$ of the order p and this means that the following system of inequalities hold:

$$\begin{cases} \beta^-_{e_1\dots e_{p-1}} < \vartheta_{e_1} + \dots + \vartheta_{e_{p-1}} + \beta^+_{e_p} < \beta^+_{e_1\dots e_{p-1}} \\ \beta^-_{e_1\dots e_{p-1}} < \vartheta_{e_1} + \dots + \vartheta_{e_{p-1}} + \beta^-_{e'_p} < \beta^+_{e_1\dots e_{p-1}} \end{cases}$$

So if m is a total number of those angles β^+_j and β^-_j, for which $\gamma + \beta^-_j \in \dot{N}^u_\gamma$ or $\gamma + \beta^+_j \in \dot{N}^u_\gamma$, then the number of the components of the order p contained in P^u_γ is not larger than $p^u_\gamma + m$. The number m does not exceed the number $n = \#E$ of similarities generating the system $\mathcal{S}$.

The number of different sets P^u_γ is not larger than the number of different sums composed of $p-1$ summands, chosen from the set $\{\vartheta_e, e \in E\}$. This number does not exceed the number of different monomials in the expansion $(x_1 + \dots + x_n)^{p-1}$, which is equal to $\dfrac{(n+p-2)!}{(p-1)!(n-1)!}$.

Then, while passing from $(p-1)$ to p, for each $u \in V$ there arises no more than $n \cdot \dfrac{(n+p-2)!}{(p-1)!(n-1)!}$ new components. Therefore the total number of the components of the order p does not exceed

$$n\left(1+\frac{n!}{1!(n-1)!}+\frac{(n+1)!}{2!(n-1)!}+\ldots+\frac{(n+p-2)!}{(p-1)!(n-1)!}\right)=\frac{(n+p-1)!}{(p-1)!(n-1)!},$$

and this number is in it's turn smaller than p^{n+1}.

2.5 The proof of the main theorem

Proof of the Theorem 1. We may suppose that the system $\mathcal{S}$ is a proper one. Since the set $\vec{F}_0$ of isolated extreme points of $\partial\widetilde{K_u}$ is at most countable, and therefore has zero Hausdorff dimension, it suffices to find the dimension of it's complement $\vec{F}' = \vec{F} \setminus \vec{F}_0$. For any p, each set F'_u may be covered by components $Q_{e_1 \ldots e_p}$, having non-empty interior in $\partial\widetilde{K_u}$. The total number of these components does not exceed p^{n+1}, while the diameter of each of them is smaller than $q^p_{\max} \cdot \max\{\mathrm{diam}(\widetilde{K_u}), u \in V\}$. Since for each $u \in V$ the box dimension [4, p.41] of the set F'_u is equal to $\dim_B F'_u = \lim\limits_{p\to\infty}\left(-\dfrac{\ln p^{n+1}}{\ln q^p_{\max}}\right) = 0$, Hausdorff dimension of the set $\vec{F}'$ is zero. Therefore Hausdorff dimension of the set $\vec{F}$ also is equal to zero. □

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TRIPLED COMMON FIXED POINT THEOREM IN FUZZY METRIC SPACE

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Key words: fuzzy metric space; Tripled coincidence point; E.A. property

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Abstract. In the present study, we introduce the notion of E. A. property for trivariate mapping F and self mapping g in fuzzy metric spaces. Also utilize these perceptions to prove a tripled coincidence point theorem for such nonlinear contractive mappings in both sense GV-fuzzy metric spaces and KM-fuzzy metric spaces. The efforts of this work extend, unify and generalize all the result of Mihet. We illustrate an example to support our results.

1 Introduction

Metric space is an important notion in analysis and the Banach contraction principle is the root of fruitful tree of fixed point theory. The custom of improving contraction conditions in the fixed point theorem is still in fashion. Many studies have been done on contractive mappings, e.g., Rhoades [20] made a comparison of various definition (more than 100 types varied from 25 basic types) of contractive mappings on complete metric space in 1977. And up to now, such study is still going on, proceeding the same tradition, the concept of coupled fixed point was initially introduced by Chang and Ma [3]. Since then, the concept has been of interest to many researchers in metrical fixed point theory.

Specifically, Bhaskar and Lakshmikanthan [3] established coupled fixed point for mixed monotone operator in partially ordered metric spaces. Afterward, Ciric and Lakshmikanthan [7] extended the results of [3] by furnishing coupled coincidence and coupled fixed point theorem for two commuting mappings. In a subsequent series, B. S. Choudhary and A. Kundu [6] introduced the concept of compatibility and proved the result of [3]. under different set of condition. Very recently, Borcut and Berinde [4] introduce tripled fixed point theorem for contractive type mapping in partially ordered metric spaces.

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The concept of fuzzy metric space was introduced by different authors (see [8, 12]) in different ways and further using these different concepts various authors (([1, 7–12, 20–25]) proved theorem which assures the existence of fixed point. Here, we use notion of fuzzy metric space established by Kramosil and Michalek [15]. was modified by George and Veeramani [9].

Aamri and ElMoutawakil [2] defined a property (E.A) which generalizes the concept of non-compatible mappings and gave some common fixed point theorems under strict contractive conditions. Recently, Mihet [18] enlarged the concept in setting of fuzzy metric space. Motivated by Bhaskar and Lakshmikanthan [3] and Borcut and Berinde [4], the purpose of present study is to investigate tripled coincidence point theorem for mappings that possess monotonicity type properties, in the context of GV-fuzzy metric space and KM-Fuzzy metric spaces which combine method of contraction principle with method of monotone iterations.

2 Preliminaries

In what follows, we collect some relevant definitions, results, examples for our further use.

Definition 1. A fuzzy set A in X is a function with domain X and values in $[0,1]$.

Definition 2. A continuous t-norm (in sense of Schweizer and Sklar [21]) is a binary operation T on $[0,1]$ satisfying the following conditions:

— T is a commutative and associative;,
— $T(a,1)=a$ for all $a \in [0,1]$;
— and $T(a,b)=T(c,d)$ whenever $a=c$ and $b=d (a,b,c,d \in [0,1])$;
— The mapping $T:[0,1]\times[0,1]\to[0,1]$ is continuous.

Remark 1. The following are classical example of continuous t-norm

— $T_M(a,b)=\min\{a,b\}$, minimum t-norm.,
— $T_H(a,b)=\begin{cases} 0 & \text{if} \quad a=b=0, \\ \dfrac{ab}{a+b+ab} & \text{otherwise}, \end{cases}$ Hamacher product.
— and $T_N(a,b)=\begin{cases} \min\{a,b\} & \text{if} \quad a+b>1, \\ 0 & \text{otherwise}, \end{cases}$ Nilpotent minimum.
— $T_L(a,b)=\max\{a+b-1,0\}$, Lukasiewict t-norm

— $T_D(a,b) = \begin{cases} b & \texttt{if} \quad a = 1, \\ a & \texttt{if} \quad b = 1, \\ 0 & \texttt{otherwise}, \end{cases}$ `Drastic` t`-norm.`

The minimum t-norm is point wise largest t-norm and the drastic t-norm is point wise smallest t-norm; that is, $T_M(a,b) = T(a,b) = T_D(a,b)$ for any t-norm t with $a, b \in [0,1]$.

Kramosil and Michalek in [16] generalized the concept of probabilistic metric space given by Menger [19] to the fuzzy framework as follows.

Definition 3. A fuzzy metric space (in sense of Kramosil and Michalek [15]) is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ such that the following axioms holds:

- `[(FM-1)]` $M(x,y,0) = 0 \ (x, y \in X)$;
- `[(FM-2)]` $M(x,y,t) = 1$ `for all` $t > 0$ `iff` $x = y$;
- `[(FM-3)]` $M(x,y,t) = M(y,x,t) \ (x, y \in X, t > 0)$;
- `[(FM-4)]` $M(x,y,\cdot) : [0,\infty) \to [0,1]$ `is left continuous for all` $x, y \in X$;
- `[(FM-5)]` $M(x,z,t+s) \geqslant M(x,y,t) * M(y,z,s)$ `for all` $x, y, z \in X$ `and` $s, t > 0$.

We will refer to these spaces as KM-fuzzy metric spaces.

Lemma 1 (see [8]). *For every $x, y \in X$, the mapping $M(x, y, \cdot)$ is nondecreasing on $(0, \infty)$.*

In order to introduce an Hausdorff topology on the fuzzy metric spaces, George and Veeramani [9] modified in a slight but appealing way the notion of fuzzy metric spaces of Kramosil and Michalek [16].

Definition 4. A fuzzy metric space (in sense of George and Veeramani [9]) is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ such that the following axioms holds:

- `[(GV-1)]` $M(x,y,t) > 0 (x, y \in X)$;
- `[(GV-2)]` $M(x,y,t) = 1$ `for all` $t > 0$ `iff` $x = y$;
- `[(GV-3)]` $M(x,y,t) = M(y,x,t) \ (x, y \in X, t > 0)$;
- `[(GV-4)]` $M(x,y,\cdot) : (0,\infty) \to (0,1]$ `is continuous for all` $x, y \in X$;
- `[(GV-5)]` $M(x,z,t+s) \geqslant M(x,y,t) * M(y,z,s)$ `for all` $x, y, z \in X$ `and` $s, t > 0$.

Notice that condition (GV-5) is a fuzzy version of triangular inequality. The value $M(x, y, t)$ can be thought of as degree of nearness between x and y with respect to t and from axiom (GV-2) we can relate the value 0 and 1 of a fuzzy metric to the notions of ∞ and 0 of classical metric respectively.

We will refer to these spaces as GV-fuzzy metric spaces.

Definition 5 (see [9]). A fuzzy metric M on X is said to be stationary if M does not depend on t, i.e., the function $M_{x,y}(t) = M(x, y, t)$ is constant.

Definition 6 (see [21]). If $(X, M, *)$ is a KM-fuzzy metric space and $\{x_n\}$, $\{y_n\}$ are sequences in X such that $x_n \to x$, $y_n \to y$, then $M(x_n, y_n, t) \to M(x, y, t)$ for every continuity point t of $M(x, y, \cdot)$. We can fuzzify example of metric space into fuzzy metric spaces in a normal way.

Example 1 (see [10]). Let (X, d) be metric space and $g : R^+ \to R^+$ is an increasing continuous function. For $m > 0$, we define the function M by

$$M(x, y, t) = \frac{g(t)}{gt + m \cdot d(x, y)} \tag{1}$$

Then for $a * b = a \cdot b, (X, M, *)$ is a GV-fuzzy metric space on X.

As a particular case if we take $g(t) = t^n$ where $n \in N$ and $m = 1$. Then (1) becomes

$$M(x, y, t) = \frac{t^n}{t^n + d(x, y)} \tag{2}$$

Then for $a * b = T_M(a, b), (X, M, *)$ is a GV-fuzzy metric space on X.

If we take $n = 1$ in (1), the well-known fuzzy metric space is obtained.

On the other hand, if we take g as a constant function in (1) i.e., $g(t) = k > 0$ and $m = 1$, we obtain

$$M(x, y, t) = \frac{k}{k + d(x, y)}$$

And so $(X, M, *)$ is a stationary GV-fuzzy metric space for $a * b = a \cdot b$ but, in general, (X, M, T_M) is not.

Definition 7. An element $(x, y, z) \in X \times X \times X$ is called tripled coincidence point of $F : X \times X \times X \to X$ and $g : X \to X$ if

$$F(x, y, z) = g(x),\ F(y, x, y) = g(y) textand\ F(z, y, x) = g(z).$$

Now, we introduce the notion of E.A. Property for trivariate mapping F and self mapping g in fuzzy metric spaces.

Definition 8. The mappings F and g where $F : X \times X \times X \to X$ and $g : X \to X$, of a fuzzy metric space $(X, M, *)$ satisfy E.A. property, if there exist sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X, such that $\lim_{n\to\infty} F(x_n, y_n, z_n) = \lim_{n\to\infty} g(x_n) = g(u)$, $\lim_{n\to\infty} F(y_n, x_n, y_n) = \lim_{n\to\infty} g(y_n) = g(v)$ and $\lim_{n\to\infty} F(z_n, y_n, x_n) = \lim_{n\to\infty} g(z_n) = g(w)$ for $u, v, w \in X$ and $t > 0$.

Let the class Φ of all mappings $\varphi : [0, 1] \to [0, 1]$ satisfying the following properties:

— `[`(φ_1)`]` φ `is continuous and nondecreasing on` $[0, 1]$;
— `[`(φ_2)`]` $\varphi(x) > x$ `for all` $x \in (0, 1)$.

We note that if $\varphi \in \Phi$, then $\varphi(1) = 1$ and that $\varphi(x) = x$ for all $x \in [0, 1]$.

3 Tripled coincidence point in KM-fuzzy metric spaces.

In this section, we prove tripled coincidence point theorem for mapping satisfying E. A. property for φ-contraction. E. A property buys containment of ranges without any continuity requirements. Moreover, E. A. property allows replacing the completeness requirement of the space with more natural condition of closeness of ranges.

Theorem 1. *Let $(X, M, *)$ be a KM-fuzzy metric space and and the mappings $F : X \times X \times X \to X$, $g : X \to X$, such that, for some $\varphi \in \Phi$ and $x, y, z, u, v, w \in X$, $t > 0$,*

$$M(F(x, y, z), F(u, v, w), t) \geqslant \varphi(\min\{M(g(x), g(u), t), M(F(x, y, z), g(x), t), M(F(u, v, w), g(u), t), M(F(y, x, y), g(v), t), M(F(z, y, x), g(w), t)\}). \quad (3)$$

If F and g satisfy E.A. property and range of g is a closed subspace of X, then F and g have a tripled coincidence point in X.

Proof. Suppose F and g satisfy E.A. property, so we can find sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X and the point u, v, w in X such that

$$\lim_{n\to\infty} F(x_n, y_n, z_n) = \lim_{n\to\infty} g(x_n) = g(u), \quad \lim_{n\to\infty} F(y_n, x_n, y_n) = \lim_{n\to\infty} g(y_n) = g(v)$$

and $\lim_{n\to\infty} F(z_n, y_n, x_n) = \lim_{n\to\infty} g(z_n) = g(w)$. Let t be continuity point of $M(gu, F(u, v, w), \cdot)$. Then by using (3), we have

$$M(F(x_n, y_n, z_n), F(u, v, w), t) \geqslant \\ \geqslant \varphi(\min\{M(g(x_n), g(u), t), M(F(x_n, y_n, z_n), g(x_n), t), \\ M(F(u, v, w), g(u), t), M(F(y_n, x_n, y_n), g(v), t), M(F(z_n, y_n, x_n), g(w), t)\}).$$

By applying limit $n \to \infty$ and using Lemma 1, we obtain the inequality

$$M(g(u), F(u, v, w), t) \geqslant \varphi(\min\{M(g(u), g(u), t), M(g(u), g(u), t), \\ M(F(u, v, w), g(u), t), M(g(v), g(v), t), M(g(w), g(w), t)\}) = \\ = \varphi(\min\{1, 1, M(F(u, v, w), g(u), t), 1, 1\} = \varphi(M(F(u, v, w), g(u), t))$$

Now, if $F(u, v, w) \neq g(u)$, then $0 < M(F(u, v, w), g(u), t_0) < 1$ for some $t_0 > 0$. As $M(gu, F(u, v, w), \cdot)$ is left continuous and the $M(gu, F(u, v, w), \cdot)$ is nondecreasing, that it has only a most countable point of discontinuity, we may suppose that t_0 is a continuity point of $M(gu, F(u, v, w), \cdot)$. Then, from condition φ_2 we obtain $\varphi(M(g(u), F(u, v, w), t_0)) > M(g(u), F(u, v, w), t_0)$, which is a contradiction to the inequality (4), which implies that $g(u) = F(u, v, w)$. Similarly, it can be proved that $F(v, u, v) = g(v)$ and $F(w, v, u) = g(w)$. Hence F and g have tripled coincidence point. □

Corollary 1. Let $(X, M, *)$ be a KM-fuzzy metric space and the mappings $F : X \times X \times X \to X$, $g : X \to X$, such that, for some $\varphi \in \Phi$ and $x, y, z, u, v, w \in X$, $t > 0$,

$$M(F(x, y, z), F(u, v, w), t) \geqslant \varphi(\min\{M(g(x), g(u), t)\}).$$

If F and g satisfy E.A. property and range of g is a closed subspace of X, then F and g have tripled coincidence point.

4 Tripled coincidence point in GV-fuzzy metric spaces.

If we suppose $(X, M, *)$ is a GV-fuzzy metric, then some of hypothesis in the preceding theorem can be relaxed.

Theorem 2. *Let $(X, M, *)$ be a GV-fuzzy metric space and the mappings $F : X \times X \times X \to X$, $g : X \to X$, such that, for some $\varphi \in \Phi$ and $x, y, z, u, v, w \in X$, $t > 0$,*

$$M(F(x,y,z),F(u,v,w),t) \geqslant \varphi(\min\{M(g(x),g(u),t),M(F(x,y,z),g(x),t),\\ M(F(u,v,w),g(u),t),M(F(y,x,y),g(v),t),M(F(z,y,x),g(w),t)\}). \quad (4)$$

If F and g satisfy E.A. property and range of g is a closed subspace of X, then F and g have tripled coincidence point.

Proof. Since F and g satisfy E.A. property, then there exist sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X and the point u, v, w in X such that

$$\lim_{n\to\infty} F(x_n,y_n,z_n) = \lim_{n\to\infty} g(x_n) = g(u),\ \lim_{n\to\infty} F(y_n,x_n,y_n) = \lim_{n\to\infty} g(y_n) = g(v)$$

and $\lim_{n\to\infty} F(z_n,y_n,x_n) = \lim_{n\to\infty} g(z_n) = g(w)$. Then, by using (3), we have

$$M(F(x_n,y_n,z_n),F(u,v,w),t) \geqslant \\ \geqslant \varphi(\min\{M(g(x_n),g(u),t),M(F(x_n,y_n,z_n),g(u),t),\\ M(F(u,v,w),g(x_n),t),M(F(y_n,x_n,y_n),g(v),t),M(F(z_n,y_n,x_n),g(w),t)\}) \quad (5)$$

Taking the limit as $n \to \infty$, ones obtain the inequality

$$M(g(u),F(u,v,w),t) \geqslant \varphi(\min\{M(g(u),g(u),t),M(g(u),g(u),t),\\ M(F(u,v,w),g(u),t),M(g(v),g(v),t),M(g(w),g(w),t)\}) \geqslant \\ \geqslant \varphi(\min\{1,1,M(F(u,v,w),g(u),t),1,1\}) = \varphi(M(F(u,v,w),g(u),t)).$$

Now, if $F(u,v,w) \neq g(u)$, then $0 < M(F(u,v,w),g(u),t) < 1$, that is,

$$\varphi(M(F(u,v,w),g(u),t)) > M(F(u,v,w),g(u),t),$$

contradicting the above inequality. This proves that $M(g(u),F(u,v,w),t) = 1$, which implies due to (GV-2), $F(u,v,w) = g(u)$. Similarly, it can be proved that $F(v,u,v) = g(v)$ and $F(w,v,u) = g(w)$. Hence F and g have tripled coincidence point. □

Remark 2. The results of [14] are deduced from the results discussed here, by choosing $f(x) = F(x,y,z)$, $f(y) = F(y,x,y)$, $f(z) = F(z,y,x)$ and setting $u = y$ and $v = x$.

Example 2. Consider the space $(X, M, *)$, where $X = [0,1]$ and $a * b = ab$. Let

$$M(x,y,t) = \frac{t}{t + |x-y|} \text{ for } x, y \in X \text{ and } t > 0.$$

Let the mapping $g : X \to X$ be defined as $g(x) = x^2$ for all $x, y \in X$. Let $F : X \times X \times X \to X$ be defined as

$$F(x,y,z) = \begin{cases} (x-y-z)^2 & \text{if } x,y \in X \text{ and } x \neq y \neq z \\ 0 & \text{if } x = y = z \end{cases}$$

It is obvious that F and g obeys E.A. property and if $\varphi : [0,1] \to [0,1]$, $\varphi = \sqrt[3]{t}$, then it is easy to satisfy all the conditions of preceding theorem (5). The tripled coincidence point of F and g is $(0,0,0)$.

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II.5. Generalized Functions and Asymptotics

(Sessions organizers: M. Oberguggenberger, S. Pilipovic)

PARADIGMATIC WELL-POSEDNESS IN SOME GENERALIZED CHARACTERISTIC CAUCHY PROBLEMS

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Key words: regularization of data, regularization of problems, distributions, nonlinear generalized functions, nonlinear problems, characteristic problems, transport equation

AMS Mathematics Subject Classification: 35A01, 35A02, 35A25, 35D99, 46F30, 46T30

Abstract. By means of convenient regularizations for an ill-posed Cauchy problem, we define an associated generalized problem and discuss the conditions for the solvability of it. To illustrate this, starting from the semilinear unidirectional wave equation with data given on a characteristic curve, we show existence and uniqueness of the solution.

1 Introduction

Many obstructions can be encountered when trying to solve a Cauchy problem for PDEs with the data given on a characteristic manifold, and, a fortiori, to obtain uniqueness or well-posedness in Hadamard sense. We can refer to many works inspired in the complex field by the ideas of Gårding, Kotake, Leray [9] and others on the continuation of holomorphic solutions and, in the real field, by the ideas of Egorov [8], Hörmander [11] and others on the distribution solutions of some Cauchy problems supported in a half space whose boundary is a characteristic hyperplane.

Here, we propose another method, based on a parametrized family of geometric transformations of the characteristic manifold, in continuation of previous ideas developed in [4–7, 12]. In order to concentrate on the methods and not on the technicalities, we consider the Cauchy problem for a simple equation, namely the transport equation (in basic form) (P_c) $\partial u/\partial t = F(.,.,u), u\mid_{\gamma}= v$ where γ of equation $x = 0$ is obviously globally characteristic for the Cauchy problem.

In order to focus only on the characteristic aspect, v and F are supposed to be smooth. Moreover F has to verify some estimates involving derivatives. Clearly (P_c) is ill-posed, but can be associated to a generalized problem $P(D)u = \mathcal{F}(u)$, $\mathcal{R}(u) = v$ well formulated in convenient algebras of nonlinear generalized functions, by means of generalized operators: $\mathcal{F}$, associated to F, and $\mathcal{R}$, obtained by replacing the characteristic curve γ by a family $(\gamma_\varepsilon)_\varepsilon$ of non-characteristic ones of equation $x = l_\varepsilon(t)$ where $(l_\varepsilon)_\varepsilon$ is a regularizing family. We can show the existence of a generalized solution in some $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebra [12] $\mathcal{A}\left(\mathbb{R}^2\right)$ based on the space of smooth functions. It is also proved that this solution does not depend on the representative of the "tempered" class $[(l_\varepsilon)_\varepsilon]$ under some additional assumption on the growth of $(l_\varepsilon)_\varepsilon$. However this generalized solution in $\mathcal{A}\left(\mathbb{R}^2\right)$ fails to be, in general, unique. We show how uniqueness may be recovered by searching a solution in the space of new tempered generalized functions $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$ based on the space of slowly increasing smooth functions [3] in which some useful tools that we use, like pointwise characterization, are still valid [14].

2 General overview on $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-type algebras

2.1 Algebraic and topological structures [12]

Let: (1) Λ be a set of indices left-filtering for a given (partial) order relation $\prec$.
(2) A be a solid subring with unity of the ring $\mathbb{K}^\Lambda$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and I_A a solid ideal of A, which means that $B = A, I_A$ has the following stability property: whenever $(|s_\lambda|)_\lambda \leqslant (r_\lambda)_\lambda$ (i.e. for any λ, $|s_\lambda| \leqslant r_\lambda$) for any pair $((s_\lambda)_\lambda, (r_\lambda)_\lambda) \in \mathbb{K}^\Lambda \times |B|$, it follows that $(s_\lambda)_\lambda \in B$, with $|B| = \{(|r_\lambda|)_\lambda,\ (r_\lambda)_\lambda \in B\}$;
(3) $\mathcal{E}$ be a sheaf of $\mathbb{K}$-topological algebras over a topological space X .

Moreover, suppose that:
(4) For any open set Ω in X, the algebra $\mathcal{E}(\Omega)$ is endowed with a family $\mathcal{P}(\Omega) = (P_i)_{i \in I(\Omega)}$ of semi-norms such that if $\Omega_1 \subset \Omega_2$ are two open subsets of X, it follows that $I(\Omega_1) \subset I(\Omega_2)$ and if ρ_1^2 is the restriction operator $\mathcal{E}(\Omega_2) \to \mathcal{E}(\Omega_1)$, then, for each $P \in \mathcal{P}(\Omega_1)$ the semi-norm $\widetilde{P} = P \circ \rho_1^2$ extends P to $\mathcal{P}(\Omega_2)$;
(5) Let $\Theta = (\Omega_h)_{h \in H}$ be any family of open sets in X with $\Omega = \cup_{h \in H} \Omega_h$. Then, for each $P \in \mathcal{P}(\Omega)$, there exists a finite subfamily $(\Omega_j)_{1 \leqslant j \leqslant n}$ of Θ and semi-norms $P_j \in \mathcal{P}(\Omega_j)$ $(1 \leqslant j \leqslant n)$ such that, for any $u \in \mathcal{E}(\Omega)$, $P(u) \leqslant \sum_{j=1}^{n} P_j\left(u|_{\Omega_j}\right)$.

Set $\mathcal{C} = A/I$ and let $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$ (resp. $\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega)$) be the set of all $(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda$ such that $((P_i(u_\lambda))_\lambda \in |A|$ (resp. $|I_A|$) for all $i \in I(\Omega)$.

From (2), it follows that $|A|$ is a subset of A and that $A_+ = \{(b_\lambda)_\lambda \in A \mid (\forall \lambda \in \Lambda)(b_\lambda \geqslant 0)\} = |A|$. The same holds for I_A. Furthermore, (2) implies also that A is a $\mathbb{K}$-algebra. Thus $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ (resp. $\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$) is a sheaf of $\mathbb{K}$-subalgebras (resp. of ideals) of the sheaf $\mathcal{E}^\Lambda$ (resp. of $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$). The factor $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$ is a presheaf with localization principle [12]. Moreover, the constant sheaf $\mathcal{H}_{(A,\mathbb{K},|\,|)}/\mathcal{J}_{(I_A,\mathbb{K},|.|)}$ is equal to the sheaf $\mathcal{C} = A/I_A$. We call *presheaf of* $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-*algebra*, the factor presheaf of the algebras $\mathcal{A} = \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$ over the ring $\mathcal{C} = A/I_A$.

Notation 1. *(i) We denote by* $[u_\lambda]$ *the class in* $\mathcal{A}(\Omega)$ *of* $(u_\lambda)_{\lambda\in\Lambda} \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$.
(ii) For a topological space T, $K \Subset T$ *means that* K *is a compact subset of* T *and* $\mathcal{O}(T)$ *denotes the set of all open sets of* T.

Example 1. (Special Colombeau Algebra [10, 13]) We consider the sheaf $\mathcal{E} = \mathrm{C}^\infty$ over $\mathbb{R}^d$, endowed with the usual family of topologies $\mathcal{P} = (\mathcal{P}_\Omega)_{\Omega\in\mathcal{O}(\mathbb{R}^d)}$. Let us recall that $\mathcal{P}_\Omega$ is defined by the family of semi-norms $(p_{K,l})_{K\Subset\Omega,l\in\mathbb{N}}$ with

$$\forall f \in \mathrm{C}^\infty(\Omega),\ p_{K,l}(f) = \sup\nolimits_{|\alpha|\leqslant l} p_{K,\alpha}(f) \ \text{ with } p_{K,\alpha}(f) = \sup\nolimits_{x\in K} |D^\alpha f(x)|. \quad (1)$$

Let A (resp. I) be the set of all $(r_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]}$ such that there exists $m \in \mathbb{N}$ (resp. for all $q \in \mathbb{N}$) with $|r_\varepsilon| = \mathrm{o}(\varepsilon^{-m})$ (resp. $|r_\varepsilon| = \mathrm{o}(\varepsilon^q)$) as $\varepsilon \to 0$.The sheaf $\mathcal{A} = \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$ is the sheaf of (special) Colombeau algebras $\mathcal{G}$. In this case, we shall write $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})} = \mathcal{X}$ and $\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})} = \mathcal{N}$.

Remark 1. (An association process) Consider Ω an open subset of X, $\mathcal{F}$ a given sheaf of topological $\mathbb{K}$-vector spaces (resp. $\mathbb{K}$-algebras) over X containing $\mathcal{E}$ as a subsheaf and $a : \mathbb{R}_+ \to A_+$ a map such that $a(0) = 1$ (for $r \in \mathbb{R}_+$, we denote $a(r)$ by $(a_\lambda(r))_\lambda$). For $(v_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$, we shall denote by $\lim_{\Lambda,\mathcal{F}(\Omega)} v_\lambda$ the limit of $(v_\lambda)_\lambda$ for the $\mathcal{F}$-topology when it exists. We recall that $\lim_{\Lambda,\mathcal{F}(\Omega)} u_\lambda|_V = f \in \mathcal{F}(V)$ iff, for each $\mathcal{F}$-neighborhood W of f, there exists $\lambda_0 \in \Lambda$ such that: $\lambda \prec \lambda_0 \Longrightarrow u_\lambda \in W$. We also assume that, for each open subset $V \subset \Omega$, we have

$$\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(V) \subset \left\{(v_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(V) \mid \lim\nolimits_{\Lambda,\mathcal{F}(\Omega)} v_\lambda = 0\right\}. \qquad (2)$$

Consider $u = [u_\lambda] \in \mathcal{A}(\Omega)$, $r \in \mathbb{R}_+$, V an open subset of Ω and $f \in \mathcal{F}(V)$. We say that u is $a(r)$*-associated to* f *in* V, denoted by $u \underset{\mathcal{F}(V)}{\overset{a(r)}{\sim}} f$, if $\lim_{\Lambda,\mathcal{F}(\Omega)}(a_\lambda(r)\, u_\lambda|_V) = f$. In particular, if $r = 0$, u and f are said to be *associated* in V.

Example 2. Take $X = \mathbb{R}^d$, $\mathcal{F} = \mathcal{D}'$, $\Lambda =]0,1]$, $\mathcal{A} = \mathcal{G}$, $V = \Omega$, $r = 0$. The usual association [10, §1.2.6] between $u = [u_\varepsilon] \in \mathcal{G}(\Omega)$ and $T \in \mathcal{D}'(\Omega)$ is defined by

$$u \sim T \Longleftrightarrow u \underset{\mathcal{D}'(\Omega)}{\overset{a(0)}{\sim}} T \Longleftrightarrow \lim\nolimits_{\varepsilon\to 0,\mathcal{D}'(\Omega)} u_\varepsilon = T.$$

The ring A and the ideal I_A are given by the asymptotic structure of the problem and constructed as follows. Let B_p a finite family of p nets in $(\mathbb{R}_+^*)^\Lambda$. Consider B the subset of elements in $(\mathbb{R}_+^*)^\Lambda$ obtained as rational fractions with coefficients in $\mathbb{R}_+^*$, of elements in B_p as variables. Define

$$A \ = \ \left\{(a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid (\exists\, (b_\lambda)_\lambda \in B)\ \ (\exists \lambda_0 \in \Lambda)\, (\forall \lambda \prec \lambda_0)\, (|a_\lambda| \leqslant b_\lambda)\right\};$$

$$I_A = \left\{ (a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid (\forall (b_\lambda)_\lambda \in B) (\exists \lambda_0 \in \Lambda) (\forall \lambda \prec \lambda_0) (|a_\lambda| \leqslant b_\lambda) \right\}.$$

It is easy to see that A (resp. I_A) is a solid subring of $\mathbb{K}^\Lambda$ (resp. A). We say that A and $\mathcal{C} = A/I_A$ are *overgenerated* by B_p.

In this paper, we consider the particular case $\mathcal{E} = \mathrm{C}^\infty$, $X = \mathbb{R}^d$ endowed with the usual topology defined in Example 1. For any choice of $\mathcal{C}$, $\mathcal{A}$ is a sheaf of differential algebras with $D^\alpha u = [D^\alpha u_\lambda]$ where $(u_\lambda)_\lambda \in u$. For $(\mathcal{C}, C^\infty, \mathcal{P})$-algebras, we have the analogue of [10, Thm 1.2.3]:

Proposition 1 (see [2]). Assume that the set B, defined above, is stable by inverse and that there exists $(a_\lambda)_\lambda \in B$ with $\lim_\Lambda a_\lambda = 0$. Consider $(u_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\mathbb{R}^d)$ such that, for all $K \Subset \mathbb{R}^d$, $(P_{K,0}(u_\lambda))_\lambda \in |I_A|$. *Then* $(u_\lambda)_\lambda \in \mathcal{J}_{(A,\mathcal{E},\mathcal{P})}(\mathbb{R}^d)$.

We shall also consider the algebra of tempered generalized functions. For $f \in \mathrm{C}^\infty(\mathbb{R}^n)$, $r \in \mathbb{Z}$ and $m \in \mathbb{N}$, we set $\mu_{r,m}(f) = \sup_{x \in \mathbb{R}^n, |\alpha| \leqslant m} (1 + |x|)^r |\mathcal{D}^\alpha f(x)|$. We say that $(f_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^n)^{(0,1]}$ belongs to $\mathcal{M}_\tau(\mathbb{R}^n)$ (resp. $\mathcal{N}_\tau(\mathbb{R}^n)$) if

$$(\forall m \in \mathbb{N}) (\exists q \in \mathbb{N}) (\exists N \in \mathbb{N}) \, text(resp. \forall p \in \mathbb{N}) \ \left(\mu_{-q,m}(f_\varepsilon) = O(\varepsilon^{-N}) \text{ (resp. } O(\varepsilon^p)) \right).$$

From [10], it follows that $\mathcal{M}_\tau(\mathbb{R}^n)$ (resp. $\mathcal{N}_\tau(\mathbb{R}^n)$) is a subalgebra (resp. ideal) of $\mathcal{O}_M(\mathbb{R}^n)^{(0,1]}$ (resp. $\mathcal{M}_\tau(\mathbb{R}^n)$). The algebra $\mathcal{G}_\tau(\mathbb{R}^n) = \mathcal{M}_\tau(\mathbb{R}^n) / \mathcal{N}_\tau(\mathbb{R}^n)$ is called the *algebra of tempered generalized functions*. The generalized derivation, defined as for $(\mathcal{C}, C^\infty, \mathcal{P})$-algebras, provides $\mathcal{G}_\tau(\mathbb{R}^n)$ with a differential algebraic structure.

Remark 2. (Simplification of notations) In the sequel, we have $d = 1$ or $d = 2$ and $\Lambda = (0, 1]$. We shall write $\mathcal{H}$ (resp. $\mathcal{J}$) instead of $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ (resp. $\mathcal{J}_{(A,\mathcal{E},\mathcal{P})}$) and use the same sheaf symbols $\mathcal{H}$, $\mathcal{J}$, $\mathcal{A} = \mathcal{H}/\mathcal{J}$ for $X = \mathbb{R}^d$ or $X = \Omega$, where $d = 1, 2$ and Ω is an open subset of $\mathbb{R}^d$.

2.2 Generalized operators and general restrictions

Let Ω be an open subset of $\mathbb{R}^2$ and $F \in \mathrm{C}^\infty(\Omega \times \mathbb{R}, \mathbb{R})$. We say that the algebra $\mathcal{A}(\Omega)$ is *stable under* F if, for all $(u_\varepsilon)_\varepsilon \in \mathcal{H}(\Omega)$ and all $(i_\varepsilon)_\varepsilon \in \mathcal{J}(\Omega)$, we have $(F(\cdot, \cdot, u_\varepsilon))_\varepsilon \in \mathcal{H}(\Omega)$ and $(F(\cdot, \cdot, u_\varepsilon + i_\varepsilon) - F(\cdot, \cdot, u_\varepsilon))_\varepsilon \in \mathcal{J}(\Omega)$. If $\mathcal{A}(\mathbb{R}^2)$ if stable under F, for $u = [u_\varepsilon] \in \mathcal{A}(\mathbb{R}^2)$, $[F(.,., u_\varepsilon)]$ is a well defined element of $\mathcal{A}(\mathbb{R}^2)$ (i.e. not depending on $(u_\varepsilon)_\varepsilon \in u$).

A simple example of stability condition is when F is *smoothly tempered*, which means that the following two conditions are satisfied:

(i) For each $K \Subset \mathbb{R}^2$, $l \in \mathbb{N}$ and $u \in \mathrm{C}^\infty(\Omega, \mathbb{R})$, there is a positive finite sequence $(C_j)_{1 \leqslant j \leqslant l}$ such that: $P_{K,l}(F(\cdot,\cdot,u)) \leqslant \sum\limits_{i=0}^{l} C_i \left(P_{K,l}(u)\right)^i$,

(ii) For each $K \Subset \mathbb{R}^2$, $l \in \mathbb{N}$, $u, v \in \mathrm{C}^\infty(\Omega, \mathbb{R})$, there is a positive finite sequence $(D_j)_{1 \leqslant j \leqslant l}$ such that: $P_{K,l}\left(F(\cdot,\cdot,v) - F(\cdot,\cdot,u)\right) \leqslant \sum\limits_{j=1}^{l} D_j \left(P_{K,l}(v-u)\right)^j$.

Definition 1 (see [5]). If $\mathcal{A}\left(\mathbb{R}^2\right)$ if stable under F, the map $\mathcal{F} : \mathcal{A}\left(\mathbb{R}^2\right) \to \mathcal{A}\left(\mathbb{R}^2\right)$ $u = [u_\varepsilon] \mapsto [F(.,.,u_\varepsilon)]$ is called the *generalized map corresponding to F*.

Consider $(l_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\mathbb{R})^\Lambda$. Set $R_\varepsilon : \mathrm{C}^\infty\left(\mathbb{R}^2\right) \to \mathrm{C}^\infty(\mathbb{R}), g \mapsto R_\varepsilon(g)$ with $R_\varepsilon(g) : \mathbb{R} \to \mathbb{R},\ t \mapsto g(t, l_\varepsilon(t))$. We say that $(l_\varepsilon)_\varepsilon$ is *compatible with the generalized restriction* if, for all $(u_\varepsilon)_\varepsilon \in \mathcal{H}(\mathbb{R}^2)$ (resp. $(i_\varepsilon)_\varepsilon \in \mathcal{J}(\mathbb{R}^2)$), $(u_\varepsilon(\cdot, l_\varepsilon(\cdot)))_\varepsilon \in \mathcal{H}(\mathbb{R})$ (resp. $(i_\varepsilon(\cdot, l_\varepsilon(\cdot)))_\varepsilon \in \mathcal{J}(\mathbb{R})$).

Definition 2 (see [5]). If the family of smooth functions $(l_\varepsilon)_\varepsilon$ is compatible with the generalized restriction, the map $\mathcal{R} : \mathcal{A}\left(\mathbb{R}^2\right) \to \mathcal{A}(\mathbb{R}),\ u = [u_\varepsilon] \mapsto [u_\varepsilon(\cdot, l_\varepsilon(\cdot))] = [R_\varepsilon(u_\varepsilon)]$ is called the *generalized restriction mapping* corresponding to the family $(l_\varepsilon)_\varepsilon$.

Definition 3 (see [10]). Let $(l_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^n)^\Lambda$. We say $(l_\varepsilon)_\varepsilon$ is *c-bounded* if for all $K \Subset \mathbb{R}^n$, there exists $L \Subset \mathbb{R}^n$ such that $l_\varepsilon(K) \subset L$ for all ε (L is independent of ε).

The following proposition establishes a link between the c-boundeness and the compatibility with restriction.

Proposition 2. Assume that $(l_\varepsilon)_\varepsilon$ belongs to $\mathcal{H}(\mathbb{R})$ and $(l_\varepsilon)_\varepsilon$ is c-bounded, then the family $(l_\varepsilon)_\varepsilon$ is compatible with generalized restriction.

3 Application to a characteristic Cauchy problem

We deal with the characteristic Cauchy problem for the transport equation formally written in characteristic coordinates: $P_c\ \ \partial u/\partial t = F(.,.,u)$, $u\mid_{\{x=0\}} = f$, where $f \in \mathrm{C}^\infty(\mathbb{R})$. We are going to formulate some assumptions which will allow us to associate to (P_c) a generalized and well-posed problem (P_g) constructed below.

3.1 From the ill-posed problem (P_c) to a well-posed formulation (P_g)

We approximate the characteristic curve $\{x = 0\}$ by a family of non-characteristic ones $\gamma_\varepsilon = \{x = l_\varepsilon(t)\}_{\varepsilon \in (0,1]}$. We assume that the family $(l_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\mathbb{R})^{]0,1]}$ tends

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Theorem 1. *Under the previous assumptions (H),* $\mathcal{A}\left(\mathbb{R}^2\right)$ *is stable under* F *and the generalized restriction operator* $\mathcal{R} : \mathcal{A}\left(\mathbb{R}^2\right) \to \mathcal{A}(\mathbb{R}), u = [u_\varepsilon] \to [u_\varepsilon(t, l_\varepsilon(t))]$ *is well defined.*

We can link to (P_c) the generalized problem : (P_g) $\partial u/\partial t = \mathcal{F}(u)$, $\mathcal{R}(u) = f$.

3.2 Existence of a solution to (P_g)

In order to solve (P_g), we begin to solve in $\mathrm{C}^\infty\left(\mathbb{R}^2\right)$ the regularized problem: (P_∞) $\partial u_\varepsilon/\partial t\,(t,x) = F(t,x,u_\varepsilon(t,x))$, $u_\varepsilon(t, l_\varepsilon(t)) = f(t)$.

Proposition 3. With the assumptions (4) and (H), the problem (P_∞) admits a unique smooth solution u_ε such that

$$u_\varepsilon(t,x) = f(l_\varepsilon^{-1}(x)) + \int_{l_\varepsilon^{-1}(x)}^{t} F(\tau, x, u_\varepsilon(\tau, x))\,\mathrm{d}\tau. \tag{5}$$

Moreover we have the estimate

$$\|u_\varepsilon\|_{\infty,K} \leqslant (\omega_{K,\beta} Q_\varepsilon + B_K a_\varepsilon M_\varepsilon)\,(\exp a_\varepsilon M_\varepsilon)^{C_K} \tag{6}$$

where $B_K = \mu_{K,0}\nu_K, C_K = \mu_{K,1}\nu_K$ depend only upon the compact set K.

The proof uses the Cauchy-Lipschitz theorem (for fixed x) for the existence and the uniqueness of a smooth solution u_ε to the problem (P_∞), which satisfies (5). Starting from this relation, the Gronwall Lemma, leads to the estimate (6).

Theorem 2. *Under Assumption (H), the problem (P_g) admits $[u_\varepsilon]_{\mathcal{A}(\mathbb{R}^2)}$ as solution where u_ε is the solution given in Proposition 3.*

The proof follows the same steps as the existence results which can be found in [6, 7]: starting from the estimate (6), an induction process on the order of the successive derivatives shows that $(u_\varepsilon)_\varepsilon$ belongs to $\mathcal{H}\left(\mathbb{R}^2\right)$.

For linear (or semi linear) problems with irregular data, a more complete theory exists, based on the functorial properties of the Colombeau type algebras [4]. Existence and uniqueness are obtained whenever the map associating the solution to the data for the classical problem is continuously temperate. Of course, this theory fails when the problem under consideration is characteristic as in the present paper. Moreover, without further assumption the solution given by Theorem 2 fails in general to be unique as shown by a counter example given in [4].

3.3 Independence of the generalized solution from the regularizing process

The solution of all the problems which are regularized by the Colombeau method depends *a priori* on the choice of the regularizing process. Indeed, in the preceding section we have built the solution u to (P_g) by making use in a crucial way of the representative $(l_\varepsilon)_\varepsilon$. So even though the map $\mathcal{R}$ itself does not depend on the representative of $l = [(l_\varepsilon)_\varepsilon]$, we need to prove that our solution is independant of this representative. A first step in this direction is done by [1] in which the purely characteristic case is studied (with regular data). Here we have an analogous result whose proof follows essentially the same lines:

Theorem 3. *In addition to the previous assumptions, suppose that* $(l_\varepsilon)_\varepsilon \in \mathcal{M}_\tau(R)$ *and* $\left(l_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{M}_\tau(\mathbb{R})$*. Then, the generalized function* $u = [u_\varepsilon]$*, where* $(u_\varepsilon)_\varepsilon$ *is given by (5), depends solely on* $l = [l_\varepsilon] \in \mathcal{G}_\tau(\mathbb{R})$ *as generalized function and not on the representatives* $(l_\varepsilon)_\varepsilon$*.*

However, we shall give the main step of the proof, as it emphasizes the difference between the case of usual Colombeau algebra and tempered generalized functions:

Lemma 1. *Let* $(l_\varepsilon)_\varepsilon \in \mathcal{M}_\tau(\mathbb{R})$ *such that for every* ε*,* l_ε *is bijective and* $\left(l_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{M}_\tau(\mathbb{R})$*. Then, for any* $(g_\varepsilon)_\varepsilon \in \mathcal{M}_\tau(\mathbb{R})$ *such that for every* ε*,* g_ε *is bijective,* $\left(g_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{M}_\tau(\mathbb{R})$ *and* $(g_\varepsilon - l_\varepsilon)_\varepsilon \in \mathcal{N}_\tau(\mathbb{R})$*, we have* $\left(l_\varepsilon^{-1} - g_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{N}_\tau(\mathbb{R})$*.*

Proof. We shall use the point values characterization [10, §1.2.4]. Let $\mathcal{M}_\mathbb{R}$ (resp. $\mathcal{N}_\mathbb{R}$) be the set of all $(x_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]}$ such that: $(\exists N \in \mathbb{N})$ $\left(|x_\varepsilon| = \mathrm{O}(\varepsilon^{-N})\right)$ (resp. $(\forall m \in \mathbb{N})\,(|x_\varepsilon| = \mathrm{O}(\varepsilon^m))$) as $\varepsilon \to 0$. We denote by $\widetilde{\mathbb{R}} = \mathcal{M}_\mathbb{R}/\mathcal{N}_\mathbb{R}$ the ring of generalized real numbers in the Colombeau setting. Let $(l_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{M}_\tau(\mathbb{R})$. Define the maps

$$G : \widetilde{\mathbb{R}} \to \widetilde{\mathbb{R}}, \tilde{x} \mapsto g(\tilde{x}) = [(g_\varepsilon(x_\varepsilon))_\varepsilon]_{\widetilde{\mathbb{R}}} \ ; H : \widetilde{\mathbb{R}} \to \widetilde{\mathbb{R}}, \tilde{x} \mapsto h(\tilde{x}) = [g_\varepsilon^{-1}(x_\varepsilon)]_{\widetilde{\mathbb{R}}}$$

where $g(\tilde{x})$ (resp. $h(\tilde{x})$) is the generalized point value of g (resp. h) at the generalized point $\tilde{x} = [(x_\varepsilon)_\varepsilon]$ and well defined from [10, Prop. 1.2.45]. It is easy to see that $G \circ H = H \circ G = id$ so that $G^{-1} = H$. In the same way, if we set $F : \widetilde{\mathbb{R}} \to \widetilde{\mathbb{R}}, \tilde{x} \mapsto l(\tilde{x}) = [l_\varepsilon(x_\varepsilon)]_{\widetilde{\mathbb{R}}}$. Then $F^{-1} : \widetilde{\mathbb{R}} \to \widetilde{\mathbb{R}}$ is defined by $F^{-1}(\tilde{x}) = \left[l_\varepsilon^{-1}(x_\varepsilon)\right]$. Then, proving the Lemma amounts to prove that $l^{-1} - g^{-1} = 0$ in $\mathcal{G}_\tau(\mathbb{R})$, and, by point value characterization [10, Prop. 1.2.47], it suffices to show that $\forall \widetilde{y} \in \widetilde{\mathbb{R}}, \left(F^{-1} - G^{-1}\right)(\widetilde{y}) = 0$. Let $\widetilde{y} = [y_\varepsilon] \in \widetilde{\mathbb{R}}$. As G is bijective there exists $\tilde{x} = [x_\varepsilon] \in \widetilde{\mathbb{R}}$ such that $\widetilde{y} = G(\tilde{x})$ and for all ε we have

$$\left(F^{-1} - G^{-1}\right)(\tilde{y}) = \left[\left(l_\varepsilon^{-1}\left(g_\varepsilon(x_\varepsilon)\right) - g_\varepsilon^{-1}\left(g_\varepsilon(x_\varepsilon)\right)\right)_\varepsilon\right] = \left[\left(l_\varepsilon^{-1}\left(g_\varepsilon(x_\varepsilon)\right) - x_\varepsilon\right)_\varepsilon\right]$$

but as $(g_\varepsilon - l_\varepsilon)_\varepsilon \in \mathcal{N}_\tau(\mathbb{R})$ we have $\left(l_\varepsilon^{-1} \circ g_\varepsilon - id\right)_\varepsilon \in \mathcal{N}_\tau(\mathbb{R})$ so that $\left[\left(l_\varepsilon^{-1}\left(g_\varepsilon(x_\varepsilon)\right) - x_\varepsilon\right)_\varepsilon\right] \in \mathcal{N}_{\mathbb{R}}$, which concludes the proof. □

Example 4. We consider the problem: (P_{char}) $\partial u/\partial t = 0$, $u \mid_{\{x=0\}} = f$ and $f \in \mathrm{C}^\infty(\mathbb{R})$. We regularize (P_{char}) by choosing $l_\varepsilon(t) = \varepsilon t$ and obtain: (P_∞) $\partial u_\varepsilon/\partial t\,(t,x) = 0$, $u_\varepsilon(t, \varepsilon t) = f(t)$. Clearly the solution to (P_∞) is the function $u_\varepsilon : (t,x) \mapsto f(x/\varepsilon)$. Then, a generalized solution u of (P_g) is $[(t,x) \mapsto f(x/\varepsilon)]_{\mathcal{A}(\mathbb{R}^2)}$. Remark that here $\mathcal{C}$ is overgenerated by the family $(\varepsilon)_\varepsilon$ showing that $\mathcal{A}\left(\mathbb{R}^2\right)$ is the simplified Colombeau algebra $\mathcal{G}\left(\mathbb{R}^2\right)$. This generalized function is neither a function nor a distribution. However, it is possible to link u to a distribution by means of the association process defined in Remark 1. Suppose that f is integrable with $\int f(x)\,\mathrm{d}x = 1$ and write $(1/\varepsilon)\, u_\varepsilon : (t,x) \mapsto 1_t \otimes (1/\varepsilon)\, f(x/\varepsilon)$. We have clearly $\lim_{\varepsilon \to 0, D'(\mathbb{R}^2)} (u_\varepsilon/\varepsilon) = 1_t \otimes \delta_x = \delta_\Gamma$, where δ_Γ is the Dirac distribution on the characteristic manifold $\Gamma = \left\{(t,x) \in \mathbb{R}^2 : x = 0\right\}$. Thus, the solution u of the generalized problem (P_g) associated to (P_{char}) satisfies $u \underset{\varepsilon}{\sim} \delta_\Gamma$. In addition, this solution is not unique but depends only on the class in $\mathcal{G}_\tau\left(\mathbb{R}^2\right)$ of $(t \mapsto \varepsilon t)_\varepsilon$.

4 The framework $\mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^2)$ and uniqueness

The natural topology of $\mathcal{O}_M$ permits to define a new algebra of tempered generalized functions, $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ [3] which differs from $\mathcal{G}_\tau\left(\mathbb{R}^d\right)$ but permits a point value characterization [14] and an extension $\mathcal{A}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ in the framework of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras [5]. As $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ is of $(\mathcal{C},\mathcal{E},\mathcal{P})$-type and endowed with the sharp topology [2], our goal is at least to recover uniqueness of the solution of (P_g) in this context, the well-posedness in Hadamard setting being the final goal.

4.1 Point values in $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$

Define $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ as the quotient algebra $\mathcal{M}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)/\mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ where

$$\begin{aligned}
\mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d) = \{&(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^d)^{(0,1]} : (\forall \varphi \in \mathcal{S}(\mathbb{R}^d))\ (\forall \alpha \in \mathbb{N}^d) \\
&(\exists M \in \mathbb{N})\,(\exists \varepsilon_0)\,(\forall \varepsilon < \varepsilon_0)\ \left(\sup\nolimits_{x \in \mathbb{R}^d} |\varphi(x)\,\partial^\alpha u_\varepsilon(x)| \leqslant \varepsilon^{-M}\right)\}\ ; \\
\mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d) = \{&(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^d)^{(0,1]} : (\forall \varphi \in \mathcal{S}(\mathbb{R}^d))\ (\forall \alpha \in \mathbb{N}^d) \\
&(\forall m \in \mathbb{N})\,(\exists \varepsilon_0)\,(\forall \varepsilon < \varepsilon_0)\ \left(\sup\nolimits_{x \in \mathbb{R}^d} |\varphi(x)\,\partial^\alpha u_\varepsilon(x)| \leqslant \varepsilon^{m}\right)\}.
\end{aligned}$$

This definition can be compared to the one of $\mathcal{G}_\tau(\mathbb{R}^d)$. On one hand, we have $\mathcal{M}_{\mathcal{O}_M}\left(\mathbb{R}^d\right) = \mathcal{M}_\tau\left(\mathbb{R}^d\right)$ [3, Prop. 3.2]. However we only have $\mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^d\right) \supsetneq \mathcal{N}_\tau\left(\mathbb{R}^d\right)$.

Example 5. Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ with supp$\psi \subseteq B(0,1)$ and $\psi(0) = 1$. Let $e \in \mathbb{R}^d$ be a unit vector. Let $u_\varepsilon(x) := \psi(x - \varepsilon^{-1}e)$ for each ε. It is easy to check that $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d)$. However $(u_\varepsilon)_\varepsilon \notin \mathcal{N}_\tau(\mathbb{R}^d)$. Indeed take $\alpha = 0$. Let $p \in \mathbb{N}$ arbitrary. Then: $\sup_{x\in\mathbb{R}^d}(1+|x|)^{-p}|u_\varepsilon(x)| \geqslant (1+\varepsilon^{-1})^{-p}|u_\varepsilon(\varepsilon^{-1})| \geqslant (2\varepsilon^{-1})^{-p}|\psi(0)| = (\varepsilon/2)^p$, so no choice of p satisfies: $(\forall m \in \mathbb{N})$ $(\exists \varepsilon_0)$ $(\forall \varepsilon \leqslant \varepsilon_0)$ $(\sup_{x\in\mathbb{R}^d}(1+|x|)^{-p}|u_\varepsilon(x)| \leqslant \varepsilon^m)$.

Thus $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ differs from $\mathcal{G}_\tau\left(\mathbb{R}^d\right)$. On the other hand, along the same lines as [3, Prop. 3.2], we get

$$\begin{aligned}\mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d) = \{(u_\varepsilon)_\varepsilon \in (\mathcal{O}_M(\mathbb{R}^d)^{(0,1]} \mid (\forall \alpha \in \mathbb{N}^d)\ (\forall m \in \mathbb{N})\ (\exists p \in \mathbb{N})\\ (\exists \varepsilon_0)\ (\forall \varepsilon < \varepsilon_0)\ (\sup_{x\in\mathbb{R}^d}(1+|x|)^{-p}|\partial^\alpha u_\varepsilon(x)| \leqslant \varepsilon^m)\}.\end{aligned}$$

By the same Taylor-argument as in [10, Thm. 1.2.25], we obtain:

Theorem 4.

$$\begin{aligned}\mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d) = \{(u_\varepsilon)_\varepsilon \in \mathcal{M}_\tau(\mathbb{R}^d) \mid (\forall m \in \mathbb{N})\ (\exists p \in \mathbb{N})\\ (\exists \varepsilon_0)\ (\forall \varepsilon < \varepsilon_0)\ (\sup_{x\in\mathbb{R}^d}(1+|x|)^{-p}|u_\varepsilon(x)| \leqslant \varepsilon^m)\}.\end{aligned}$$

We refer to generalized points and point values as developed in [10, §1.2.4]. We recall that $\widetilde{\mathbb{K}} = \mathcal{M}_\mathbb{K}/\mathcal{N}_\mathbb{K}$ is the ring of Colombeau generalized numbers ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) and similarly $\widetilde{\mathbb{K}^d} = \widetilde{\mathbb{K}}^d$ the set of generalized points.

Definition 4. An element $\widetilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}^d$ is of slow scale if for all $n \in \mathbb{N}$ there exists ε_0 such that, for all $\varepsilon < \varepsilon_0$, we have $|x_\varepsilon| \leqslant \varepsilon^{-1/n}$.

Theorem 5. *Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^d)$ and let $\tilde{x} = [(x_\varepsilon)_\varepsilon]$ be of slow scale. Then the point value $u(\tilde{x}) := [(u_\varepsilon(x_\varepsilon))_\varepsilon] \in \widetilde{\mathbb{C}}$ is well-defined.*

Proof. Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d) = \mathcal{M}_\tau(\mathbb{R}^d)$ be a representative of u. By [10, Prop. 1.2.45], $(u_\varepsilon)_\varepsilon \in \mathcal{M}_\tau(\mathbb{R}^d)$ implies that $(u_\varepsilon(x_\varepsilon))_\varepsilon \in \mathcal{M}_\mathbb{R}$, and that $(u_\varepsilon(x_\varepsilon) - u_\varepsilon(x'_\varepsilon))_\varepsilon \in \mathcal{N}_\mathbb{R}$ if $(x'_\varepsilon)_\varepsilon$ is another representative of $\tilde{x}$. It remains to show that the definition of the point value does not depend on the choice of representative of u. So let $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d)$. Let $m \in \mathbb{N}$. Choose $p \in \mathbb{N}$ as in the statement of theorem 4. Then for sufficiently small ε,

$$|u_\varepsilon(x_\varepsilon)| \leqslant \varepsilon^m(1+|x_\varepsilon|)^p \leqslant \varepsilon^m(2|x_\varepsilon|)^p \leqslant \varepsilon^m(2\varepsilon^{-1/p})^p = 2^p\varepsilon^{m-1}.$$

Since $m \in \mathbb{N}$ is arbitrary, $(u_\varepsilon(x_\varepsilon))_\varepsilon \in \mathcal{N}_\mathbb{C}$. □

Theorem 6. *Let $u \in \mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^d)$. Then $u = 0$ iff $u(\tilde{x}) = 0$ for each slow scale point $\tilde{x}$.*

Proof. If $u = 0$, then clearly $u(\tilde{x}) = 0$ for each slow scale point (since the definition of point values does not depend on the representative of u). Conversely, let $u(\tilde{x}) = 0$ for each slow scale point $\tilde{x}$. We first show by contradiction that

$$(\forall m \in \mathbb{N})\ (\exists n \in \mathbb{N})\ (\exists \varepsilon_0)\ (\forall \varepsilon < \varepsilon_0) big(\sup_{|x| \leqslant \varepsilon^{-1/n}} |u_\varepsilon(x)| \leqslant \varepsilon^m\big). \tag{7}$$

Assuming the contrary, we find $M \in \mathbb{N}$, a decreasing sequence $(\varepsilon_n)_n$ tending to 0 and $x_{\varepsilon_n} \in \mathbb{R}^d$ with $|x_{\varepsilon_n}| \leqslant \varepsilon_n^{-1/n}$ and $|u_{\varepsilon_n}(x_{\varepsilon_n})| > \varepsilon_n^M$, for each nLet $x_\varepsilon := 0$ if $\varepsilon \notin \{\varepsilon_n : n \in \mathbb{N}\}$. Then $\tilde{x} := [(x_\varepsilon)_\varepsilon]$ is of slow scale and $(u_\varepsilon(x_\varepsilon))_\varepsilon \notin \mathcal{N}_\mathbb{R}$, contradicting $u(\tilde{x}) = 0$.

Now let $m \in \mathbb{N}$ arbitrary. Choose n as in equation ((7)). Since $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d) = \mathcal{M}_\tau(\mathbb{R}^d)$, there exists $N \in \mathbb{N}$ such that for small ε, $\sup_{x \in \mathbb{R}^d}(1 + |x|)^{-N}|u_\varepsilon(x)| \leqslant \varepsilon^{-N}$. Let $p := nm + nN + N$. Then, for small ε,

$$\begin{aligned}
&\sup_{x \in \mathbb{R}^d}(1 + |x|)^{-p}|u_\varepsilon(x)| = \\
&\quad = \max\Big(\sup_{|x| \leqslant \varepsilon^{-1/n}}(1 + |x|)^{-p}|u_\varepsilon(x)|, \sup_{|x| \geqslant \varepsilon^{-1/n}}(1 + |x|)^{-p}|u_\varepsilon(x)|\Big) \leqslant \\
&\leqslant \max\Big(\sup_{|x| \leqslant \varepsilon^{-1/n}} |u_\varepsilon(x)|, \sup_{x \in \mathbb{R}^d}(1 + |x|)^{-N}|u_\varepsilon(x)| \sup_{|x| \geqslant \varepsilon^{-1/n}}(1 + |x|)^{N-p}\Big) \leqslant \\
&\qquad\qquad \leqslant \max\big(\varepsilon^m, \varepsilon^{-N}(\varepsilon^{-1/n})^{N-p}\big) = \varepsilon^m.
\end{aligned}$$

Hence $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d)$ by Theorem 4. □

4.2 The main theorem

We start by two technical lemmas, the proof of the first one being a simple adaptation of [10, Thm 1.2.29].

Lemma 2. *Let (f_ε), (g_ε), $(\tilde{f}_\varepsilon)$, $(\tilde{g}_\varepsilon) \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R})$ be such that $[f_\varepsilon] = [\tilde{f}_\varepsilon]$, $[g_\varepsilon] = [\tilde{g}_\varepsilon]$. We have that $[f_\varepsilon \circ g_\varepsilon] = [f_\varepsilon \circ \tilde{g}_\varepsilon]$. If $[g_\varepsilon]$ preserves slow scale points then $[\tilde{f}_\varepsilon \circ g_\varepsilon] = [f_\varepsilon \circ g_\varepsilon]$.*

Lemma 3. *Consider $(f_\varepsilon)_\varepsilon$, $(g_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R})$ with f_ε and g_ε bijective, $(f_\varepsilon - g_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$ and $(f_\varepsilon^{-1})_\varepsilon$, $(g_\varepsilon^{-1})_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R})$. Suppose that $\left[g_\varepsilon^{-1}\right]$ preserves slow scale points. Then $(f_\varepsilon^{-1} - g_\varepsilon^{-1})_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$.*

Proof. We have $(f_\varepsilon^{-1} - g_\varepsilon^{-1}) \circ g_\varepsilon = f_\varepsilon^{-1} \circ g_\varepsilon - Id \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$ because $g_\varepsilon - f_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$ which implies that $[f_\varepsilon^{-1} \circ g_\varepsilon] = [f_\varepsilon^{-1} \circ f_\varepsilon] = [Id]$. Note that $f_\varepsilon^{-1} - g_\varepsilon^{-1} =$

$\left((f_\varepsilon^{-1} - g_\varepsilon^{-1}) \circ g_\varepsilon\right) \circ g_\varepsilon^{-1}$ and that $[g_\varepsilon^{-1}] \in \mathcal{G}_{O_M}(\mathbb{R})$ preserves slow scale points. Using the preceding Lemma, we find that $f_\varepsilon^{-1} - g_\varepsilon^{-1} \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$. □

Theorem 7. *Suppose that* $(l_\varepsilon)_\varepsilon$ *belongs to the subset* $\mathcal{L}_{\mathcal{O}_M}(\mathbb{R})$ *in* $\mathcal{M}_{\mathcal{O}_M}(\mathbb{R})$ *of families* $(g_\varepsilon)_\varepsilon$ *such that:* $g'_\varepsilon > 0$, $\left[g_\varepsilon^{-1}\right]_\varepsilon \in \mathcal{G}_{\mathcal{O}_M}(\mathbb{R})$ *preserves slow scale points and* $\lim_{\varepsilon \to 0, \mathcal{D}'(\mathbb{R})} g_\varepsilon = 0$. *Then if* $f \in \mathcal{O}_M(\mathbb{R})$ *and* $F = 0$, *the generalized function* $u = \left[1_t \otimes f \circ l_\varepsilon^{-1}\right]_{\mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^2)}$ *depends only on* $l = [l_\varepsilon]_{\mathcal{G}_{\mathcal{O}_M}(\mathbb{R})}$. *Moreover* u *is the unique solution to* (P_g) *in* $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$.

Proof. Take $(l_\varepsilon)_\varepsilon, (h_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R})$ such that $[l_\varepsilon] = [h_\varepsilon]$ and let $u = [u_\varepsilon]$, $v = [v_\varepsilon]$ (with $(u_\varepsilon)_\varepsilon, (v_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^2)$) be the corresponding solutions of (P_g). For all ε, we have

$$\begin{cases} u_\varepsilon(t,x) = f(l_\varepsilon^{-1}(x)) + \mu_\varepsilon(l_\varepsilon^{-1}(x)) + \int_{l_\varepsilon^{-1}(x)}^t i_\varepsilon(\tau,x)d\tau \\ v_\varepsilon(t,x) = f(h_\varepsilon^{-1}(x)) + \nu_\varepsilon(h_\varepsilon^{-1}(x)) + \int_{h_\varepsilon^{-1}(x)}^t j_\varepsilon(\tau,x)d\tau \end{cases}$$

where $(i_\varepsilon)_\varepsilon, (j_\varepsilon)_\varepsilon, (\mu_\varepsilon)_\varepsilon, (\nu_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$. First we know that $l_\varepsilon^{-1} - h_\varepsilon^{-1} \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$ and $f \in \mathcal{O}_M(\mathbb{R})$ so that $f \circ l_\varepsilon^{-1} - f \circ h_\varepsilon^{-1} \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$. Furthermore, as $\mu_\varepsilon, \nu_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$, $\left[l_\varepsilon^{-1}\right], \left[h_\varepsilon^{-1}\right] \in \mathcal{G}_{OM}(\mathbb{R})$ and they preserve slow scale points, we have that $\mu_\varepsilon \circ l_\varepsilon^{-1}, \nu_\varepsilon \circ h_\varepsilon^{-1} \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$. To finish the proof, we have to check that

$$\int_{l_\varepsilon^{-1}(x)}^{t} i_\varepsilon(\tau,x)d\tau - \int_{h_\varepsilon^{-1}(x)}^{t} j_\varepsilon(\tau,x)d\tau \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^2\right).$$

We will do it only for the first integral part, as they are almost identical. First we set, for all ε, $k_\varepsilon(t,x) = \int_{l_\varepsilon^{-1}(x)}^t i_\varepsilon(\tau,x)d\tau$. Let $(t_\varepsilon, x_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}^2$ be a slow scale point. Then $x_\varepsilon \in \widetilde{\mathbb{R}}$ is a slow scale point and $y_\varepsilon = l_\varepsilon^{-1}(x_\varepsilon)$ is also a slow scale point. We have $\forall \varepsilon, \exists c_\varepsilon \in [y_\varepsilon, t_\varepsilon], k_\varepsilon(t_\varepsilon, x_\varepsilon) = \int_{y_\varepsilon}^{t_\varepsilon} i_\varepsilon(\tau, x_\varepsilon)d\tau = (t_\varepsilon - y_\varepsilon)i_\varepsilon(c_\varepsilon, x_\varepsilon)$ but as $|c_\varepsilon| \leqslant \max(|y_\varepsilon|, |t_\varepsilon|)$, (c_ε) is also a slow scale point. But then $(c_\varepsilon, x_\varepsilon)$ is a slow scale point of $\mathbb{R}^2$ so that $(i_\varepsilon(c_\varepsilon, x_\varepsilon))_\varepsilon \in \mathcal{N}_\mathbb{R}$ and finally $(k_\varepsilon(t_\varepsilon, x_\varepsilon))_\varepsilon \in \mathcal{N}_\mathbb{R}$. □

Remark 3. However, we cannot prove the existence of a solution to (P_g) in $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$ if $F \neq 0$ as can be seen by taking $F(.,.,u) = u$. Indeed the regularized problem becomes: (P_∞) $\partial u_\varepsilon / \partial t\,(t,x) = u_\varepsilon(t,x)$, $u_\varepsilon(t, \varepsilon t) = v(t)$ whose solution is $u_\varepsilon(t,x) = v(x/\varepsilon)\exp(-x/\varepsilon)\exp(t)$ which clearly is not in $\mathcal{M}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$.

5 The well-posedness

Classically, in Hadamard sense, the well-posedness for a Cauchy problem asks for existence, uniqueness of solution to the problem and in addition, its continuous dependence on the data. Sharp topologies and functorial properties are extended to the case of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebra in [2]. Thus, one can expect here the following Hadamard setting: Let $u\,(v,\mathcal{R})$ be the solution given by Theorem 7 to the generalized problem $\partial u/\partial t = 0$, $\mathcal{R}\,(u) = v$ with $v \in \mathcal{O}_M\,(\mathbb{R}) \subset \mathcal{G}_{\mathcal{O}_M}\,(\mathbb{R})$. Then, at least in a neighborhood of v, the map $\mathcal{G}_{\mathcal{O}_M}\,(\mathbb{R}) \to \mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^2\right),\ v \mapsto u\,(v,\mathcal{R})$ is continuous for the corresponding sharp topologies.

For this result, which is left to a forthcoming paper, we shall build $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ with a unique parameter, the one used to de-characterize the problem, in contrast to previous works in which a parameter is used for the singular data, and a different one is introduced for each regularization procedure. The ring $\mathcal{C} = A/I_A$ will be the same for $d = 1,\ 2$.

But to obtain a good continuity result in this setting will require great care for choosing the type of tempered class of regularizations used to de-characterize the problem.

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AN INTRODUCTION TO NONLINEAR GENERALIZED FUNCTIONS AND TOPICAL DEVELOPMENTS

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Abstract. This is a general introduction to the nonlinear generalized functions aimed at a wide audience of nonspecialists. It also includes forthcoming developments.

The following calculations are the core of the Schwartz Impossibility Result 1954 [7,13,17]. Let H denote the Heaviside function defined by $H(x) = 0$ if $x < 0$, $H(x) = 1$ if $x > 0$ and $H(0)$ unspecified.

$$H^2 = H \Rightarrow \int_{-\infty}^{+\infty} (H^2 - H)H' dx = 0. \tag{1}$$

On the other hand "formal calculations" mimicking classical calculations on $\mathcal{C}^\infty$ functions give

$$\int_{-\infty}^{+\infty} (H^2 - H)H' dx = \left[\frac{H^3}{3} - \frac{H^2}{2}\right]_{-\infty}^{+\infty} = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}. \tag{2}$$

The trouble is that $0 \neq -\frac{1}{6}$! Indeed Schwartz carried similar calculations on some continuous functions instead of H to have a more convincing result but the situation is exactly the same and the above gives a maximum of clearness.

Schwartz adopted $H^2 = H$ as obvious (more precisely he adopted the classical product of continuous functions which was the assumption in his more elaborate calculation [7, 13, 17]) and therefore he claimed that the formal calculations are wrong because they give a wrong result. From the introduction of [18] "Multiplication of distributions is impossible in any mathematical context possibly different from distribution theory". In his mind this result was positive in that it showed that the absence of a general multiplication in distribution theory was not due to a defect of the distributions, but to the foundations of mathematics.

But since nearly one century our basic knowledge of the World has been based on formal calculations (including the above one, and a lot of considerably more complicated formal calculations): the field quantization by Heisenberg–Pauli (1927), the Renormalized Quantum Electrodynamics by Feynman–Tomonaga–Schwinger (1947), the Electroweak Model by Weinberg-Salam (1968,69 and Nobel Prize in 1979), Quantum Chromodynamics and finally the to-day Standard Model. Since about 1955 an enormous amount of effort was devoted to the search of a formulation of Quantum Field Theory that could make sense within the distributions (Axiomatic Field Theories). The failure was recognized after 1980.

I had been seeking for long time how to give a mathematical sense to the formal calculations in QFT (postponing the understanding of the Schwartz Impossibility Result since I believed physics was more basic than a paradox in pure mathematics) when I found a guideline from a study of spaces of $\mathcal{C}^\infty$ functions over the locally convex spaces of distribution theory. This guideline led to "Nonlinear Generalized Functions" [4–6]. When going back to the Schwartz Impossibility Result I noticed with amazement a "miracle": in these nonlinear generalized functions one has both $H^2 \neq H$ (needed for formal calculations) and "$H^2 = H$" (needed from intuition)! An explanation is in order. In the nonlinear generalized functions there are "nonzero infinitesimal functions" (i.e. functions that look null in distribution theory: in a natural sense their integral with test functions appear null) and $H^2 - H$ is precisely such an infinitesimal function. Since in 1954 mathematicians had not in mind the possibility of existence of nonzero infinitesimals in mathematics, these infinitesimal functions (to be considered as "approximately null but not exactly null") were considered as "exactly null", and clearly this is the origin of the impossibility claim. Therefore the Schwartz Impossibility Result could now be formulated as: "infinitesimals are needed to multiply the distributions".

The understanding of the paradox is now quite clear: in a way similar to the fact that a real number (with an infinite sequence of unpredictable digits) is an idealization of its approximations using a finite increasing number of its digits, a nonlinear generalized function is an idealization of a sequence of $\mathcal{C}^\infty$ functions: an Heaviside function H is an idealization of a sequence of smooth functions whose jump from the value 0 to the value 1 takes place on a 0-neighborhood of size ϵ tending to 0. Therefore since the approximations of H^2 are slightly different from the approximations of H in the interval of size ϵ in which the jump takes place, it appears that the sequence of approximations of H^2 and H are slightly different which explains that $H^2 - H$ is infinitesimal but not zero. In short $H^2 - H$ has the status of a limit $=0$, not of a fixed value 0. H' is infinitely large ($\approx \frac{1}{\epsilon}$) in the region of the jump of length ϵ. Finally the integral in (1,2) appears as some undeterminate limit of the conventional form $0 \times \infty$. Nonlinear generalized functions replace the 0 by a lot of possible infinitesimal functions and the ∞ by a lot of possible infinite functions: for any choice of one of them in each category this gives the possibility to resolve the indeterminacy. This is exactly the method taught by school teachers in mathematics to pupils who meet the concept of limit for the first time: replace 0 and ∞ by explicit formulas and seek a simplification that would resolve the ambiguity. The nonlinear generalized functions are based on a reproduction of this perfectly standard method in mathematical analysis!

In all calculations that make sense within the distributions infinitesimals are only multiplied by bounded quantities, for example $\int (H^2 - H)\psi dx, \psi$ a continuous test function. Therefore infinitesimals remain infinitesimals and do not give rise to new finite results. In short they serve to nothing and so they are conveniently replaced by $0_{\mathcal{D}'}$ (the 0 of distribution theory). In short the calculations in nonlinear generalized functions always give exactly the classical results modulo infinitesimals: there is a perfect coherence with distribution theory and classical mathematics. Schwartz believed this coherence impossible because he did not perceive how to reconcile $H^2 = H$ with $H^2 \neq H$.

Are these infinitesimals a mathematical trick or are they really present and observable in the physical world? The evidence of their presence in nature can be given by an observation of the elastoplastic shock waves [7, 8]. Imagine a metallic bar on one side of which you knock with a hammer. If you knock strongly enough you create an elastoplastic shock wave that propagates in the bar. An elastoplastic shock wave is made of an elastic region first, then a plastic region, put side by side on a certain width, the total width of the shock wave, of the order, for instance, of about one hundred crystal sizes. The velocity and density of the solid vary troughout the width of the shock wave. Another variable, the stress, increases in the elastic region, then it reaches some critical value for which the links that maintain the crystalline structure disappear: the plastic region is attained; then the stress remains constant equal to this critical value in the plastic region. In this region the solid behaves like a fluid: it changes its shape definitively and can break. The jump of the Heaviside function representing the stress takes place only in the elastic region while the jumps of the Heaviside functions representing the density and the velocity take place throughout the width of the shock wave encompassing both the elastic and the plastic regions. Therefore it is obvious even qualitatively that at least two kinds of very different Heaviside functions are requested to describe an elastoplastic shock wave. This has been observed by experimental physicists. This has also been observed from numerical tests and from explicit calculations of these shock waves, see [7, 8]. This phenomenon occurs in everyday's life (definitive deformations or breakings).

Nonlinear generalized functions in physics have many applications that escape from the domain of distribution theory.

• They give a rigorous mathematical sense to formal calculations involving nonlinear functions of distributions.

• They permit to correct mistakes such as those implied by erroneous simplifications: it is common in the physical litterature to find expressions such as $H^2 = H$ or $\sqrt{\delta} = 0$, where δ is the Dirac delta function. When these are followed by nonlinear calculations such as respectively multiplication by H', or elevation to the square, the formal calculations so obtained lead to wrong results.

• The nonlinear generalized functions permit to state physics in a way precise enough so as to resolve ambiguities in form of products of distributions, for instance in products of the form $H \times \delta$. Indeed if the notation $\approx$ means "differ by an infinitesimal" then

$$H^2 \approx H \Rightarrow HH' \approx \frac{1}{2}H'$$

and

$$H^3 \approx H \Rightarrow H^2H' \approx \frac{1}{3}H'$$

since both sides of $\approx$ can be differentiated freely. This gives examples of two products of the form $H \times \delta$ that are respectively equal to $\frac{1}{2}\delta$ and $\frac{1}{3}\delta$, modulo infinitesimals.

Examples of removal of ambiguities by a deeper formulation of the equations on physical ground are provided by models in nonconservative form. Consider a system made of one or several conservation laws (stating the balance of mass, momentum, total energy) and of a state law in nonconservative form, for instance a differential form of Hooke's law [7, 8], that we state for convenience in the form

$$u_t + (u^2)_x" = "\sigma_x, sigma_t + u\sigma_x" = "u_x.$$

The difficulty comes from the term $u\sigma_x$ which is of the form $H \times \delta$ in the case of shock waves. How to state this system in the nonlinear generalized functions in order to have well defined shock wave solutions?

• If both equations of the system are stated with the equality in nonlinear generalized functions then arbitrary nonlinear calculations on the equations would be allowed; they give contradictions in the case of shock waves therefore this statement has no shock wave solutions.

• If both equations are stated with the association i.e.

$$u_t + (u^2)_x \approx \sigma_x, sigma_t + u\sigma_x \approx u_x,$$

then there are shock wave solutions, but too many!: there is an infinity of possible jump conditions depending on an arbitrary real parameter while one needs a precisely well defined jump condition playing the role of the classical Rankine-Hugoniot jump formula of conservation laws.

• If one equation is stated with $=$ and the other one with $\approx$ one has existence of shock wave solutions and a well defined jump condition, which is satisfactory from a qualitative viewpoint. From a quantitative viewpoint one wonders which statement, if any, should be chosen since the two jump formulas are different. One has the choice between the two statements

$$u_t + (u^2)_x = \sigma_x, sigma_t + u\sigma_x \approx u_x$$

and

$$u_t + (u^2)_x \approx \sigma_x, sigma_t + u\sigma_x = u_x.$$

The choice is easy from the following reasoning on physical ground: it is known in physics that shock waves have a (very small) width. In this width one can state the basic conservation laws while the state laws are rather unknown in a situation of fast deformation and entropy increase inside a shock wave. This imposes the statement

$$u_t + (u^2)_x = \sigma_x, sigma_t + u\sigma_x \approx u_x.$$

Not only everything makes sense mathematically but also one has obtained nonambiguous jump conditions that have given satisfactory results.

Now we sketch various closely related presentations of the nonlinear generalized functions, giving slightly different differential algebras of nonlinear generalized functions, all of them denoted abusively by the generic symbol $\mathcal{G}(\Omega), \Omega$ any open set in $\mathbb{R}^n$.

• The original presentation [4–6] is directly issued from distribution theory by following a guideline. One has the inclusions

$$\mathcal{C}^\infty(\Omega) \subset \mathcal{C}^0(\Omega) \subset \mathcal{D}'(\Omega) \subset \mathcal{G}(\Omega).$$

The partial derivatives in $\mathcal{G}(\Omega)$ induce the partial derivatives in $\mathcal{D}'(\Omega)$ which is a vector subspace of $\mathcal{G}(\Omega)$. $\mathcal{C}^\infty(\Omega)$ is a faithful subalgebra of $\mathcal{G}(\Omega)$, while $\mathcal{C}^0(\Omega)$ is only a subalgebra "modulo infinitesimals" as explained above. The situation is very clear and expected by mathematicians (after one has understood the role of infinitesimals). There is a privileged Heaviside function and a privilegied Dirac delta function (those in $\mathcal{D}'(\Omega)$). This situation is useless in physics in which the various Heaviside functions and the various Dirac delta functions (functions in each category differ between each other by infinitesimal functions) have to be treated on an equal footing. Further the fact that these inclusions are canonical is paid by a certain amount of quantifiers in the statements of definitions. This version of $\mathcal{G}(\Omega)$ is often refered to as "full algebra". It is the richest from a viewpoint of pure mathematics.

• Then a simplified presentation has been introduced in which the inclusion $\mathcal{D}'(\Omega) \subset \mathcal{G}(\Omega)$ is no longer canonical and depends on the arbitrary choice of a mollifier ρ which is a smooth function with integral one and a few other properties [13]. The advantage is a simplification in definitions (less quantifiers).

• Simplifying more one obtains a "more objective" setting from the viewpoint of physics: there is no well defined inclusion $\mathcal{D}'(\Omega) \subset \mathcal{G}(\Omega)$: several elements of $\mathcal{G}(\Omega)$ have the "macroscopic aspect" of distributions, which is still denoted by the symbol $\approx$ which then connects an element of $\mathcal{G}(\Omega)$ with a distribution, and by extension connects two elements of $\mathcal{G}(\Omega)$ whose difference has the macroscopic aspect of the distribution 0, i.e. whose difference is an infinitesimal function. This is the simplest presentation and presumably the most objective for use in physics since it is free from mathematical complications due to a distinction of privilieged objects among those that have the macroscopic aspect of distributions.

Various other concepts have been introduced by different authors. First there is a clear connection with Nonstandard Analysis due to the presence of infinitesimals (but the infinitesimals were introduced here only for the problem of multiplication of distributions; they do not play a role inside distribution theory). Then the quotient (which is present in the constuction of the three concepts defined above) can be dropped and one obtains the concept developped by Egorov [12]. This quotient stems from distribution theory: it permits to identify objects that play the same role in all multiplications of distributions. When restricted to applications to the concept of solutions of nonlinear PDEs the concept of a "weak asymptotic method" defined by Danilov, Omelyanov and Shelkovich [11] is very well adapted to the search of weak solutions in a sense of generalized functions. In short the theory is not fixed. The theory of nonlinear generalized functions has been extended to manifolds and applied to General Relativity by Vickers, Grosser, Kunzinger, Steinbauer,... [20]. However we will note that these various concepts are not really competing since they manipulate the same ideas, with only superficial mathematical variations due to various degrees of simplicity according to the taste of the author or for the adaptation to a particular problem. But the presence of all these variations can be a serious drawback for the divulgation of nonlinear generalized functions since it can give a superficial impression of confusion.

Now we give an idea of the third concept quoted above: the "most objective" one from the viewpoint of physics and the simplest one from the viewpoint of mathematics due to the absence of a well defined embedding of distributions. The construction of this simple version of $\mathcal{G}(\Omega)$ is done by defining a large algebra called "Reservoir of representatives", an ideal of it called "Ideal of null representatives", and then $\mathcal{G}(\Omega)$ is the quotient of the Reservoir by the Ideal of null representatives.

In what follows $x \in \Omega$ and $\epsilon > 0$.
•Reservoir of representatives$=\{(x,\epsilon) \longmapsto R(x,\epsilon)\mathcal{C}^{\infty}\ / forall \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n,\ |\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} R(x,\epsilon)| \leqslant \frac{const}{\epsilon^{N(\alpha)}}\}$ uniformly on compact sets in Ω, where $N(\alpha)$ is an integer which may depend on α (and on the compact subset of Ω).
•Ideal of null representatives$=\{R / \forall \alpha \in \mathbb{N}^n, forall q \in \mathbb{N}\ |\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} R(x,\epsilon)| \leqslant const(q)\epsilon^q\}$ for $\epsilon > 0$ small enough, uniformly on compact sets in Ω, where $const(q)$ is a constant that depends on q (and on the compact subset of Ω).

Therefore an element of the quotient has a specific behavior when $\epsilon \to 0$. The ideal has been chosen so as to identify maps R that have a close enough behavior when $\epsilon \to 0$ so that they need not be distinguished to resolve ambiguities in multiplication of distributions. An element of $\mathcal{G}(\Omega)$ is said to be infinitesimal iff it has a representative R (then all representatives) that tends to zero in the sense of distributions when $\epsilon \to 0$, i.e. $\int R(x,\epsilon)\psi(x)dx \to 0$ for any test function $\psi \in \mathcal{C}_c^{\infty}(\Omega)$. An element of $\mathcal{G}(\Omega)$ is said to have the macroscopic aspect of a distribution T iff it has a representative R (then all representatives) such that $\int R(x,\epsilon)\psi(x)dx \to < T, \psi >$ when $\epsilon \to 0$, for any test function $\psi \in \mathcal{C}_c^{\infty}(\Omega)$.

The more elaborate concepts point out a privilieged inclusion of the vector space of distributions into the algebra of nonlinear generalized functions, which is nicer from the mathematical viewpoint: it simplifies statements of theorems since one can speak of distributions as contained in the nonlinear generalized functions.

Now we recall the two steps in the historical development of distribution theory.
• First step: Sobolev 1936 [19].
Definition 1. A sequence $(\varphi_n)_n,\ \ \varphi_n \in \mathcal{C}_c^k(\Omega)$, is said to be null iff support(φ_n) is contained in a fixed compact set (independent on n) and if this sequence tends to zero in sup norm as well as all derivatives up to order k.
Definition 2. A distribution T (of order k) is a linear map on the vector space $\mathcal{C}_c^k(\Omega)$ (without topology) such that $< T, \varphi_n > \to 0$ as soon as $(\varphi_n)_n$ is a null sequence.
• Second step: Schwartz 1945 introduced locally convex topologies on these vector spaces and defined a distribution as a linear continuous map. The topology gives an exactly equivalent definition and Schwartz developped the theory using locally convex vector space topologies [18]. He had a very great success which permitted a wide divulgation of distribution theory.

As the Sobolev theory the nonlinear theory presented up to now is of algebraic nature. One could try to find an analog of the Schwartz theory based on locally convex algebras. A natural topology-but not a vector space topology: some kind of ultra-metric topology - has been pointed out at the very beginning [2, 3]. It has been used in [2] to state the well posedness of Cauchy problems. It is now called "sharp topology" following Scarpalezos [16] and it has been studied and used by many authors: Scarpalezos, Aragona, Juriaans, Vernaeve, ... There appears an obstruction to the existence of a Hausdorff locally convex algebra topology $\mathcal{T}$ on $\mathcal{G}(\Omega)$ or suitable subalgebras. Indeed let $f_n(x) = \sqrt{n}$ if $0 < x < \frac{1}{n}, f_n(x) = 0$ elsewhere. Clearly for a natural topology $\mathcal{T}$, one should have $f_n \to 0$ since this is the case in the classical L^1 space (one privileges the classical L^p spaces in view of the applications; intuitively note that the family (f_ϵ) represents a $\sqrt{\delta}$ function) and $(f_n)^2 \to \delta$, the Dirac δ function. Since $\mathcal{T}$ has to be a topological algebra $f_n \to 0 \Rightarrow (f_n)^2 \to 0$. Then since $\mathcal{T}$ is Hausdorff one has $\delta = 0$, which is absurd. We are confronted with another "Impossibility Result", now of a topological nature!

I am starting to propose a solution (still unpublished and only announced in [9]).

Theorem 1. *There exists subalgebras of $\mathcal{G}(\mathbb{R}^n)$ (the simplest one, i.e. the third concept as described above) with the following properties:*

i) these subalgebras are Hausdorff locally convex algebras in which all bounded sets are relatively compact.

ii) they contain "most" distributions on Ω (in particular L^p spaces, distributions with compact support) with continuous inclusions; therefore bounded sets of distributions are relatively compact in these subalgebras.

iii) all partial derivatives are linear continuous from any such algebra into itself.

As far as I know up to now $\mathcal{G}(\mathbb{R}^n)$ is far from being covered by the union of these subalgebras. No attempt has been done concerning the two richer concepts of $\mathcal{G}(\mathbb{R}^n)$ described above. Although non metrizable these topological algebras have optimal properties: they are complete, Schwartz (i.e. a compactness property on the 0-neighborhoods), nuclear. They are completely different from the spaces of distribution theory. Their construction and study use very deeply the classical theories of locally convex spaces and nuclear spaces.

As an application these topological subalgebras permit the passage to the limit in nonlinearities: for these topologies

$$u_n \to u, v_n \to v \Rightarrow u_n.v_n \to u.v.$$

Such results are well known to be wrong for the weak convergence in classical spaces of distributions, but one has to keep in mind that the products here are those in $\mathcal{G}(\mathbb{R}^n)$, which differ from the classical products by infinitesimal quantities (recall the paradox raised by the Schwartz impossibility result considered at the beginning of this introduction) and that the situation is original. The redaction of this material and deeper investigations are in preparation.

In conclusion the nonlinear theory of generalized functions is used in physics: in particular

• in continuum mechanics for systems in nonconservative form such as elastoplasticity, one pressure model in multifluid flows [7].
• in general relativity, see the research-expository paper [20], in particular see works of Vickers, Grosser, Kunzinger, Steinbauer,... that extend the nonlinear theory of generalized functions to manifolds and apply it to nonlinear functions of distributions in General Relativity.
• in quantum field theory it permits to give a mathematical sense to the canonical Hamiltonian formalism, [10].

It is compatible with mathematics: $0'_{\mathcal{D}}$ splits into infinitesimals, which permits to resolve ambiguities in multiplications of distributions as exposed at the beginning of the paper. This theory is used for linear PDEs with discontinuous coefficients and for nonlinear PDEs with distributional initial data, see [14,15] and recent works in arXiv.org of Oberguggenberger, Pilipovic, Kunzinger, Hoermann, Garetto, ... I know between 500 and 800 papers on this theory therefore I give only a few introductory papers.

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ASSOCIATED HOMOGENEOUS DISTRIBUTIONS IN CLIFFORD ANALYSIS

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Abstract. We introduce Ultrahyperbolic Clifford Analysis (UCA) as a motivation for studying Associated Homogeneous Distributions (AHDs). UCA can be regarded as a higher-dimensional function theory that generalizes the theory of complex holomorphic functions. In UCA, the algebra of complex numbers is replaced with a Clifford algebra $Cl_{p,q}$ and the classical complex Cauchy-Riemann equation is replaced with a Clifford algebra-valued equation, having physical relevance.

The convolution kernel in Cauchy's integral formula from complex analysis, $\frac{1}{2\pi i}z^{-1}$, becomes in UCA a (non-trivial) AHD. In the theoretical development of UCA and also for its practical application, it is necessary that we can convolve and multiply AHDs. The aim of this talk is to show that UCA can be founded on classical distribution theory, so that it is not necessary to use a more general generalized function algebra for this purpose. This is achieved by using a new convolution and isomorphic multiplication algebra of (one-dimensional) AHDs developed earlier by the author, entirely within the setting of Schwartz' distributions.

1 Introduction

Ultrahyperbolic Clifford Analysis (UCA) is a particular generalization of complex analysis to hypercomplex analysis. Let $p, q \in \mathbb{N}$, $n \triangleq p + q$, P the canonical quadratic form of signature (p, q), $\mathbf{R}^{p,q} \triangleq (R^n, P)$ the inner product space with inner product induced by P and $Cl_{p,q}$ the Clifford algebra generated by $\mathbf{R}^{p,q}$. Then, UCA can be regarded as the study of a particular subset of functions from $R^n \to Cl_{p,q}$. A physical interpretation of UCA is that of a theory of functions defined on a generalized Lorentzian space with an arbitrary number of time (p) and space dimensions (q). UCA generalizes Hyperbolic Clifford Analysis (HCA), corresponding to $p = 1$ or $q = 1$, and Elliptic Clifford Analysis (ECA), corresponding to $p = 0$ or $q = 0$. ECA is about 30 years old and now a mature part of analysis, [2,4]. HCA and UCA are still under development.

The set of Associated Homogeneous Distributions (AHDs) with support in R, denoted by $\mathcal{H}'(R)$, is the distributional analogue of the set of power-log functions with domain in R, [11,18,25]. $\mathcal{H}'(R)$ contains the majority of the (one-dimensional) distributions one typically encounters in physics applications, such as δ, $\eta \triangleq \frac{1}{\pi}x^{-1}$ (a normalized Cauchy's principal value $\mathrm{Pv}\,\frac{1}{x}$), the Heaviside step distributions $1_{\pm}$, pseudo-functions generated by taking Hadamard's finite part of certain divergent integrals, associated Riesz kernels, generalized Heisenberg distributions, all their generalized derivatives and primitives, etc.

There is a close relationship between UCA and AHDs. First, the development of UCA requires us to study AHDs since the latter appear as cornerstone objects in the formulation of UCA. In addition, one needs their properties, e.g. for solving Boundary Value Problems (BVPs) and Riemann-Hilbert Problems (RHPs).

In particular, HCA with $p = 1$ and $q = 3$ appears to be a very suitable mathematical tool for solving physics applications, e.g. in Electromagnetism (EM) and Quantum Physics (QP). The latter physical relevance explains why AHDs appear so often in applications.

In earlier work, I constructed a convolution algebra and an isomorphic multiplication algebra of AHDs on R within Schwartz' distribution theory, [11]– [17]. We will see that higher dimensional versions of these algebras on R^n, obtained as pullbacks along the quadratic form P, play a key role in UCA. Consequently, UCA can be founded on Schwartz' distribution theory and it is thus not necessary to use a more general generalized function algebra for its construction.

2 Ultrahyperbolic Clifford Analysis

For an in depth overview of Clifford analysis, see [2–4, 9, 10].

2.1 Clifford algebras

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ denote an orthogonal basis for R^n. The universal (real) Clifford algebra $Cl_{p,q}$ over $\mathbf{R}^{p,q}$ is defined by

$$\begin{aligned} \mathbf{e}_1^2 &= \ldots = \mathbf{e}_p^2 = +1 \text{ and } \mathbf{e}_{p+1}^2 = \ldots = \mathbf{e}_n^2 = -1, && (1)\\ \mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i &= 0, i \neq j, && (2) \end{aligned}$$

together with linearity over $\mathbb{R}$ and associativity. Clifford showed how to turn an n-dimensional linear space into an 2^n-dimensional algebra. Essential is that his algebra is not closed for vectors, but is closed for all anti-symmetric tensors which can be generated from the underlying linear space. These anti-symmetric tensors represent oriented subspaces of the original n-dimensional linear space. A (real) Clifford number (also called "multivector") x is therefore a hypercomplex number over $\mathbb{R}$ of the form

$$x = \underbrace{a1}_{1} + \underbrace{a^i \mathbf{e}_i}_{\binom{n}{1}} + \frac{1}{2!}\underbrace{a^{i_1 i_2}\left(\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2}\right)}_{\binom{n}{2}} + \ldots + \underbrace{a^{1,\ldots,n}\left(\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n\right)}_{\binom{n}{n}}. \tag{3}$$

This can be regarded as a direct sum of $n+1$ grades $x = \oplus_{k=0}^{n} [x]_k$, making $Cl_{p,q}$ a graded linear space of dimension 2^n. We have the embeddings: $\mathbb{R} \hookrightarrow Cl_{p,q}$ by the grade 0 part and $\mathbf{R}^{p,q} \hookrightarrow Cl_{p,q}$ by the grade 1 part. A Clifford number of pure grade k has the geometrical interpretation of an oriented k-dimensional subspace. E.g., $x = [x]_1$ represents an oriented line segment (a vector), $x = [x]_2$ represents an oriented surface segment, etc. Clifford himself called his algebras geometrical algebras, because they are the natural choice when doing geometrical meaningful calculations with oriented subspaces of a given n-dimensional linear space.

The Clifford product of two numbers of pure grade, $x = [x]_k$ and $y = [y]_l$, is given by

$$xy = \sum_{i=|k-l|,2}^{k+l} [xy]_i \,. \tag{4}$$

In particular, the Clifford product of two vectors $\mathbf{v}$ and $\mathbf{w}$ decomposes into the sum of the inner and outer products,

$$\mathbf{vw} = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \wedge \mathbf{w}, \tag{5}$$

wherein the grade 0 part contains information about the angle between the vectors and the grade 2 part expresses that two vectors also span an oriented parallelogram.

Familiar Clifford algebras are: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (Hamilton's quaternions), P (Pauli's algebra), M (Majorana's algebra), and the time-space algebra $Cl_{1,3}$. Clifford algebras have been found to be very well-suited to formulate physical problems, [1,20,21,24].

2.2 Generalized Cauchy-Riemann equation

Introduce the $Cl_{p,q}$-valued nabla operator $\partial \triangleq \sum\limits_{i=1}^{n} \mathbf{e}_i \partial_i$, called Dirac operator, and let Ω be a domain in R^n .

Definition 1. Ultrahyperbolic Clifford Analysis is the study of functions satisfying

$$\partial F = -S, \tag{6}$$

with $F \in C^{\infty}(\Omega, Cl_{p,q})$ and for given $S \in C_c^{\infty}(R^n, Cl_{p,q})$, together with a boundary condition for F at infinity and possibly integrability conditions on S.

If $S = 0$, eq. (6) is a particular generalization of the Cauchy-Riemann equation from complex analysis and then defines functions F called (left) holomorphic.

2.3 Physical interpretation

Let us restrict eq. (6) to the Clifford algebra $Cl_{1,3}$, choose for S a smooth compact support multivector function having as only non-vanishing grades 1 and 3 (i.e., $S = [S]_1 + [S]_3$) and restrict F to be of pure grade 2 (i.e., $F = [F]_2$). Then eq. (6) reproduces the Maxwell-Heaviside equations for the EM field F, generated by a given electric monopole current density source $[S]_1$ and a given magnetic monopole current density source $[S]_3$, [19, 23, 26]. Hence, HCA of signature $(1, 3)$ (and with additional grade restrictions) is a mathematical function theory that models physical EM fields. This identification now leads to the correspondences summarized in Table 1.

Table 1

Correspondences between CA and EM

CA	**EM**
Cauchy-Riemann eq.	Equation of EM
Clifford-valued functions	Generalized EM fields
Holomorphy	Holography
Singularities, Residues	Source fields
Cauchy/Integral theorems	Reciprocity theorems
Riemann-Hilbert problems	Scattering problems
etc.	etc.

The above physical interpretation can be readily generalized. Choose any Clifford algebra $Cl_{p,q}$, let F be a general $Cl_{p,q}$-valued function and S a given smooth compact support $Cl_{p,q}$-valued function. Then eq. (6) becomes a model for a generalized EM in a universe with p time dimensions and q space dimensions!

2.4 Cauchy kernels

Of central importance in the formulation of UCA are the Cauchy kernels. The Cauchy kernel C_{x_0} in UCA is a vector-valued distribution, which derives from a scalar distribution $g_{x_0} \in \mathcal{D}'$ as

$$C_{x_0} = \partial g_{x_0}. \tag{7}$$

The scalar distribution g_{x_0} is a fundamental solution of the *Ultrahyperbolic Equation* (UE) (i.e., the wave equation) in $\mathbf{R}^{p,q}$

$$\Box_{p,q} g_{x_0} = \delta_{x_0}. \tag{8}$$

The point $x_0 \in R^n$ will eventually play the role of calculation point in the generalized Cauchy's integral theorem in UCA, but can here be thought of as parametrizing a family of distributions.

Introduce the shorthands, $P_{x_0} \triangleq P\left(x - x_0\right)$ and $A_{n-1} \triangleq 2\pi^{n/2}/\Gamma\left(n/2\right)$. A (real) fundamental solution of the UE for general (p, q) with $2 \leqslant n$ is, [5, 6, 8, 22],

(i) for $n > 2$

$$g_{x_0} = -\frac{1}{\left(n-2\right) A_{n-1}} \frac{1}{2}\left(e^{iq\frac{\pi}{2}}\left(P_{x_0} + i0\right)^{1-\frac{n}{2}} + e^{-iq\frac{\pi}{2}}\left(P_{x_0} - i0\right)^{1-\frac{n}{2}}\right), \tag{9}$$

(ii) for $n = 2$

$$\begin{aligned} g_{x_0} &= \frac{1}{4\pi}\frac{1}{2}\left(e^{iq\frac{\pi}{2}} \ln(P_{x_0} + i0) + e^{-iq\frac{\pi}{2}} \ln(P_{x_0} - i0)\right), && (10)\\ &= \frac{1}{4}\left(\cos(q\pi/2)\frac{1}{\pi}\ln\left|P_{x_0}\right| - \sin(q\pi/2) 1_-(P_{x_0})\right). && (11) \end{aligned}$$

The distributions g_{x_0} are readily seen to be pullbacks along P_{x_0} of one-dimensional AHDs. This is how AHDs enter in the formulation of UCA.

3 Associated Homogeneous Distributions on R

3.1 Definition

Definition 2. HD. A distribution $f_0^z \in \mathcal{D}'$ is called a (positively) homogeneous distribution of degree of homogeneity $z \in \mathbb{C}$ iff it satisfies for any $r > 0$,

$$\left\langle f_0^z, \varphi\left(x/r\right)\right\rangle = r^{z+1}\left\langle f_0^z, \varphi\left(x\right)\right\rangle, \forall \varphi \in \mathcal{D}. \tag{12}$$

Definition 3. AHD. A distribution $f_m^z \in \mathcal{D}'$ is called an associated (positively) homogeneous distribution of degree of homogeneity $z \in \mathbb{C}$ and order of association $m \in \mathbb{Z}_+$, iff there exists a sequence of associated homogeneous distributions f_{m-l}^z of degree of homogeneity z and associated order $m - l$, $\forall l \in \mathbb{Z}_{[1,m]}$, not depending on r and with $f_0^z \neq 0$, satisfying,

$$\langle f_m^z, \varphi(x/r)\rangle = r^{z+1}\left\langle f_m^z + \sum_{l=1}^{m} \frac{(\ln r)^l}{l!} f_{m-l}^z, \varphi(x)\right\rangle, \forall \varphi \in \mathcal{D}. \tag{13}$$

For a more detailed overview of AHDs, see [11, 18, 25].

3.2 Preliminaries

We will use hereafter the following terminology.

Definition 4. A partial distribution is a linear and continuous functional that is only defined on a proper subset $\mathcal{D}_r \subset \mathcal{D}$.

Definition 5. A $f_m^z \in \mathcal{H}'(R)$ has a critical degree of homogeneity at $z = z_c$ iff $f_m^{z_c}$ exists as a partial distribution.

Definition 6. An extension f_ε from $\mathcal{D}_r$ to $\mathcal{D}$, of a partial distribution f, is a distribution $f_\varepsilon \in \mathcal{D}'$, defined $\forall \varphi \in \mathcal{D}$, such that $\langle f_\varepsilon, \psi\rangle = \langle f, \psi\rangle$, $\forall \psi \in \mathcal{D}_r \subset \mathcal{D}$.

Definition 7. A regularization of a partial distribution $f_m^{z_c} \in \mathcal{H}'(R)$ is any extension $(f_m^{z_c})_e$ in $\mathcal{H}'(R)$ of $f_m^{z_c}$ from $\mathcal{D}_r$ to $\mathcal{D}$.

Definition 8. (i) The convolution product of any two AHDs on R of degrees $a-1$ and $b-1$ is called a critical convolution product, iff the resulting degree $a+b-1 \triangleq k \in \mathbb{N}$.

Definition 9. (ii) The multiplication product of any two AHDs on R of degrees a and b is called a critical multiplication product, iff the resulting degree $a+b \triangleq -l \in \mathbb{Z}_-$.

3.3 Definition of the products

3.3.1 Convolution

Let $\mathcal{D}'_R$ denote the distributions based on R with support bounded on the left and $\mathcal{D}'_L$ denote the distributions based on R with support bounded on the right. A structure theorem for AHDs states that any AHD on R is the sum of an AHD in $\mathcal{D}'_L$ and an AHD in $\mathcal{D}'_R$. To define a convolution product on $\mathcal{H}'(R)$ we must consider three cases.

Case 1. The factors have one-sided support, bounded at the *same side*.

In this case we can use for any degree of the factors the standard definition involving the *direct product* (the standard convolution integral). This case is an example of the method of retarded distributions.

Case 2. The factors have one-sided support, bounded at *different sides*, and the resulting degree of homogeneity is *not a natural number*.

In this case, the convolution $f * g$, with $f \in \mathcal{D}'_L$ and $g \in \mathcal{D}'_R$, can not straightforwardly be defined in terms of a direct product, because $\operatorname{supp}(f * g) \cap \operatorname{supp}(\varphi \in \mathcal{D}\left(R^2\right))$ is generally non-compact. This case is handled in two steps:

(i) First in $T \triangleq \left\{(a,b) \in \mathbb{C}^2 : 0 < \operatorname{Re}(a),\ 0 < \operatorname{Re}(b) \text{ and } \operatorname{Re}(a+b) < 1\right\}$ we use the standard convolution integral.

(ii) Then we extend by *analytic continuation* to $R \triangleq \left\{(a,b) \in \mathbb{C}^2 : a + b - 1 \notin \mathbb{N}\right\}$.

Case 3. The factors have one-sided support, bounded at *different sides*, and the resulting degree of homogeneity $a + b - 1$ is *a natural number* k (critical product). It was observed that:

(a) Any critical convolution product $f^{a-1} * f^{b-1}$ is in general a partial distribution only defined on $\mathcal{S}_{\{k\}}$. $S_{\{k\}}$ is the subspace of S whose members have zero k-th order moment.

(b) A particular extension of the partial distribution $f^{a-1} * f^{b-1}$ from $\mathcal{S}_{\{k\}}$ to $\mathcal{S}$ can be realized as *an* analytic finite part.

(c) This finite part, being a limit in $\mathbb{C}^2$, is in general non-unique.

(d) *Fortunately*, it turned out that this non-uniqueness only involves an arbitrary term of the form cx^k, $c \in \mathbb{C}$ arbitrary.

A critical convolution product, only existing as a partial distribution $f^{a-1} * f^{b-1}$, is then *defined as any extension* in $\mathcal{H}'\left(R\right)$ and so obtains meaning as a distribution.

3.3.2 Multiplication

Let $f^a, g^b \in \mathcal{H}'\left(R\right)$ of degree a and b. The multiplication of AHDs is defined in terms of the convolution product by

$$f^a \,.\, g^b \triangleq \mathcal{F}\left(\left(\mathcal{F}^{-1} f^a\right) * \left(\mathcal{F}^{-1} g^b\right)\right). \tag{14}$$

3.4 Properties of the products

The constructed algebras of AHDs on R have the following properties.

A. Non-commutativity

(i) Non-critical products are always commutative.

(ii) Critical products are generally non-commutative.

(iii) Deviation from commutative is by a term of the form cx^k (convolution) or $c\delta^{(l)}$ (multiplication), $k \in \mathbb{N}$, $l \in \mathbb{Z}_+$ and $c \in \mathbb{R}$ arbitrary.

B. Non-associativity

(i) Non-critical triple products are always associative.

(ii) Critical triple products are generally non-associative.

(iii) Deviation from commutative is by a term of the form cx^k (convolution) or $c\delta^{(l)}$ (multiplication), $k \in \mathbb{N}$, $l \in \mathbb{Z}_+$ and $c \in \mathbb{R}$ arbitrary.

4 Conclusion

The here presented connection between UCA and AHDs clearly reveals the importance of this rather small subset of Schwartz distributions. On the one hand, they appear as crucial building blocks in the construction of advanced higher dimensional analysis. On the other hand, and essentially because of their role in UCA, they appear ubiquitous in physics applications.

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