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ABSTRACTS

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Steklov Mathematical Institute of the RAS
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Конференция посвящена классическим разделам теории дифференциальных уравнений и различным видам нелокальных взаимодействий: обыкновенные дифференциальные уравнения, динамические системы, уравнения в частных производных, полугруппы операторов, нелокальные пространственно-временные системы, функционально-дифференциальные уравнения, приложения.

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On Some Problem for Control System with Delays

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We consider a linear control system with delay described by the following differential-difference equations:

$$\sum_{m=0}^{M} A_m y'(t-m\tau) + \sum_{m=0}^{M} B_m y(t-m\tau) = u(t), \quad 0 < t, \tag{1}$$

$$y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}, u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix},$$

where A_m and B_m are $n \times n$ constant matrices, A_0 is a nonsingular matrix, the delay $\tau > 0$ is constant, and $u(t)$ is a control vector-function.

A prehistory of the system is defined by the initial condition

$$y(t) = \varphi(t), \ t \in [-M\tau, 0], \tag{2}$$

where $\varphi(t) = \begin{pmatrix} \varphi_1(t) \\ \vdots \\ \varphi_n(t) \end{pmatrix}$ is a given vector-function.

In [1], N. N. Krasovskii considered the damping problem for a control system with aftereffect described by differential-difference equations of retarded type. He reduced this problem to the boundary value problem for a system of differential-difference equations with deviating argument in lower order terms. The Krasovskii damping problem was generalized in [2] to systems described by equations of neutral type.

Now we consider the damping problem for a multidimensional control system with several delays. We establish the relationship between the variational problem for a nonlocal functional and the boundary value problem for a system of differential-difference equations and study solvability of this boundary value problem [3].

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Kinetic Models of Integration-Fragmentation in the Becker–Döring Case: Derivation of Equations and the H-Theorem

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We examined the interconnections between the kinetic equations of evolution of particles distinguishing by masses (numbers of molecules forming them) or by other property in the Becker–Döring case and the H-theorem for them.

The Becker–Döring case [1] is the model when the only one molecule attaches or separates. We generalize this suggestion, replacing one molecule by a particle with size not greater than specified.

The formation of solids is well described by division into several stages - firstly primary particles emerge and grow, and then the stage of aggregation or the formation of agglomerates of the primary particles as a result of their collisions with each other comes, and then the formation of aggregates of them occurs, etc. Although all stages go at the same time, but first dominated the process of formation of primary particles, then their aggregates, then aggregates of the secondary agglomerates. Therefore, these processes are well modeled by generalizations of the Becker–Döring case that is considered by us.

From the continuum the integration-fragmentation equations, we derived a new equation which we call the continuous Becker–Döring equation. The basic assumption that defines this equation is that only particles with size not more than the specified can attach. From this equation we obtained the Becker–Döring system of equations and the continuum equation of the Fokker-Planck type (or of the Einstein-Kolmogorov type, or of diffuse approximation). We clarified the form of the obtained equations basing on the physical sense of these conclusions.

Also we have derived the Becker–Döring system of equations (on distribution function of particles by sizes) from the generalized kinetic Boltzmann equation on distribution function of the bodies by sizes and velocities of their centers of masses, i.e., moved from a multi-parameter description to reduced one.

We proved that, generally speaking, the H-theorem is incorrect for the equations of the Fokker-Planck type, but it's valid for a partially implicit discretization on time of the Becker–Döring system of equations. For the implicit discretization in time, this was proved earlier for general equations of physico-chemical kinetics [2] (in the case of fulfillment of the Stuckelberg Batishcheva–Pirogov condition [3, 4]). The obtained results are important for computer simulation: partially implicit discretization is more convenient for the discrete simulation, and the system of equations of the Becker–Döring type is more preferable than difference schemes of equations of the Fokker–Planck type.

Due to unity of the kinetic approach, the present work may be useful for specialists of various specialties who study the evolution of structures with differing properties.

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On Multiple Lebesgue Functions

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We introduce a notion of being a k-fold Lebesgue function for measure preserving transformations, where any 2-fold Lebesgue function is just ordinary Lebesgue. We discuss how this new metric isomorphisms invariant of a dynamical system is related to others classical notions in ergodic theory, mostly focusing on its spectral aspects. In particular, for transformations with sufficiently many multiple Lebesgue functions, we treat the corresponding multiple analogs of very well-known problems of Banach and Rokhlin.

On Nonlinear Delay Parabolic Equations

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In delay differential equations, the presence of the delay term causes the difficulties in analysis of the equation. Therefore, researchers recompense in numerical methods the lack of theoretical studies. One of the main methods used in this area is the finite difference method. Lu [1] studies monotone iterative schemes for finite-difference solutions of reaction-diffusion systems with time delays and gives modified iterative schemes by combing the method of upper-lower solutions and the Jacobi method or the Gauss–Seidel method. Gu and Wang [2] construct a linearized Crank–Nicolson difference scheme to solve a type of variable coefficient delay partial differential equations. Ashyralyev and Sobolevskii [3] consider the initial-value problem for linear delay partial differential equations of the parabolic type and give a sufficient condition for the stability of the solution of this initial-value problem. Ashyralyev and Agirseven [4]–[8] investigated stability of the initial boundary value problems for delay parabolic equations and of difference schemes for the approximate solutions to delay parabolic equations.

In present paper, we consider the initial value problem for a nonlinear differential equation,

$$\begin{cases} \frac{du}{dt} + Au(t) = f(u(t), u(t-w)), t \geqslant 0, \\ u(t) = \varphi(t), -w \leqslant t \leqslant 0, \end{cases}$$

in a Banach space E with a strongly positive operator A. A theorem on the existence of a unique bounded solution to this problem is established. The first and second accuracy order difference schemes for solutions of nonlinear time delay parabolic equations are presented. Theoretical results are supported by numerical experiments.

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On a Problem with Oblique Derivatives

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The paper is devoted to the boundary value problem for the three-dimensional Laplace equation with nonlocal boundary conditions, which, in particular, leads to the oblique derivative problem. We note that the predetermined inclination to the border may be tangential to the boundary at any number of points, lines, and surfaces.

The research method is as follows. The basic relations are constructed with the help of the fundamental solution of the equation. Necessary conditions are selected from these relations (the maximum number of linearly independent conditions). These conditions include singular integrals containing non general singularities. They have a special way of regularization due to the authors. This regularization is carried out with the use of the given boundary conditions.

The obtained regular expressions lead us to a sufficient condition for the Fredholm property of the given boundary value problems for elliptic (Cauchy-Riemann equations, Laplace etc.), parabolic, hyperbolic, composite, mixed type (and atypical) equations [1].

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Stationary Problem of Complex Heat Transfer in a System of Semitransparent Bodies with Boundary Conditions of Reflection and Refraction of Radiation

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We consider the boundary value problem

$$-\operatorname{div}(\lambda(x,u)\nabla u) + 4\pi \int_0^\infty \varkappa_\nu k_\nu^2 h_\nu(u)\,d\nu = \int_0^\infty \varkappa_\nu \int_\Omega I_\nu\,d\omega\,d\nu + f, \quad x \in G, \tag{1}$$

$$\omega \cdot \nabla I + (\varkappa_\nu + s_\nu) I_\nu = s_\nu \mathcal{S}_\nu(I_\nu) + \varkappa_\nu k_\nu^2 h_\nu(u), \quad (\omega, x) \in \Omega \times G, \tag{2}$$

$$\lambda(x,u)\nabla u \cdot n = 0, \quad x \in \partial G, \tag{3}$$

$$I_\nu|_{\Gamma^-} = \mathfrak{B}_\nu(I_\nu|_{\Gamma^+}), \quad (\omega, x) \in \Gamma^-,\ 0 < \nu < \infty, \tag{4}$$

describing complex (radiation-conductive) heat transfer in a system of semitransparent bodies $G = \bigcup_{j=1}^{m} G_j$, $G_j \subset \mathbb{R}^3$, separated by vacuum. The physical meaning of the sought-for functions $u(x)$ and $I_\nu(\omega, x)$ is the absolute temperature at a point x and the radiation intensity at frequency ν in the direction $\omega \in \Omega = \{\omega \in \mathbb{R}^3 \mid |\omega| = 1\}$.

Hear $0 < \lambda(x,u)$, $0 \leqslant \varkappa_\nu$, $0 \leqslant s_\nu$, and $1 < k_\nu$ are the heat conduction coefficient, absorbtion coefficient, scattering coefficient, and refractive index. The function $h_\nu(u)$ for $u > 0$ represents Planck's spectral distribution: $h_\nu(u) = \dfrac{2\nu^2}{c_0^2} \dfrac{\hbar\nu}{\exp(\hbar\nu/(ku)) - 1}$. In radiation transfer equation (2), $\mathcal{S}_\nu$ denote the scattering operator:

$$\mathcal{S}_\nu(\varphi)(\omega, x) = \int_\Omega \theta_{j,\nu}(\omega' \cdot \omega)\varphi(\omega', x)\,d\omega', \quad (\omega, x) \in \Omega \times G_j, \quad 1 \leqslant j \leqslant m.$$

Boundary condition (4) describes reflection and refraction of radiation on the boundary ∂G. Here $\Gamma^- = \{(\omega, x) \in \Omega \times \partial G \mid \omega \cdot n(x) < 0\}$, $\Gamma^+ = \{(\omega, x) \in \Omega \times \partial G \mid \omega \cdot n(x) > 0\}$.

In [1] and [2], the existence of a unique weak solution to problem (1)–(4) is established, the comparison theorem is proved, estimates for the weak solution are derived, and its regularity is established.

This work was supported by the Russian Science Foundation (project No. 14-11-00306).

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Error Estimates for Solutions of Lower Dimensional Obstacle Problems

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We consider elliptic variational inequalities generated by obstacle type problems with thin obstacles. The exact mathematical formulation is as follows. Let Ω be an open, connected and bounded domain in $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$, and let $\mathcal{M}$ be a smooth $(n-1)$-dimensional manifold in $\mathbb{R}^n$, which divides Ω into two Lipschitz subdomains Ω_+ and Ω_-. For given functions $\psi : \mathcal{M} \to \mathbb{R}$ and $\varphi : \partial\Omega \to \mathbb{R}$ satisfying $\varphi \geqslant \psi$ on $\mathcal{M} \cap \partial\Omega$, we minimize the functional

$$J(v) = \frac{1}{2} \int\limits_{\Omega} |\nabla v|^2 dx \tag{1}$$

over the closed convex set $\mathbb{K} = \left\{ v \in H^1(\Omega) : \quad v \geqslant \psi \text{ on } \mathcal{M} \cap \Omega, \quad v = \varphi \text{ on } \partial\Omega \right\}$. Here, $\varphi \in H^{1/2}(\partial\Omega)$ and the function ψ is supposed to be smooth. The existence of a unique minimizer $u \in \mathbb{K}$ is well known.

Problem (1) is called the *thin obstacle problem* associated with the thin obstacle ψ. This kind of unilateral problem appears in miscellaneous applications. One can find it, e.g., in boundary temperature control problems and in analysis of flow through semi-permeable walls subject to the phenomenon of osmosis, as well as in financial mathematics if the random variation of an underlying asset changes discontinuously. In many respects, problem (1) differs from the classical obstacle problem where the constrain $v \geqslant \psi$ is imposed on the entire domain Ω.

Thin obstacle problems have been actively studied from the early 1970s. These studies were mainly focused either on regularity of minimizers or on properties of the respective free boundaries.

It should be pointed out that we are concerned with a different question. Our analysis is focused not on properties of the exact minimizer, but on estimates of the distance (measured in terms of the natural energy norm) between u and any function

$v \in \mathbb{K}$. In other words, we wish to obtain estimates able to detect which neighborhood of u contains a function v (considered as an approximation of the minimizer). These estimates are fully computable, i.e., they depend only on v (which is assumed to be known) and on the data of the problem (u and the respective exact coincident set $\{u = \psi\}$ do not enter the estimate explicitly). For the classical obstacle problem (which solution is bounded in Ω from above and below by two obstacles) estimates of such a type was obtained in ([1]).

The talk is based on the results from [2] obtained in collaboration with Sergey I. Repin. This work was partly supported by the Russian Foundation for Basic Research (RFBR) through grant 15-01-07650.

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New Regularity Results for Nonlinear Parabolic Systems. The $A(t)$-Caloric Approximation Method

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We discuss the latest regularity results for second order quasilinear and nonlinear parabolic systems. The partial Hölder continuity of weak solutions and their gradients is proved under the optimal assumptions on the principal nondiagonal matrix, namely, the integral VMO-continuity in the space variables and just boundedness in time. The results are based on the A(t)-caloric approximation method (the matrix A(t) has the entries from L^{∞}-space on the time interval).

This approach was suggested and applied to parabolic systems of different classes in joint works by Arkhipova A. A. and her colleagues O. John and J. Stara from Charles University (Prague).

This research has been financially supported by the Russian Foundation for Basic Research (RFBR), Grant 15-01-07650.

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The Stoke Phenomenon for an Irregular Gelfand–Kapranov–Zelevinsky System Associated with a Rank One Lattice

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A keystone in the analytic theory of ordinary differential equations is a construction called the Riemann-Hilbert correspondence. It is the correspondence between a system of linear differential equations and the monodromy data which characterize the behavior of solutions near singular points of the system. The construction of the monodromy data in the case of a regular singular point differs from that one in the case of an irregular singular point.

The theory of irregular singularities has been generalized in the recent papers by T. Mochizuki, C. Sabbah.

Even in the case of dimension one, which is the case of an ODE, not so many examples of explicit description of the Stokes phenomenon in the irregular case are known. Despite some artificial examples, one can list the Jordan–Pochgammer equation, the parabolic cylinder equation, the equation for the Eiri functions, and the equation for the generalized hypergeometric functions associated with a series $F_{p,q}$. At the same time, there are no examples of explicit description of the Stokes phenomenon in the multidimensional case.

In the present paper, we give such description for an irregular Gelfand–Kapranov–Zelevinsky (GKZ for short) system associated with a lattice of rank 1. The key step in the investigation of this GKZ system is the fact that its solutions can be expressed through the generalized hypergeometric functions of the scalar argument, associated with the series $F_{p,q}$.

This example is interesting not only as an example of description of the Stokes phenomenon in the multidimensional case but also as a complete description of singular behaviour for some class of GKZ systems.

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On the Solvability of a System of Forward-Backward Linear Equations with Unbounded Operator Coefficients

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Consider the linear system of forward-backward evolution equations

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & -A^* \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \begin{matrix} x(0) = x_0 \\ y(T) = Gx(T) \end{matrix} \quad t \in [0, T].$$

Here A is an accretive (unbounded) operator, while B, C, and G are non-negative self-adjoint (unbounded) operators.

Both the solvability of the system and the solvability of the related differential operator Riccati equation are proved in a collection of Banach spaces.

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Stable Difference Schemes for a Third Order Partial Differential Equation

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The nonlocal boundary-value problem for a third order partial differential equation

$$\begin{cases} \frac{d^3u(t)}{dt^3} + A\frac{du(t)}{dt} = f(t), \quad 0 < t < 1, \\ u(0) = \gamma u(\lambda) + \varphi, \quad u'(0) = \alpha u'(\lambda) + \psi, |\gamma| < 1, \\ u''(0) = \beta u''(\lambda) + \xi, \quad |1 + \beta\alpha| > |\alpha + \beta|, 0 < \lambda \leqslant 1 \end{cases} \tag{1}$$

in a Hilbert space H with a self-adjoint positive definite operator A is considered.

A function $u(t)$ is a solution of problem (1) if the following conditions are satisfied:

(i) $u(t)$ is thrice continuously differentiable on the interval $(0, 1)$ and twice continuously differentiable on the segment $[0, 1]$.

(ii) The element $u'(t)$ belongs to $D(A)$ for all $t \in [0, 1]$, and the function $Au'(t)$ is continuous on $[0, 1]$.

(iii) $u(t)$ satisfies the equation and nonlocal boundary conditions (1).

We are interested in studying the stable difference schemes for the approximate solution of problem (1). Three-step difference schemes generated by Taylor's decomposition on four points for the approximate solution of problem (1) are presented [1]. Applying operator approach of [2], theorems on stability of these difference schemes are established. In applications, the stability estimates for the solution of difference schemes of approximate solution of three nonlocal boundary value problems for third order partial differential equations are obtained. Numerical results are provided.

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Integro-Differential Equations of Convolution Type with Monotone Nonlinearity

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Methods of the theory of maximal monotone operators [1] are used to prove global theorems on the existence and uniqueness of solutions for various classes (cf. [2]) of nonlinear convolution type integro-differential equations in the real space of 2π-periodic functions $L_p(-\pi,\pi)$.

Hereafter we assume that a given function $F(x,u)$ generating nonlinearity in the considered equations is defined for $x \in [-\pi,\pi]$ and $u \in \mathbb{R}$, has period 2π w.r.t. x and satisfies the Caratheodory conditions, namely, it is measurable in x for each fixed u and is continuous in u for almost all x. We denote by $L_p^+(-\pi,\pi)$ the set of all non-negative functions in $L_p(-\pi,\pi)$.

For example, we have the following theorem.

Theorem 1. *Let $1 < p < \infty$, $f(x) \in L_{p'}(-\pi,\pi)$, $p' = p/(p-1)$, $h(x) \in L_1(-\pi,\pi)$, and*

$$h_s(n) = \int_{-\pi}^{\pi} h(t) \cdot \sin(nt)dt \geqslant 0 \quad \forall n \in \mathbb{N}.$$

If for almost all $x \in [-\pi,\pi]$ and all $u \in \mathbb{R}$ the nonlinearity $F(x,u)$ satisfies the conditions

1. $|F(x,u)| \leqslant a(x) + d_1|u|^{p-1}$, where $a(x) \in L_{p'}^+(-\pi,\pi)$, $d_1 > 0$;

2. $F(x,u)$ is a nondecreasing function in the argument u for almost all $x \in [-\pi,\pi]$;

3. $F(x,u)\cdot u \geqslant d_2|u|^p - D(x)$, *where* $D(x)\in L_1^+(-\pi,\pi)$, $d_2>0$, *then the nonlinear integro-differential equation*

$$F(x,u(x)) + \int_{-\pi}^{\pi} h(x-t)u'(t)dt = f(x)$$

has a solution $u(x)\in L_p(-\pi,\pi)$ *with* $u'(x)\in L_{p'}(-\pi,\pi)$. *This solution is unique if* $F(x,u)$ *strictly increases in* u *for almost all* $x\in[-\pi,\pi]$ *fixed.*

The proof is based on the use of the equality

$$\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} h(x-t)\,u'(t)dt\right)u(x)dx = 4\pi\sum_{n=1}^{\infty} n\,h_s(n)\,|u_n|^2,$$

where

$$u_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\,\exp(-inx)\,dx.$$

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On Hyers–Ulam Stability of Third Order Linear Differential Equations Having Constant Coefficients

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Consider the following third order linear differential equations:

$$y^{(3)}(x) + ay''(x) + by'(x) + cy(x) = 0 \tag{1}$$

and

$$y^{(3)}(x) + ay''(x) + by'(x) + cy(x) = f(x), \tag{2}$$

where $y\in C^3([a,b])$, $f\in C([a,b])$, and $a,b,c\in\mathbb{R}$.

The stability of equation (2) has already been studied by M. R. Abdollahpour and A. Najati in [1]. This time, we employ a straightforward strategy similar to Xue [2] to prove the Hyers-Ulam stability of (2). We first apply the strategy in proving the stability of (1).

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Mixed Boundary Value Problem for Parabolic Systems on the Plane and Boundary Integral Equations

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We consider the mixed boundary value problem for one-dimensional (with respect to the spatial variable) parabolic systems with Dini-continuous coefficients in a curvilinear domain with nonsmooth lateral boundaries. The boundary condition of the first kind is posed on one lateral boundary, and the boundary condition of the second kind is posed on the second lateral boundary. By the method of boundary integral equations, developed in [1] for second-order parabolic systems on the plane, we construct a regular solution of this problem. The obtained solution is represented as a sum of simple layer parabolic potentials. We study the smoothness of this solution in the space of functions that are continuous together with the spatial derivative and the fractional derivative of order 1/2 with respect to "time" variable in the closed domain. We also establish estimates for the higher derivatives of the solution.

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The Komogorov $1/3$ Law and the 1984 Kato Theorem in the Zero Viscosity Limit

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The two is based on the two following observations. First, the notion of week convergence is the deterministic avatar of the notion of average in the statistical theory of turbulence. Second, in most cases even for some generation of homogeneous and isotropic turbulence, the basic effects are due to the boundary.

And it turns out that it is a theorem of T. Kato of 1984 that allows one to make through its diverse generalizations the more natural connection between these points of view.

In particular, it allows one to connect the loss of regularity with the anomalous energy dissipation and with the Onsager conjecture.

Stationary Solutions of the Flat Vlasov–Poisson System

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The flat Vlasov–Poisson system describes the evolution of the so-called "flat" galaxies (such as our Milky Way) which, with sufficient accuracy, can be considered to be spread in a plain. Stationary solutions satisfy the system

$$U \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0,$$

$$U(x) = -\int_{\mathbb{R}^2} \frac{\rho(y)}{|x-y|} dy, \quad \rho(x) = \int_{\mathbb{R}^2} f(x,v) dv, \quad x, v \in \mathbb{R}^2.$$

Here $f = f(x,v) \geqslant 0$ is the distribution function, $\rho = \rho(x) \geqslant 0$ the local density and $U = U(x) \leqslant 0$ the Newtonian potential. We prove the existence of wide classes of spherically symmetric solutions (f, ρ, U) with the property that ρ depends on $r = |x|$ and f has the from $f(x,v) = q(-E_0 - E)$, where $E_0 \geqslant 0$ is a constant and $E := U(x) + \frac{v^2}{2}$ is the local energy.

The first result is: *Every function $\rho = \rho(x)$ which is strictly decreasing on its support $[0, R]$ (and satisfies some other technical assumptions) can be considered as the local density of a stationary solution (f, ρ, U) of the type described above.*

The second question is: *For which function $q : \mathbb{R} \to [0, \infty)$, $q = 0$ on $(-\infty, 0]$, $q > 0$ on $(0, \infty)$ exist a constant $E_0 \geqslant 0$ and functions ρ and U such that $(f(x,v) := q(-E_0 - E)$, q, $U)$ is a stationary solution.*

We develop a numerical scheme to calculate ρ by approximations on a compact interval $[0, R]$.

This is a joint work with E. Jörn and Y. Li.

Analyticity in Time and Space for a Semilinear Cauchy Problem

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We consider the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u}(x,t) + \mathbf{P}\left(x, t, \dfrac{1}{\mathrm{i}}\dfrac{\partial}{\partial x}\right)\mathbf{u}(x,t) = \mathbf{f}\left(x, t; \mathbf{u}(x,t)\right) & \text{for } (x,t) \in \mathbb{R}^N \times (0,T), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases}$$

for a parabolic system of semilinar partial differential equations with an initial value $\mathbf{u}_0 \in \mathbf{B}^{s;p,p}(\mathbb{R}^N) \cap \mathbf{L}^2(\mathbb{R}^N)$. We make use of an abstract nonlinear Cauchy problem with an operator that satisfies the maximal L^p–regularity property to show the existence and uniqueness of the solution. The abstract case also yields the analyticity of the solution with respect to the time variable. We establish an a priori estimate for the solution of the Cauchy problem for the system of partial differential equations. With the help of this estimate we can construct a holomorphic extension of the solution to a complex domain and we obtain analyticity of the solution with respect to the space variable, as well. The results generalize known results for linear systems of partial differential equations introduced by P. Takáč in [1].

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Fast Growing Solutions of Linear Differential Equations With Dominant Coefficient of Lower μ_φ-Order

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In this paper, we deal with the growth of entire solutions of higher order linear differential equations with entire coefficients when the dominant coefficient is of μ_φ-order. The results presented in this paper mainly improve the corresponding results announced in the literature.

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Dynamics of Travelling Waves in the Parabolic Problem with Transformed Argument

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We consider the equations

$$u_t + u = \mu u_{xx} + K(1 + \gamma \cos Qu(x,t)),\ Qu(x,t) = u(x+h,t),\ t > 0, \tag{1}$$

with periodic boundary condition for $x \in [0, 2\pi]$. Here, $0 < \mu \ll 1 > 0$, $K > 0$, $0 < \gamma < 1$, $h = 2\pi p/q$, where $p > 0$, $q > 0$ are integer numbers and, in addition, q is even. We fix the smooth branches $w = w(K)$ of solutions of the equation $w = K(1+\gamma \cos w)$. Let $\Lambda(K) = -K\gamma \sin w(K) < -1$. There exist integer $m^+ < m^-$, $m^+ + m^- = q$ such that

$$\min_{0 \leqslant k \leqslant q} \cos(kh) = \cos(m^{\pm} h) = -\cos \pi/q.$$

Let us determinate $\widehat{K}$ from the condition $-1 + \Lambda(\widehat{K}) \cos(m^+ h) = 0$. We denote $w(\nu) = w(\widehat{K}+\nu)$, where $\nu > 0$ is the small parameter. The stability of the state $w(\nu)$ is determined by the roots of the characteristic equation for the corresponding linearized problem given by $\lambda_m(\mu, \nu) = -1 - \mu m^2 + (\widehat{\Lambda} - \nu)\exp(imh)$. We have the following infinite dimensional critical cases for these conditions: $\Im\lambda_{s^{\pm}}(0,0) = \pm\widehat{\Lambda} \sin m^+ h = \pm\omega_0$, $s^{\pm} = m^{\pm} + sq$, $s = 0, 1, 2, \ldots$, $\Re\lambda_{s^{\pm}}(\mu, \nu) = -\mu(m^{\pm} + sq)^2 - \widehat{\Lambda}^{-1}\nu$.

Theorem 1 (see [1])**.** *Let the relation* $\Re\lambda_{k^{\pm}}(\mu,\nu) > 0$ *hold. Then problem* (1) *has the following solutions:* $u_{k^{\pm}} = w(\nu) + 2\frac{\Re\lambda_{k^{\pm}}(\mu,\nu)}{-\Re c}\cos(\Im\lambda_{k^{\pm}} t + (m^{\pm} + kq)x) + O(|(\mu,\nu)|)$. *Here,* $c = \frac{1}{2}\exp(im^+h)\big(-\widehat{\Lambda} + (\widehat{\Lambda}\cot\widehat{w})^2(2(1 - \widehat{\Lambda}^{-1} + \exp(2im^+h)b^{-1})\big)$ *and* $b = 2i\omega_0 + 1 - \widehat{\Lambda}\exp(2im^+h)$. *We denote*

$$A_{k^+,s^-} = \begin{pmatrix} \Re\lambda_{s^-} - 2\Re\lambda_{k^+} - i\Im c\rho_{k^+} & \bar{c}\rho_{k^+} \\ c\rho_{k^+} & \Re\lambda_{((2k+s+1))^+} - 2\Re\lambda_{k^+} + i\Im c\rho_{k^+} \end{pmatrix}.$$

$$A_{k^+,s^+} = \begin{pmatrix} \Re\lambda_{s^+} - 2\Re\lambda_{k^+} - i\Im c\rho_{k^+} & \bar{c}\rho_{k^+} \\ c\rho_{k^+} & \Re\lambda_{(2k-s)^+} - 2\Re\lambda_{k^+} + i\Im c\rho_{k^+} \end{pmatrix}.$$

The solution $u_{k^{\pm}}$ *are asymptotically orbitally stable if and only if*

1. *for any* $s \geqslant 0$ *the matrix* $A_{k^{\pm},s^{\mp}}$ *is stable;*

2. *for any* $0 \leqslant s < k$ *the matrix* $A_{k^{\pm},s^{\pm}}$ *is stable.*

In accordance with this result, we have the interaction of travelling waves. It follows from this phenomenon that the number of stable travelling waves increases as μ/ν decreases.

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Probabilistic Interpretation of Solutions to the Cauchy Problem for Quasilinear Parabolic Systems

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Probabilistic approach reveals the intrinsic nature of systems of nonlinear parabolic equations. In other words, it allows one to treat these systems as macromodels of some phenomena and gives a way to construct the underlying micromodels. We consider the Cauchy problem for forward and backward quasilinear parabolic systems and investigate various types of their solutions, namely, classical, generalized, and viscosity solutions. We show connections between nonlinear parabolic systems and stochastic processes and construct probabilistic representations of solutions to the Cauchy problem. The probabilistic approach to systems of PDEs allows one to classify these systems as follows:

1. a class including advection-diffusion-reaction systems with diagonal principal parts and equal diffusion coefficients (corresponding to systems considered in [1]);
2. a class consisting of systems with diagonal entrance of the second order terms and different diffusion coefficients in each equation and with non diagonal terms of zero order (reaction-diffusion systems);
3. a class including both systems with diagonal second order terms having different coefficients along with non diagonal first order terms and systems with non diagonal second order terms (convection-diffusion-reaction systems and systems with cross-diffusion [2]).

To demonstrate the simplest of our results, let us consider the Cauchy problem

$$\frac{\partial u_m}{\partial s} + \frac{1}{2}\sum_{ijk=1}^{d} F_{ij}^{m}(x,u)\frac{\partial^2 u_m}{\partial x_i \partial x_j} + \sum_{i=1}^{d} a_i^m(x,u)\frac{\partial u_m}{\partial x_i} + \sum_{l=1}^{d_1} q_{lm}(x,u)u_l = 0, \tag{1}$$

$u_m(T) = u_{0m}, \quad m = 1,\dots,d_1$. We show that under certain conditions the function $u(s,x,m) = E[u_0(\xi_{s,x}(T),\gamma_{s,m}(T))]$ satisfies (1), where $\xi(\theta)$ solves the SDE

$$d\xi(\theta) = a^u(\xi(\theta),\gamma(\theta))d\theta + A^u(\xi(\theta),\gamma(\theta)dw(\theta), \quad \xi(s) = x \in R^d.$$

Here $a^{u,m}(x) = a_m(x, u(t,x)) \in R^d, A^{u,m}(x) = A_m(x, u(t,x)) \in R^d \otimes R^d$, $F_{ij} = \sum_k A_{ki}A_{jk}$, $w(t) \in R^d$ is a standard Wiener process and $\gamma(t) \in \{1, \ldots, d_1\}$ is a Markov chain with generator $Q = (q_{lm})_{l,m=1}^{d_1}$. Besides, we present examples of each of the above mentioned classes of quasilinear parabolic equations (see [3]–[5]).

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Stationary Solutions of the Vlasov System under External Magnetic Field in the Half-Space

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The Vlasov–Poisson equations play a significant role in contemporary science. These equations have applications to such areas as thermonuclear fusion, modeling and studying high-temperature rarefied plasma, radiation, neutral particles transportation, atmosphere optics etc. A mathematical model for evolution of the density distribution of ions and electrons in the two-component high temperature rarefied plasma is described by the Vlasov–Poisson system. One of the most important direction in investigation of the Vlasov–Poisson plasma system is related to stationary solutions. They were studied by J. Batt, K. Fabian, W. Faltenbacher, E. Horst, V. V. Vedenyapin, A. L. Skubachevskii and others.

We consider the stationary Vlasov–Poisson system in the half-space:

$$-\Delta\varphi(x) = 4\pi e \int\limits_{\mathbb{R}^3} \sum_\beta \beta f^\beta(x,p)\,dp, \qquad x \in \mathbb{R}^3_+, \tag{1}$$

$$\frac{1}{m_\beta}(p, \nabla_x f^\beta) + \beta e \left(-\nabla_x\varphi + \frac{1}{m_\beta c}[p, B(x)], \nabla_p f^\beta\right) = 0, \tag{2}$$
$$x \in \mathbb{R}^3_+,\ p \in \mathbb{R}^3,\ \beta = \pm 1$$

with the Dirichlet boundary condition

$$\varphi(x)|_{x_1=0} = 0, \qquad x' = (x_2, x_3) \in \mathbb{R}^2. \tag{3}$$

Here $\varphi = \varphi(x,t)$ is the potential of the self-consistent electric field, $f^\beta = f^\beta(x,p)$ is the density distribution function of either positively charged ions (for $\beta = +1$) or electrons (for $\beta = -1$) at a point x and with impulse p; ∇_x and ∇_p are the gradients with respect to x and p respectively; m_{+1} and m_{-1} are the ion and electron masses; e is the electron charge; c is the velocity of light; B is the external magnetic field induction; $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^3$; $[\cdot,\cdot]$ is the vector product in $\mathbb{R}^3$ and $\mathbb{R}^3_+ = \{x = (x_1, x_2, x_3 \in \mathbb{R}^3) : x_1 > 0\}$.

In order to provide plasma confinement, it is important to consider the two-component plasma model under an external magnetic field. Since the supports of charged-particle density distributions must not intersect the boundary, the external magnetic field ensures the existence of solutions to the Vlasov–Poisson equations supported at some distance from the boundary of the domain. Thus, in [1] and [2], for the case of an infinite cylinder, a stationary solution was constructed for which the charged-particle density distributions were supported in a strictly interior cylinder. Solutions were constructed as products of two cut-off functions.

For the case of the half-space and for a sufficiently large induction of the external magnetic field, we have constructed stationary solutions with the trivial potential of the electric field, supported at some distance from the hyperplane $x_1 = 0$.

We would like to emphasize that stationary solutions in [1, 2] did not depend on x_3. This fact can be interpreted in such a way that the plasma has infinite mass. Unlike the above solutions, here we construct the stationary solution compactly supported in the half-space, which means that the plasma mass is finite. In this case, the solution is constructed as the product of five cut-off functions whose arguments are first integrals describing the general solution of the Vlasov equation.

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Analytic Continuation of the Lauricella Function and Integration of the Associated System of PDEs

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One of generalizations of the Gaussian hypergeometric function $F(a,b;c;z)$ to the case of several complex variables $(z_1,\dots,z_N) =: \mathbf{z}$ is the Lauricella function $F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})$, which is defined by the N-multiple series (see [1], [2])

$$F_D^{(N)}(\mathbf{a};b,c;\mathbf{z}) := \sum_{|\mathbf{k}|=0}^{\infty} \frac{(b)_{|\mathbf{k}|}(a_1)_{k_1}\cdots(a_N)_{k_N}}{(c)_{|\mathbf{k}|}k_1!\cdots k_N!} z_1^{k_1}\cdots z_N^{k_N},$$

where b and $c \notin \mathbb{Z}^-$ are some scalar (complex-valued) parameters, $\mathbf{a} = (a_1, \ldots, a_N)$ is some vector-valued parameter, and $\mathbf{k} = (k_1, \ldots, k_N)$ is the multi-index with non-negative components; this series converges in the unit polydisk $\mathbb{U}^N$. Function $F_D^{(N)}$ satisfies to a system of N second order linear partial differential equations.

The problem of analytic continuation of the Lauricella function consists in representing this function outside the polydisk $\mathbb{U}^N$ in the form of a linear combination of particular solutions of the above mentioned system,

$$F_D^{(N)}(\mathbf{a};\, b,\, c;\, \mathbf{z}) \,=\, \sum_{j=0}^{N} A_j\, \mathcal{U}_{p,\,j}^{(1,\infty)}\, (\mathbf{a};\, b,\, c;\, \mathbf{z}),$$

where the coefficients A_j do not vanish simultaneousely and the expression on the right-hand side defines the function $F_D^{(N)}$ near the point $(\underbrace{1, \ldots, 1}_{p}, \underbrace{\infty, \ldots, \infty}_{(N-p)})$. We have solved this problem, found explicit analytic representation for the functions $\mathcal{U}_{p,\,j}^{(1,\infty)}$, and have proved that they constitute the complete system of solutions of the above mentioned system of partial differential equations.

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A Decomposition Result for Vector Fields in $\mathbb{R}^d$

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Given a vector field $\rho(1, \mathbf{b}) \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}^{d+1})$ such that $\mathrm{div}_{t,x}(\rho(1, \mathbf{b}))$ is a measure, we consider the problem of uniqueness of the representation η of $\rho(1, \mathbf{b})\mathcal{L}^{d+1}$ as a superposition of characteristics $\gamma : (t^-_\gamma, t^+_\gamma) \to \mathbb{R}^d$, $\dot{\gamma}(t) = \mathbf{b}(t, \gamma(t))$. We give conditions in terms of a local structure of the representation η on suitable sets in order to prove that there is a partition of $\mathbb{R}^{d+1}$ into disjoint trajectories $\wp_{\mathfrak{a}}$, $\mathfrak{a} \in \mathfrak{A}$, such that the PDE

$$\mathrm{div}_{t,x}\big(u\rho(1, \mathbf{b})\big) \in \mathcal{M}(\mathbb{R}^{d+1}), \qquad u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d),$$

can be disintegrated into a family of ODEs along $\wp_{\mathfrak{a}}$ with measure r.h.s. The decomposition $\wp_{\mathfrak{a}}$ is essentially unique. We finally show that $\mathbf{b} \in L^1_t(\mathrm{BV}_x)_{\mathrm{loc}}$ satisfies this local structural assumption and this yields, in particular, the renormalization property for nearly incompressible BV vector fields.

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On Differential Invariants of Quasilinear Second Order Ordinary Differential Equations

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Consider explicit ordinary differential equations $y^{(n)} = F(x, y, y', \ldots, y^{(n-1)})$. Following Lie, any ODE of the first order is point equivalent to $y' = 0$ and any ODE of the second order is contact equivalent to $y'' = 0$ in a neighborhood of a regular point. Thus the substantial issue is the classification of ODEs of the second order with respect to the action of point transformations. Such classification was obtained by A. Tresse [3] who calculated the algebra of differential subinvariants (the so-called differential parameters) and B. Kruglikov [2] who calculated the algebra of absolute differential invariants. Second order differential equations were also studied in many other works.

But there are some singular classes of ODE which can not be classified in terms of Tresse and Kruglikov. The most interesting one is the class of equations having a cubic polynomial with respect to the first derivative on the right-hand side. These equations are remarkable due to their relations with various geometrical issues such as projective connections, geodesics, and others. Such equations were intently studied in many works. This is due to the fact, in particular, that the solutions of these equations provide examples of special functions. Among them there are also the Painleve equations.

In his work we consider quasilinear second order differential equations, i.e. equations of the type $y'' + A(x, y)y' + B(x, y) = 0$. We will use the technique of differential invariants and jets (details can be found, for example, in [1]).

Theorem 1. *The symmetry group G of the quasilinear second order differential equations consists of transformations $x \mapsto \xi(x)$, $y \mapsto \eta(x)y + \zeta(x)$, where ξ, η, $\zeta \in C^\infty(\mathbb{R})$.*

Theorem 2. *The field of differential invariants for the action of the symmetry group $\widehat{G}$ on the quasilinear second order differential equations is generated by five differential invariants $J_1, \ldots, J_5$ of order 3 and two invariant derivations ∇_1 and ∇_2.*

Now consider the restrictions of the invariants $J_1, \ldots, J_5$ and their invariant derivatives $\nabla_j J_i$ on a given pair (A, B) of the coefficients of the differential equation $y'' + A(x, y)y' + B(x, y) = 0$. Then there are the following dependencies between them:

$$J_i = \mathcal{F}_i(J_1, J_2), \quad \nabla_j J_i = \mathcal{F}_{ij}(J_1, J_2). \tag{1}$$

Theorem 3. *Two non-degenerated quasilinear differential equations $y'' + A(x, y)y' + B(x, y) = 0$ and $y'' + \widetilde{A}(x, y)y' + \widetilde{B}(x, y) = 0$ are locally equivalent iff the sets of dependencies (1) coincide: $\mathcal{F}_i = \widetilde{\mathcal{F}}_i$ and $\mathcal{F}_{ij} = \widetilde{\mathcal{F}}_{ij}$.*

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A Priori Estimates and Ground States of Solutions of Some Emden–Fowler Equations with Gradient Term

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Here we consider the nonnegative solutions of equations in a punctured ball $B(0, R) \setminus \{0\} \subset \mathbb{R}^N$ or in $\mathbb{R}^N$, of the two types

$$-\Delta u = u^p |\nabla u|^q \tag{1}$$

where $p + q > 1$, and

$$-\Delta u = u^p + M |\nabla u|^q, \tag{2}$$

where $p, q > 1$ and $q > 1$ and $M \in \mathbb{R}$. We give new a priori estimates for the solutions and their gradients, and also present the Liouville-type results. We use the Bernstein technique and Osserman's or Gidas–Spruck's type methods. The most interesting cases correspond to $0 \leqslant q < 1$ for equation (1) and $q = 2p/(p + 1)$ for equation (2).

Convergence of the Solution of the Stefan Problem with Two Small Parameters $\varepsilon,\ \varkappa$ as $\varepsilon,\ \varkappa \to 0$

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We consider the two-phase multidimensional Stefan problem with two small parameters $\varepsilon > 0$ and $\varkappa > 0$ at the principal terms in the condition posed on the free boundary. This problem is studied in the Hölder spaces.

It is shown that the boundary layer does not appear as $\varepsilon \to 0,\ \varkappa \to 0$. The convergence of the solution of the perturbed problem with $\varepsilon > 0,\ \varkappa > 0$ as $\varepsilon \to 0,\ \varkappa > 0$; $\varepsilon > 0,\ \varkappa \to 0$; $\varepsilon \to 0,\ \varkappa \to 0$ to the solutions of the problems with $\varepsilon = 0,\ \varkappa > 0$; $\varepsilon > 0,\ \varkappa = 0$; $\varepsilon = 0,\ \varkappa = 0$ is proved without loss of smoothness of the given functions. Moreover, the partially or fully unperturbed problems are not being solved but their solutions are obtained from the original perturbed problem with $\varepsilon > 0,\ \varkappa > 0$.

Dynamical Averaging with Respect to a Non-Invariant Measure

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The classical Birkhoff ergodic theorem in its most popular version says that the time average along a single typical realization of a Markov process is equal to the space average with respect to the ergodic invariant distribution. This result is one of the cornerstones of the entire ergodic theory and its numerous applications. Two questions related to this subject will be addressed. First, how large is the set of typical realizations, in particular, in the case where there are no invariant distributions? Second, how is the answer connected to properties of the so called natural measures (limits of images of "good" measures under the action of the system)?

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On Some Properties of Nodal Solutions for Quasilinear Elliptic Problems

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Let Ω be a ball or a spherical shell in $\mathbb{R}^N$, $N \geqslant 2$. Consider the boundary value problem

$$-\Delta_p u = f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $p > 1$. The nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is assumed to be superlinear and subcritical (see [2]), with the model case $f(u) = |u|^{q-2}u$, where $1 < p < q < p^*$.

Problem (1) corresponds to the C^1 energy functional $E : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined as

$$E[u] = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \int_\Omega F(u)\, dx, \quad \text{where } F(s) := \int_0^s f(t)\, dt.$$

By definition, weak solutions of (1) are critical points of E.

We are interested in studying properties of *least energy sign-changing* (equivalently, *nodal*) *solutions* of (1). Such solutions can be defined as minimizers of the energy functional E over the *nodal Nehari set*

$$\mathcal{M} := \{u \in W_0^{1,p}(\Omega) : \, u^\pm \not\equiv 0, \; E'[u]u^\pm = 0\},$$

where $u^\pm := \max\{\pm u, 0\}$. It was proved in [1] that any least energy sign-changing solution of (1) is nonradial whenever $p = 2$. We generalize this fact to any $p > 1$ using different arguments based on shape optimization techniques.

Theorem 1 (see [2]). *Any minimizer of E over $\mathcal{M}$ is nonradial and has precisely two nodal domains.*

This result appears to be a corollary of a more general fact about monotonicity of energy levels of least energy sign-changing solutions of (1) in eccentric spherical shells with respect to the eccentricity.

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Current Challenges in Mathematical Modelling of Immune Processes

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Immunology as a scientific discipline studies the response of an organism to antigenic invasion, the recognition of self and nonself, and all the biological, chemical and physical aspects of immune processes. Nonlinearity, threshold effects, feedback control loops, delays, compartmental organization, multi-scale regulation inherent in the immune processes call for application of mathematics in modern immunology studies. Complex human diseases such as human immunodeficiency virus (HIV) infection require the development of multi-scale models of the virus-host interaction [1]. An integrative model of HIV infection based on a combination of ODEs, PDEs and agent-based description of cells, viruses and cytokines is presented. It is applied to study the sensitivities of infection dynamics to biophysical parameters characterizing the transport and interaction processes, providing a basis for designing efficient therapies.

Optimal treatment of virus infections requires application of multiple drugs which affects both the virus and the host organism physiology. Control approach for models of virus infections formulated with the help of delay differential equations on the base of optimal disturbances is described. The mathematical model of experimental infection of mice with lymphocytic choriomeningitis virus (LCMV), which is a gold standard in immunology, is considered. The state space of the model represents observable characteristics of the infection, i.e., the virus, precursor and effector T cells populations, and the cumulative viral load at time t: $U(t) = [V(t), E_p(t), E_e(t), W(t)]^T$. According to the model [2], their evolution is described by a system of nonlinear delay differential equations

$$\frac{d}{dt}U(t) = F(U(t), U(t-\tau), U(t-\tau_A)), \quad t \geqslant 0. \tag{1}$$

The initial value problem (IVP) requires that the $U(t)$ is specified for $-\tau_A \leqslant t \leqslant 0$. System (1) is positively invariant. The solution of the IVP exists and is unique globally on $[0, +\infty)$. A new method for constructing the multi-modal impacts on the immune system in the chronic phase of a viral infection, based on the mathematical models with delayed argument, is formulated. The so-called “optimal disturbances,” widely used in the aerodynamic stability theory for models without delays, are constructed for perturbing the steady states of the dynamical system in order to maximize the perturbation-induced response. An algorithm for computing the optimal disturbances is proposed. The concept of optimal disturbances is generalized to the systems with the delayed argument.

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Behavior of Strong Solutions to the Oblique Derivative Problem for Elliptic Second-Order Equations in a Domain with Boundary Conical Point

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We study the behavior of strong solutions to the oblique derivative problem for second-order elliptic equations in a neighborhood of a conical boundary point of an n-dimensional bounded domain.

Let $G \subset \mathbb{R}^n$ be a bounded domain with boundary ∂G that is a smooth surface everywhere except at the origin $\mathcal{O} \in \partial G$ and near $\mathcal{O}$ it is a conical surface. We consider the following semi-linear

$$\begin{cases} a^{ij}(x)u_{x_ix_j} + a^i(x)u_{x_i} + a(x)u(x) = h(u) + f(x), & x \in G, \\ h(u) = a_0(x)u(x)|u(x)|^{q-1}, \quad q \in (0,1), & \\ \frac{\partial u}{\partial \vec{n}} + \chi(\omega)\frac{\partial u}{\partial r} + \frac{1}{|x|}\gamma(\omega)u(x) = g(x), & x \in \partial G \setminus \mathcal{O}, \end{cases} \tag{SL}$$

and quasi-linear

$$\begin{cases} a^{ij}(x,u,u_x)u_{x_ix_j} + a(x,u,u_x) = 0, \ a^{ij} = a^{ji}, & x \in G, \\ \frac{\partial u}{\partial \vec{n}} + \chi(\omega)\frac{\partial u}{\partial r} + \frac{1}{|x|}\gamma(\omega)u = g(x), & x \in \partial G \setminus \mathcal{O}, \end{cases} \tag{QL}$$

elliptic problems, where $\vec{n}$ denotes the unit exterior normal vector to $\partial G \setminus \mathcal{O}$ and (r,ω) are spherical coordinates in $\mathbb{R}^n$ with pole $\mathcal{O}$; repeated indices are understood as summation from 1 to n. If $q = 1$, then problem (SL) becomes a linear one (see [1], [2]).

For problems (SL) and (QL), we establish exponents of decreasing rates of solutions near the conical boundary point, i.e. we show that $|u(x)| = O\left(|x|^{\lambda}\right)$ with exact exponents λ (see [3], [4]).

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Mathematical Theory Problems of Diffraction on Bodies with Metal and Dielectric Edges

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It is well known that the presence of salient points in the boundary can lead to singularities of solutions and to difficulties in the use of numerical methods.

Elliptic boundary value problems describing the electromagnetic field in waveguides having edges on their boundaries were studied in [1], where an asymptotic representation of the solution was obtained by the method first offered by V.A. Kondrat'ev for solutions of elliptic differential equations [2].

In this work, the method of construction of asymptotic representation for the solution in a vicinity of a silent point is applied to the problem of electromagnetic diffraction on a dielectric wedge structure. Let the permeability of the medium filling the waveguide be $\mu(x,y) \equiv 1$, while the permittivity $\varepsilon(x,y)$ be a piecewise constant real function. The field satisfies altogether the Maxwell equations system

$$\begin{cases} \mathrm{rot}\mathbf{E} = ik\mathbf{H}, \\ \mathrm{rot}\mathbf{H} = -ik\varepsilon\mathbf{E}, \end{cases} \tag{1}$$

where $k = \dfrac{\omega}{c}$ is the wave vector, and the conjugation conditions are the following:

$$\begin{array}{ll} [(\mathbf{H}\cdot\mathbf{n})]|_C = 0, & [(\mathbf{H}\times\mathbf{n})]|_C = 0, \\ [(\varepsilon\mathbf{E}\cdot\mathbf{n})]|_C = 0, & [(\mathbf{E}\times\mathbf{n})]|_C = 0, \end{array} \tag{2}$$

where C is the permittivity discontinuity plane and $\mathbf{n}$ is the normal vector to the plane of discontinuity. The field is represented in the form of the sum of the incident plane wave and the diffracted field,

$$\mathbf{E} = \mathbf{E_0} + \widetilde{\mathbf{E}}, \quad \mathbf{H} = \mathbf{H_0} + \widetilde{\mathbf{H}}.$$

Sommerfeld's conditions for the diffracted field at infinity and Meixner's conditions for the whole field in the vicinity of the edge are applied.

The radial component of an electric field asymptotic representation is

$$E_r(r,\varphi) = \sum_{0<\nu_k<1} r^{\nu_k-1}\left\{C_k\cos\left[(\pi-\varphi)\nu_k\right] + D_k\cos\left[(\pi-|\omega_0-\varphi|)\,\nu_k\right]\right\} + \Re(r,\varphi),$$

where ν_k are the roots of the equations $\sin\pi\nu_k \pm \alpha\sin(\pi\nu_k - \nu_k\omega_0) = 0$ and ω_0 is the angle of the dielectric edge.

An asymptotic representation for electromagnetic field in the vicinity of the metal-dielectric edge in a waveguide has also been constructed.

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Method of Discrete Sources for Scattering Problem in Plane Channel with Sharp Corners

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Considered waveguides with sharp corners, it is very difficult to find a suitable numerical method due to singularities of an electromagnetic field at the corners. The most of the methods and commonly used numerical codes (packages) are limited in accuracy. The method of discrete sources (MDS) seems to be the most promising one for channels with sharp corners.

In the MDS, the unknown is sought as a linear combination of the Green (source) functions for some envelope domain (with respect to the domain considered). Plugging such a representation into the boundary conditions, we arrive at a set of linear algebraic equations (SLAE). The resulting matrices of SLAE for sharp boundaries are often very large, that is why the most of numerical algorithms are time consuming and their accuracy may be limited. Effectiveness of the MDS depends essentially on the source placement. A new "dipole" source allocation in the neighbourhood of the sharp points is suggested. It requires substantially smaller matrices and leads to significantly higher accuracy in comparison with other methods.

The model plane scattering problem in a strip region with a sharp ledge is considered. It is governed by the Helmholtz equation together with the Neumann condition on the boundary and the radiation condition at infinity. Numerical results demonstrate the effectiveness of the proposed idea in comparison with some other commonly used numerical codes (packages).

Lie Group Analysis of an Optimization Problem for a Portfolio with an Illiquid Asset

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Working in Merton's optimal consumption framework with continuous time, we consider an optimization problem for a portfolio with an illiquid, a risky and a risk-free asset. Our goal in this paper is to carry out a complete Lie group analysis of PDEs describing value function and investment and consumption strategies for a portfolio with an illiquid asset that is sold in an exogenous random moment of time T with a prescribed liquidation time distribution. The problem of such type leads to the three-dimensional nonlinear Hamilton–Jacobi–Bellman (HJB) equations. Such equations are not only tedious for analytical methods but are also quite challenging from the numeric point of view. To reduce the three-dimensional problem to a two-dimensional one or even to an ODE, one usually uses some substitutions, yet the methods used to find such substitutions are rarely discussed by the authors.

We use two types of utility functions: general HARA type utility and logarithmic utility. We carry out the Lie group analysis of the both three-dimensional PDEs and are able to obtain the admitted symmetry algebras. Then we prove that the algebraic structure of the PDE with logarithmic utility can be seen as a limit of the algebraic structure of the PDE with HARA-utility as $\gamma \to 0$. Moreover, this relation does not depend on the form of the survival function $\overline{\Phi}(t)$ of the random liquidation time T. We find the admitted Lie algebra for a broad class of liquidation time distributions in the cases of HARA and log utility functions and formulate corresponding theorems for all these cases.

We use the Lie algebras found to obtain reductions of the studied equations. Some of similar substitutions were used in other papers before, whereas others are new to our knowledge. This method gives us the possibility to provide a complete set of non-equivalent substitutions and reduced equations.

We also show in [1] that if and only if the liquidation time defined by a survival function $\overline{\Phi}(t)$ is distributed exponentially, then for both types of the utility functions we get an additional symmetry. We prove that both Lie algebras admit this extension, i.e. we obtain the four-dimensional L_4^{HARA} and L_4^{LOG}, correspondingly, for the case of exponentially distributed liquidation time. We list reduced equations and corresponding optimal policies that tend to the classical Merton policies as illiquidity becomes small.

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On Spectral Gaps for Some Periodic Operators in a Strip

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Let $x = (x_1, x_2)$ be Cartesian coordinates in $\mathbb{R}^2$, $\Pi := \{x : 0 < x_2 < \pi\}$, and ε be a sufficiently small positive number. We denote $\square_\varepsilon := \{x : |x_1| < \varepsilon\pi,\, 0 < x_2 < \pi\}$ and $\mathcal{L}_\varepsilon$ a symmetric operator in $L_2(\square_\varepsilon)$ bounded for each considered value of ε. Since the restrictions of the functions in $L_2(\Pi)$ to $\square_\varepsilon$ belong to $L_2(\square_\varepsilon)$, the operator $\mathcal{L}_\varepsilon$ can be applied to the functions in $L_2(\Pi)$. The result of the action is then extended by zero outside $\square_\varepsilon$. After such continuation, the operator $\mathcal{L}_\varepsilon$ can be regarded as acting in $L_2(\Pi)$. Let $\mathcal{S}(n)$ be the shift operator in $L_2(\Pi)$: $(\mathcal{S}(n)u)(x) = u(x_1 - 2\varepsilon\pi n, x_2)$. By $\mathcal{V}_\varepsilon$ we denote the following operator in $L_2(\Pi)$:

$$\mathcal{V}_\varepsilon := \sum_{n\in\mathbb{Z}} \mathcal{S}(-n)\mathcal{L}_\varepsilon\mathcal{S}(n).$$

This operator is symmetric, bounded and periodic; the latter is understood in the sense of the identity

$$\mathcal{V}_\varepsilon\mathcal{S}(p) = \mathcal{S}(p)\mathcal{V}_\varepsilon, \quad p \in \mathbb{Z}.$$

We consider the periodic operator in the strip Π

$$\mathcal{H}_\varepsilon := -\Delta + \mathcal{V}_\varepsilon \tag{1}$$

subject to the Dirichlet condition. It is treated as an unbounded operator in $L_2(\Pi)$ on the domain $\mathring{W}_2^2(\Pi)$, which is the Sobolev space of the functions in $W_2^j(\Pi)$ vanishing on $\partial\Pi$. This operator is self-adjoint and lower-semi-bounded.

We denote

$$\omega_{\mathcal{L}_\varepsilon} := \sup_{\substack{u\in L_2(\square_\varepsilon)\\ \|u\|_{L_2(\square_\varepsilon)}=1}} (\mathcal{L}_\varepsilon u, u)_{L_2(\square_\varepsilon)} - \inf_{\substack{u\in L_2(\square_\varepsilon)\\ \|u\|_{L_2(\square_\varepsilon)}=1}} (\mathcal{L}_\varepsilon u, u)_{L_2(\square_\varepsilon)}.$$

Let $\sigma(\cdot)$ denote the spectrum of an operator and $[\cdot]$ an integer part of a number.

Our main result is as follows.

Theorem 1. *Let* $\varepsilon \leqslant \varepsilon_0$, $\varepsilon^2\omega_{\mathcal{L}_\varepsilon} \leqslant b_0$, *where*

$$6\varepsilon_0 + \frac{\pi b_0}{4} \leqslant 2A_0, \quad A_0 := \frac{3}{128} - \frac{5\sqrt{14}}{1792}.$$

Denote

$$K_\varepsilon := \frac{A_0 + \sqrt{A_0^2 - \frac{\pi}{2}b_0\varepsilon - 4\varepsilon^2}}{2\varepsilon}.$$

Then the part of the spectrum

$$\left(-\infty, \frac{([K_\varepsilon]+1)^2}{\varepsilon^2}\right] \cap \sigma(\mathcal{H}_\varepsilon)$$

of the operator $\mathcal{H}_\varepsilon$ *contains no internal gaps.*

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Pseudo-Suspensions and Anosov–Katok Fast Approximation Method

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In our talk, we present a method of pseudo-suspensions generalizing classical suspension flows. Combined with Handel's application of the Anosov-Katok fast approximation method, this method allows one to construct the first example of hereditarily indecomposable continuum supporting homeomorphism of finite non-zero entropy. This way we solve the long time open question and provide at the same time a new tool in construction of dynamical systems with invariant sets of complicated structure. If time permits, we also comment on other possible applications of this technique.

Spectral Stability of Solitary Waves in the Nonlinear Dirac Equation in the Nonrelativistic Limit

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We consider the nonlinear Dirac equation in $\mathbb{R}^n$, $n \geqslant 1$, with the scalar self-interaction, known as the Soler model [4, 6]:

$$\mathrm{i}\partial_t \psi = D_m \psi - f(\psi^* \beta \psi)\beta\psi, \qquad \psi(x,t) \in \mathbb{C}^N, \quad x \in \mathbb{R}^n,$$

where the Dirac operator is $D_m = -\mathrm{i}\alpha \cdot \nabla + \beta m$, $m > 0$, with the self-adjoint $N \times N$ Dirac matrices $\alpha = (\alpha^\jmath)_{1 \leqslant \jmath \leqslant n}$ and β chosen to have $D_m^2 = -\Delta + m^2$. The nonlinearity is represented by a real-valued function $f \in C^1(\mathbb{R}\backslash\{0\})$ such that $f(\tau) = |\tau|^k + O(|\tau|^K)$ for $\tau \to 0$, with $0 < k < K$.

We study the point spectrum of the linearization at a solitary wave solution $\phi_\omega(x)e^{-\mathrm{i}\omega t}$, focusing on the spectral stability, that is, the absence of eigenvalues with nonzero real part. For $n = 1$ and $n \geqslant 3$, we prove the spectral stability of solitary waves in the nonrelativistic limit $\omega \lesssim m$ for the charge-subcritical cases $k \lesssim 2/n$ and for the "charge-critical case" $k = 2/n$ (if $K > 4/n$). For technical reasons, we can not consider the values $k \gtrsim 0$, and we only have partial results in the dimension $n = 2$.

An important part of the stability analysis is the proof of the absence of bifurcations of nonzero-real-part eigenvalues from the embedded threshold points at $\pm 2m\mathrm{i}$. Our approach is based on constructing a new family of exact bi-frequency solitary wave solutions $\phi_\omega(x)e^{-\mathrm{i}\omega t}+\chi_\omega(x)e^{\mathrm{i}\omega t}$ in the Soler model, on using this family to determine the multiplicity of $\pm 2\omega\mathrm{i}$ eigenvalues of the linearized operator, and on the analysis of the behaviour of "nonlinear eigenvalues" being characteristic roots of holomorphic operator-valued functions [5].

The talk is based on papers [1, 2] and preprint [3].

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Phase-lock in the Josephson Effect Model and Double Confluent Heun Equations

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The talk is focused at the family of double confluent Heun equations

$$z^2E''+(nz+\mu(1-z^2))E'+(\lambda-\mu nz)E=0; \quad n,\lambda,\mu\in\mathbb{C},\ \mu\neq 0. \tag{1}$$

This is a classical family of linear differential equations in complex time z that have only two singular points (zero and infinity): both of them are irregular. The restriction of this family to the real parameters and real time τ, $z=e^{i\tau}$, is the linearization of the family of non-linear differential equations on the two-torus

$$\omega\dot{\phi}=-\sin\phi+l\omega+2\mu\omega\cos\tau, \quad \dot{\tau}=1 \tag{2}$$

used in the model of the overdamped Josephson effect in superconductivity. Here

$$l=n-1,\ 4\omega^2(\lambda+\mu^2)=1,\ v(z)=e^{-\mu z}E(z),\ 2i\omega z v'(z)e^{i\phi(\tau(z))}=v(z).$$

The parameter ω has the physical meaning of the exterior force frequency. We restrict this family to a hypersurface $\omega = const > 0$ and consider the function $\rho = \rho(l, \mu)$ that associates to any real values of the parameters the rotation number of the corresponding flow of system (2). The *phase-lock areas* are the level sets of the function ρ that have non-empty interiors. The problem to describe the structure of the phase-lock areas is motivated by physical problems on the structure of the super-current through the Josephson junction. For general families of dynamical systems on the two-torus, the problem to describe the boundaries of the phase-lock areas is known to be very complicated. In the present talk, we will describe the results on the phase-lock areas obtained via methods of complex analysis that essentially use specific properties of the underlying real dynamics of system (2). In this system, the phase-lock areas exist only for integer values of the rotation number. The complement to them is an open set U. The restriction $\rho|_U$ is an analytic submersion that induces its fibration by analytic curves. Our main result is an explicit transcendental functional equation relating the monodromy eigenvalues and the parameters of the Heun equations (1). Its proof is based on studying the corresponding linear three-term relations and applying ideas of hyperbolic dynamics to them. As an application, we obtain the description of the union of boundaries of the phase-lock areas as solutions of an explicit functional equation obtained from the previous one. We formulate open questions and conjectures, including questions about physically important asymptotics, as $\omega \to 0$.

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The Space of Symmetric Squares of Hyperelliptic Curves: Infinite-Dimensional Lie Algebras and Polynomial Integrable Dynamical Systems on $\mathbb{C}^4$

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We construct Lie algebras of vector fields on universal bundles of symmetric squares of hyperelliptic curves of genus $g = 1, 2, \ldots$, see [1]. For each of these Lie algebras, the Lie subalgebra of vertical fields has two commuting generators, while the generators of the Lie subalgebra of projectable vector fields determines the canonical representation of the Lie subalgebra with generators L_{2q}, $q = -1, 0, 1, 2, \ldots$, of the Witt algebra. The vertical vector fields yield two commuting integrable polynomial dynamical systems on $\mathbb{C}^4$, while the projectable fields provide us with the Lie algebra of derivations of the solutions with respect to the curve parameters. The method can be extended to higher symmetric powers and more general algebraic curves.

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Conditional Measures of Determinantal Point Processes: the Gibbs Property and the Lyons–Peres Conjecture

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Determinantal point processes arise in many different problems: spanning trees and Gaussian zeros, random matrices and representations of infinite-dimensional groups. How does the determinantal property behave under conditioning? The talk will first address this question for specific examples such as the sine-process, where one can explicitly write the analogue of the Gibbs condition in our situation. We will then consider the general case, where, in joint work with Yanqi Qiu and Alexander Shamov, proof is given of the Lyons–Peres conjecture on completeness of random kernels.

The talk is based on the preprint `arXiv:1605.01400` as well as on the preprint `arXiv:1612.06751` joint with Yanqi Qiu and Alexander Shamov.

Integral Transforms Framework to Solve the Field Equations of the Space-Time[1]

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Equivalence spaces modulo congruence relations on field solutions are constructed through integral transforms whose cohomology class $H^{\bullet}(\mathbb{M}, \mathcal{O})$ establishes equivalences between algebraic objects of an adequate $Oper$, on the Lie algebra $\tilde{\mathfrak{g}}$, and geometrical objects such as holomorphic complex bundles as moduli stacks of different physical phenomena of the space-time [1]. Likewise, the field solutions are constructed using specifically sheaves corresponding of differential operators on coherent $D-$ modules to generalize some specific cohomologies as for example, the

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generalizing of the $BRST-$ cohomology in quantum field theory, where the obtained development includes complexes of $D-$ modules of infinite dimension. Then the integrable systems class could be extended to the field equations and their solution through integral transforms for the space-time whose cohomology is an extension of $H^\bullet(\mathbb{M}, \mathcal{O})$ *(hypercohomology* [2]), and with it the measurement of many of their field observables [1]. Using the analysis on flat and curved differential operators, the integrability notion to these operators complete a classification of the differential operators for the different field equations on the base of *Verma modules* that are classification spaces of $SO(1, n+1)$, where elements of the Lie algebra $\mathfrak{sl}(1, n+1)$, are the differential operators of the equations to the space-time [3], [4]. The cosmological problem that exists is to reduce the number of field equations that are resoluble under the same gauge field *(Verma modules)* and to extend the gauge solutions to other fields using the topological groups symmetries that define their interactions. Likewise, is designed and determine a Penrose transform to achieve an isomorphism of type $H^\bullet(\mathbb{M}, \Omega^n \otimes \mathcal{L}) \cong Ker(U, \bar{\partial} + Q)$, where $\bar{\partial} + Q$ is the new connection considering the corresponding Langlands parameter to regularize the connection given by the global Langlands category of sheaves $Loc_{L_G}(\mathcal{D}^\times)$, to elements $Q \in \mathcal{D}_{BRST}$, in the Penrose transform problem when we want obtain the functor $\Phi + Geometrical\ \ Hypothesis$, to the extension problem of solutions [4]. From a point of view of the Deligne theorem, is to find a connection that involves the *Deligne connection* adding other connection such that ∇_s, satisfies $\nabla_{Deligne} + \nabla_s = Q_{BRST}(Oper)$, to certain $Oper$, calculated by a local geometrical Langlands correspondence, designed this, by the same integral transforms acting on ramifications of a category of Hecke $\mathcal{H}_{G,\lambda}, \forall\lambda \in \mathfrak{h}^*$, on G/B. The differential operators can be considered to this case, twisted differential operators on the moduli Bun_G.

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Sharp Spectral Stability Estimates for Higher Order Elliptic Differential Operators

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We consider the eigenvalue problem for the operator

$$Hu = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha \left(A_{\alpha\beta}(x) D^\beta u \right), \quad x \in \Omega,$$

subject to homogeneous Dirichlet or Neumann boundary conditions, where $m \in \mathbb{N}$, Ω is a bounded open set in $\mathbb{R}^N$ and the coefficients $A_{\alpha\beta}$ are real-valued Lipschitz continuous functions satisfying $A_{\alpha\beta} = A_{\beta\alpha}$ and the uniform ellipticity condition

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x) \xi_\alpha \xi_\beta \geqslant \theta |\xi|^2$$

for all $x \in \Omega$ and for all $\xi_\alpha \in \mathbb{R}, |\alpha| = m$, where $\theta > 0$ is the ellipticity constant.

We consider open sets Ω for which the spectrum is discrete and can be represented by means of a non-decreasing sequence of non-negative eigenvalues of finite multiplicity $\lambda_1[\Omega] \leqslant \lambda_2[\Omega] \leqslant \dots \leqslant \lambda_n[\Omega] \leqslant \dots$ Here each eigenvalue is repeated as many times as its multiplicity and $\lim\limits_{n\to\infty} \lambda_n[\Omega] = \infty$.

The aim is sharp estimates for the variation $|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]|$ of the eigenvalues corresponding to two open sets Ω_1, Ω_2 with continuous boundaries, described by means of the same *fixed* atlas $\mathcal{A}$.

There is vast literature on spectral stability problems for elliptic operators. However, very little attention has been devoted to the problem of spectral stability for higher order operators and in particular to the problem of finding explicit qualified estimates for the variation of the eigenvalues.

Our analysis comprehends *operators of arbitrary even order*, *with homogeneous Dirichlet or Neumann boundary conditions*, and *open sets admitting arbitrarily strong degeneration*.

Three types of estimates will be under discussion: for each $n \in \mathbb{N}$ there exists some $c_n > 0$ depending only on n, $\mathcal{A}, m, \theta$, and the Lipschitz constant L of the coefficients $A_{\alpha\beta}$, such that

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leqslant c_n d_{\mathcal{A}}(\Omega_1, \Omega_2),$$

where $d_{\mathcal{A}}(\Omega_1, \Omega_2)$ is the so-called *atlas distance* of Ω_1 to Ω_2;

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leqslant c_n \omega(d_{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2)),$$

where $d_{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2)$ is the so-called *lower Hausdoff–Pompeiu deviation* of the boundaries $\partial\Omega_1$ and $\partial\Omega_2$ and ω is the common modulus of continuity of $\partial\Omega_1$ and $\partial\Omega_2$; and, under certain regularity assumptions on $\partial\Omega_1$ and $\partial\Omega_2$,

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leqslant c_n \operatorname{meas}(\Omega_1 \Delta \Omega_2),$$

where $\Omega_1 \Delta \Omega_2$ is the symmetric difference of Ω_1 and Ω_2.

On Equivariant Boundary Value Problems and Some Applications

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Let Ω be an arbitrary bounded domain in the space $\mathbf{R}^n$ with the boundary $\partial\Omega$ and $\mathcal{L} = \sum_{|\alpha|\leqslant m} a_\alpha(x)D^\alpha, D^\alpha = (-i\partial)^{|\alpha|}/\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}$, $\alpha \in \mathbf{Z}_+^n, |\alpha| = \sum_k \alpha_k$ be some formally self-ajoint differential operation with smooth complex matrix coefficients $a_\alpha(x)$, i.e. their elements belong to $C^\infty(\overline{\Omega})$. Let L_0 and $L = (L_0)^*$ be the minimal and the maximal operators for $\mathcal{L}$ with domains $D(L_0)$ and $D(L)$, $C(L) = D(L)/D(L_0)$ be the boundary space, and $\Gamma : D(L) \to C(L)$ the factor-mapping. The boundary value problem $Lu = f, \Gamma u \in B \subset C(L)$ is called well-posed if the corresponding expansion $L_B = L|_{D(L_B)}$, $D(L_B) = \Gamma^{-1}B$ has a continuous two-sided inverse operator.

Let G be a Lie group acting smoothly in the closed domain $\overline{\Omega}$ and on the boundary $\partial\Omega$, and let this action preserve the volume of the domain. Let the differential operation $\mathcal{L}$ be invariant with respect to the group action, that is $g(\mathcal{L}u) = \mathcal{L}(gu)$. Then the spaces $D(L)$, $D(L_0), C(L)$ are invariant with respect to the action of the group G. The boundary value problem $Lu = f$, $\Gamma u \in B$, is called G-invariant if the space B is invariant with respect to the action of G. If the group G is compact, then, as is well known, the Hilbert representation space is decomposed into the direct sum of finite-dimensional invariant subspaces of irreducible representations. And if the group is also commutative, then such representations are one-dimensional. Let the representation space of the group G be the boundary space $C(L)$. If the group is compact, then we have the decompositions

$C(L) = \sum_{k=0}^\infty \oplus \tilde{C}^k$, $C(\ker L) = \sum_{k=0}^\infty \oplus C^k(\ker L)$, $B = \sum_{k=0}^\infty \oplus B^k$.

If our G-invariant boundary value problem is well-posed, then the decomposition into the direct sum $C(L) = C(\ker L) \oplus B$ appears as decompositions into the direct sums $C^k := C^k(\ker L) \oplus B^k = \sum_l \tilde{C}^{k_l}$ with finite-dimensional projectors $\Pi^k : C^k \to C^k(\ker L)$ along B^k, and thus the check of **well-posedness of the G-invariant boundary boundary value problem** is reduced to **verification of the following two properties**:

$$1)\ C^k(\ker L) \cap B^k = 0; \quad 2)\ \exists \varkappa > 0,\ \forall k,\ \|\Pi^k\|_{C^k} < \varkappa.$$

About applications. We investigate the spectrum of the general well-posed SO-equivariant boundary value problem for the Poisson equation in a disk and in a ball, distinguishing violation of well-posedness of the same problem for the Helmholz equation as violation of exactly the first property. For the fulfilment of the second one is a consequence of well-posedness of the problem for the Poisson equation. One more application is related to the quantum mechanics. We consider the Schrödinger equation for a hydrogen-like atom with Coulomb potential and non-point ball nucleus. The eigenvalues and eigenfunctions of the operator given by an arbitrary rotation-invariant boundary value problem on the spherical boundary of the nucleus are found. The eigenvalues prove to be independent on the choice of any such boundary value problem, being the same as for the point nucleus, although the corresponding eigenfunctions are essentially different.

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Chernoff Approximation of Solutions of Integro-Differential Evolution Equations Corresponding to Subordinate Markov Processes

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An evolution semigroup $\left(e^{tL}\right)_{t\geqslant 0}$ with a given generator L (on a given Banach space), on the one hand, allows one to solve an initial (or initial-boundary) value problem for the corresponding evolution equation $\frac{\partial f}{\partial t} = Lf$ and, on the other hand, defines the transition probability $P(t,x,dy)$ of an underlying Markov process $(\xi_t)_{t\geqslant 0}$ (if there is any) through $e^{tL}f(x) = \mathbb{E}^x[f(\xi_t)] = \int f(y)P(t,x,dy)$. One of the ways to construct a strongly continuous semigroup is given by the procedure of *subordination*. From two ingredients: an original strongly continuous contraction semigroup $(T_t)_{t\geqslant 0}$ and a convolution semigroup $(\eta_t)_{t\geqslant 0}$ (of measures) supported by $[0,\infty)$, this procedure produces the so-called *subordinate semigroup* $(T_t^\eta)_{t\geqslant 0}$ with $T_t^\eta := \int_0^\infty T_s\eta_t(ds)$. If the semigroup $(T_t)_{t\geqslant 0}$ corresponds to a Markov process $(X_t)_{t\geqslant 0}$, then the subordinate semigroup $(T_t^\eta)_{t\geqslant 0}$ corresponds to the random time-change of $(X_t)_{t\geqslant 0}$ by an independent a.s. increasing Lévy process (subordinator) related to $(\eta_t)_{t\geqslant 0}$. For example, spherically symmetric stable processes can be obtained via subordination of Brownian motion, and hence fractional Laplacians are generators of the corresponding subordinate heat semigroups.

If $(T_t)_{t\geqslant 0}$ is not known explicitly, then $(T_t^\eta)_{t\geqslant 0}$ is also unknown. Moreover, the generator L^η of $(T_t^\eta)_{t\geqslant 0}$ is not known explicitly too since L^η is given through the semigroup $(T_t)_{t\geqslant 0}$ itself by means of the identity

$$L^\eta f = -\sigma f + \lambda L_0 f + \int_{0+}^{\infty}(T_s f - f)\mu(ds),$$

where L_0 is the generator of $(T_t)_{t\geqslant 0}$ and (σ,λ,μ) are the Lévy characteristics of the subordinator. Therefore, L^η is an integro-differential operator, and the semigroup $(T_t^\eta)_{t\geqslant 0}$ resolves the corresponding integro-differential evolution equation $\frac{\partial f}{\partial t} = L^\eta f$.

In the talk, it is planned to present the technique of approximating subordinate semigroups $(T_t^\eta)_{t\geqslant 0}$ in the case when the original semigroups $(T_t)_{t\geqslant 0}$ are not known but are already approximated by means of the Chernoff Theorem. This theorem provides conditions for a family of bounded linear operators $(F(t))_{t\geqslant 0}$ to approximate the considered semigroup $\left(e^{tL}\right)_{t\geqslant 0}$ via the formula $e^{tL} = \lim_{n\to\infty}[F(t/n)]^n$. This formula is called *the Chernoff approximation of the semigroup* $\left(e^{tL}\right)_{t\geqslant 0}$ *by the family* $(F(t))_{t\geqslant 0}$. If families $(F(t))_{t\geqslant 0}$ are given explicitly, the expressions $[F(t/n)]^n$ can be directly used for calculations and hence for simulations of underlying stochastic processes. Moreover, if all operators $F(t)$ of a given family $(F(t))_{t\geqslant 0}$ are integral operators with elementary kernels (or pseudo-differential operators with elementary symbols), then the identity $e^{tL} = \lim_{n\to\infty}[F(t/n)]^n$ leads to a representation of the semigroup $\left(e^{tL}\right)_{t\geqslant 0}$ by limits of n-fold iterated integrals of elementary functions when n tends to infinity. Such representations are called *Feynman formulae*; the limits in the Feynman formulae usually coincide with functional (path) integrals with respect to probability measures (Feynman–Kac formulae) or with respect to Feynman pseudomeasures (Feynman path integrals). Therefore, the method of Chernoff approximation allows us also to establish new path-integral-representations for solutions of evolution equations; different Chernoff approximations (in the form of Feynman formulae) for the same semigroup allow us to establish relations between different path integrals. One further advantage is that the method is applicable for a broad class of evolution semigroups corresponding to different types of dynamics on different geometrical structures. For examle, the presented technique allows us to construct Chernoff approximations and, in particular, Feynman formulae for subordinated diffusions and subordinate Feller type jump processes in $\mathbb{R}^d$, metric graphs, and Riemannian manifolds.

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How HPV Causes Immune Suppression to Inflict Papilloma — A Control Based Approach

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Cervical cancer results when epithelial cells of cervix were infected by Human Papilloma Virus (HPV). Healthy Cervical cell is exposed to the free HPV virus during infection, although progression of cancer develops when the infected cells perform immunosuppression by escaping the immune vigilence. Dendritic cells play a crucial role in controlling the early infection of virus by performing Cytotoxic T Lymphocytes (CTL) generation with successful triggering of viral signatures to efficiently killing of the infected cells. New virus are produced from the dead infected cells upon lysis. In fact, the cancerous cells cannot produced new virus. However, dead cancerous cells were scavenged by the lysogenic Langerhans cells (LCs), a larger form of DCs, which phagocytose the dead cells present in epithelial layer of the cervix and stimulates the reservoir DCs for maturation and enhancing the CTL production. HPV can escape such regulation by hiding within host cells to express infectivity later.

To explore the implications, we have formulated our mathematical model system and studied the model by both analytical and numerical approaches consisting of six compartments to describe the interactions between Human Papilloma Virus and three classes of cervical cells (susceptible, infected, cancerous), Dendritic cells (DC) and cytotoxic T lymphocyte (CTL). In our formulated system, we carried out local stability analysis and estimate the basic reproduction ratio(R0). Numerical studies in this system carried out to screen the regulatory role of DC in containing the cancer progression by enhancing the CTL autolytic activity.

Our results reflects new insights on the disease progression and how we could restrict the cancer development by enhancing CTL cells through interaction with DCs. Here we extend our work and formulate a set of differential equations to study the effect of chemotherapy on cancer cells of an infected individuals and simultaneously enhance the CTL killing effect. We incorporate a control strategy through drug treatment, which reduces the infected cell population studied both in implicit and explicit forms. Our study reveals that imposing the control function can reduces the cancer cell population much more efficiently.

On Homogenization of Random Attractors

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In this talk, we consider autonomous and non-autonomous 3D Navier-Stokes systems and we assume that the right–hand sides $g\left(x, \frac{x}{\varepsilon}, \omega\right)$ or $g\left(x, \frac{t}{\varepsilon}, \omega\right)$ of the systems are random functions rapidly oscillating with respect to the spatial or time variables. Here ω is an element of the standard probability space $(\Omega, \mathcal{B}, \mu)$. The parameter $\varepsilon > 0$ characterizes the oscillation frequency.

In the second part of the talk, we study asymptotic behavior of trajectory attractors of autonomous reaction-diffusion systems with randomly oscillating terms (the right-hand side and the reaction coefficient).

Along with such systems, we also consider the corresponding homogenized 3D Navier–Stokes system with external force $g^{hom}(x)$, where $g^{hom}(x)$ is the mathematical expectation of $g\left(x, \frac{x}{\varepsilon}, \omega\right)$ or $g\left(x, \frac{t}{\varepsilon}, \omega\right)$ as $\varepsilon \to 0$, and the respective homogenized reaction-diffusion system with similar terms.

We prove that the trajectory attractor $\mathfrak{A}_\varepsilon$ of the system with randomly oscillating term converges almost surely as $\varepsilon \to 0$ to the trajectory attractor $\overline{\mathfrak{A}}$ of the homogenized system in an appropriate functional space.

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Multidimensional Shock Waves, Free Boundary Problems and Nonlinear PDEs of Mixed Type

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In this talk, we discuss some of the most recent developments in the analysis of multidimensional shock waves and related free boundary problems through several longstanding fundamental shock problems in continuum mechanics. The mathematical analysis of these free boundary problems involves dealing with several core difficulties we have to face in the analysis of partial differential equations (PDEs). These include nonlinear PDEs of mixed hyperbolic-elliptic type, nonlinear degenerate elliptic PDEs, nonlinear degenerate hyperbolic PDEs, corner singularities (especially when free boundaries meet the fixed boundaries where the nonlinear PDEs experience their degeneracy), among others. These difficulties also arise in many further fundamental problems in continuum mechanics, differential geometry, mathematical physics, materials science, and other areas. Some further developments, open problems, and mathematical challenges in this direction are also addressed.

Asymptotic Expansion for the Number of Points Moving along a Metric Tree

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Let us consider a finite compact metric tree and the following dynamical system (see [1]) on it. Let one point move along the graph at the initial moment of time. If k points come to the interior vertex of valence n at the same time, then n points are released, i.e. one point corresponds to one edge. Reflection occurs in vertices of valence one. Time for passing each individual edge is fixed. The problem is to analyze the asymptotic behavior of the number $N(t)$ of such points on the graph as time t increases (see [2] for details). Such dynamical system emerges while considering the Cauchy problem for the time-dependent Schrödinger equation on hybrid spaces (see [3] and references therein). We find an asymptotic expansion for $N(t)$ using Barnes' multiple Bernoulli polynomials (also known as Todd polynomials, see [4, 5, 6] and reference therein). We prove that the second term is given by a quadratic form of edge travel times. Moreover, the tree structure is uniquely determined by this form.

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Generalized Green Operator of Boundary-Value Problem for Matrix Differential-Algebraic Equation

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We study the problem of constructing solutions [1, 2, 3]

$$Z(t) \in \mathbb{C}^1_{\alpha\times\beta}[a,b] := \mathbb{C}^1[a,b] \otimes \mathbb{R}^{\alpha\times\beta}$$

to the matrix differential-algebraic linear boundary value problem

$$\mathcal{D}Z(t) = \mathcal{A}Z(t) + F(t), \quad \mathcal{L}Z(\cdot) = \mathfrak{A}, \quad \mathfrak{A} \in \mathbb{R}^{\mu\times\nu}. \tag{1}$$

Here

$$\mathcal{D}Z(t) := \sum_{i=1}^{p} S_i(t)Z'(t)R_i(t), \quad \mathcal{A}Z(t) := \sum_{j=1}^{q} \Phi_j(t)Z(t)\Psi_j(t)$$

are matrix linear operators, $S_i(t), \Phi_i(t) \in \mathbb{R}^{\alpha\times\beta}$, $R_i(t), \Psi_j(t) \in \mathbb{R}^{\gamma\times\delta}$, and $F(t)$ are continuous matrices, $\mathcal{L}Z(\cdot)$ is a boundary linear functional such that

$$\mathcal{L}Z(\cdot) : \ \mathbb{C}^1_{\alpha\times\beta}[a;b] \to \mathbb{R}^{\mu\times\nu}$$

and, moreover, $\alpha, \beta, \gamma, \delta, \mu, \nu$ are arbitrary natural numbers. Matrix differential-algebraic equation (1) generalizes conventional statements of problems for matrix differential equations [2, 3] as well as differential-algebraic equations [4, 5]. On the other hand, matrix differential-algebraic boundary value problem (1), (2) generalizes conventional statements of Noetherian boundary value problems for systems of ordinary differential equations [1].

We set forth solvability conditions and the construction of the generalized Green operator for a matrix differential-algebraic linear boundary value problem. Sufficient conditions of reducibility of a generalized matrix differential-algebraic operator to a conventional differential-algebraic equation with the unknown column vector are established.

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Existence of Stationary Solutions in Population Dynamics and Their Optimization

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It is quite natural to expect that population dynamics under a given stationary exploitation mode converges to some stationary state. That is clear, for instance, in the simplest case of a logistic model [1]. On the other hand, such a convergence could be non-obvious and needs justification for models with more complicated dynamics taking population structure, the law of emergence of a new generation, or some other natural population characteristics into account (see, for example, [2], [3]).

We study essentially nonlinear models for dynamics of an exploited population. The models consider competition between individuals in the integral form, which could be symmetric or hierarchical, and also nonlinear influence of population density and competition level on reproduction. Our models are very far generalizations of the McKendrick-von Foerster model [4].

For considered models we prove the existence of nontrivial stationary solutions for a given stationary exploitation mode. We also show the existence of a mode that provides maximum profit over all admissible modes on some of the respective stationary solutions [5], [6]. The illustrated examples are also presented.

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Constructive Solution of the Observation Problem for the Wave Equation

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Consider the following initial boundary value problem for the wave equation:

$$u_{tt} - u_{xx} - u_{yy} = 0, \quad (x, y) \in \mathbb{R}^2_+, \ t \in \mathbb{R}, \tag{1}$$

$$u|_{y=0} = 0, \quad u|_{t=0} = u_0, \quad u_t|_{t=0} = 0, \tag{2}$$

where $\mathbb{R}^2_+ := \{(x, y) | \, x \in \mathbb{R}, y \geqslant 0\}$, and $u_0 \in C^3(\mathbb{R}^2_+)$ satisfies $u_0\big|_{y=0} = (u_0)_{yy}\big|_{y=0} = 0$. We deal with the observation problem, in which the function u_0 is to be determined from the function $v := u_y\big|_U$, where U is a bounded subset of the plane $\{y = 0\}$. Taking into account the Dirichlet boundary condition in problem (1), (2), we see that the function v determines the Cauchy data on U. Reconstruction of the solution u (which is equivalent to reconstruction of the corresponding initial data u_0) from the Cauchy data given on a bounded subset of the plane $\{y = 0\}$ is known to be an ill-posed problem. We propose an algorithm which allows one to determine $u_0(x_0, y_0)$, $y_0 \geqslant 0$, provided that the function v is given on the set

$$U := \left\{ (x, t) \,\middle|\, |x - x_0| \leqslant D\left(\sqrt{y_0^2 - t^2}\right), 0 \leqslant t \leqslant y_0 \right\}, \quad D(z) := \frac{z}{\sqrt{1 + 2\alpha z}},$$

where $\alpha \geqslant 0$. Namely, the following relation holds true:

$$u_0(x_0, y_0) = \lim_{h \to 0+} \int_U K_h(x - x_0, y_0, t)\, v(x, t)\, dxdt, \tag{3}$$

where the kernel K_h is defined by

$$K_h(x, y_0, t) :=$$
$$\frac{1}{\pi^{3/2}} \operatorname{Re} \left[\frac{1}{\sqrt{h(1 + i\alpha x)}} \int_0^{\pi/2} \exp\left(-\frac{1}{4h(1 + i\alpha x)} \left(x + i\sqrt{y_0^2 - t^2} \cdot \sin s \right)^2 \right) ds \right]$$

(we assume $\operatorname{Re}\sqrt{1 + i\alpha x} > 0$). The set U depends on the coordinates x_0, y_0, and the parameter α, which means that one can use the Cauchy data given on various sets to determine $u_0(x_0, y_0)$.

A constructive solution of the observation problem for the wave equation was first proposed by R. Courant. In contrast to (3), the scheme of R. Courant involves the derivatives of all orders of the data. We also mention the inversion formula obtained by A. S. Blagoveshchensky and F. N. Podymaka requiring the Cauchy data on the entire space-time boundary rather than on a compact set.

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Isomorphism Theorems for a Class of Quasielliptic Operators

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The paper is devoted to the theory of quasielliptic operators. We consider a class of matrix homogeneous quasielliptic operators $\mathcal{L}(D_x)$ with lower terms in the whole space $\mathbb{R}^n$. This class belongs to the class of quasielliptic operators introduced by L. R. Volevich [1]. Our aim is to study isomorphism properties of these operators in function spaces. We consider a special scale of weighted Sobolev spaces $W^l_{p,q,\sigma}(\mathbb{R}^n)$ [2, 3] and investigate mapping properties of the operators $\mathcal{L}(D_x)$ in these spaces. We indicate conditions for unique solvability of the quasielliptic systems in these spaces, obtain estimates for solutions and formulate isomorphism theorems for the quasielliptic operators. In order to obtain these results, we construct special regularizators for the quasielliptic operators.

The first isomorphism theorems for matrix homogeneous elliptic operators were proved by Y. Choquet–Bruhat and D. Christodoulou [4], R. B. Lockhart and

R. C. McOwen [5]. The first isomorphism theorems for matrix homogeneous quasielliptic operators were established in [6, 7]. The investigations of [6, 7] were continued by G. N. Hile [8]. We consider a more general class of quasielliptic operators in this paper.

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The Inverse Magnetoencephalography Problem and Its Flat Approximation

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Contrary to the prevailing opinion about the incorrectness of the inverse MEEG-problem, we prove that it possesses a unique solution in the framework of the electrodynamic system of Maxwell equations [1]. The solution of this problem is the distribution $\mathbf{y} \mapsto \mathbf{q}(\mathbf{y})$ of current dipoles of brain neurons occupying a region $Y \subset \mathbb{R}^3$. It is uniquely determined by the non-invasive measurements of the electric and magnetic fields induced by the current dipoles of neurons on the patient's head. The solution can be represented in the form $\mathbf{q} = \mathbf{q}^* + \rho\delta\Big|_{\partial Y}$, where $\mathbf{q}^*$ is a usual function defined in Y, and $\rho\delta\Big|_{\partial Y}$ is the δ-function on the boundary of the domain Y with a certain density ρ. However, in the case where the conductivity is assumed to be the same everywhere (in the brain, skull, ambient air) and, in addition, the electric and magnetic inductions are impossible to record *in time*, it is impossible to find $\mathbf{q}$

completely. Nevertheless, it is still possible to obtain partial information about the distribution of $\mathbf{q} : Y \ni \mathbf{y} \mapsto \mathbf{q}(\mathbf{y})$. This question is considered in detail in a flat model situation.

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Boundary Value Problem for Second Order Quadratic Ordinary Differential Pencil with Integral Boundary Conditions

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In this talk we study a regular second order quadratic ordinary differential equation with weighted integral boundary conditions. Under some conditions on the weight functions which accure in the integral boundary conditions, expressed in terms of the values at the interval endpoints, we prove that the resolvent has a maximal growth. Furthermore, the studied operator generates an analytic semigroup in $L_P(0,1)$, $p > 1$. The obtained results are then applied to the study of a nonlocal parabolic partial differential equation with regular integral boundary conditions.

Necessary Conditions in Optimal Control Problems with Integral Equations on a Variable Time Interval

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We study an optimal control problem with a Volterra-type integral equation,

$$x(t) = x(t_0) + \int_{t_0}^{t} f(t, s, x(s), u(s))\, ds,$$

considered on a non-fixed time interval $[t_0, t_1]$, subject to endpoint constraints of equality and inequality type, and the cost in the Mayer form. We obtain first-order necessary optimality conditions for an *extended weak minimum*, the notion of which is a natural generalization of the notion of weak minimum with account of time variation. Following the tradition, we call them *stationarity conditions*. Their novelty, as compared with those for problems on a fixed time interval is that the costate equation and transversality condition with respect to time variable involve nonstandard terms that are absent in problems with ODEs.

To be more precise, assuming that a process $x^0(t), u^0(t),\ t \in [\hat{t}_0, \hat{t}_1]$ provides the extended weak minimum and introducing the costate variables ψ_x and ψ_t corresponding to x and t, respectively, we obtain the costate equations

$$\dot{\psi}_x(s) = \psi_x(s)\, f_x(s, s, x^0(s), u^0(s)) + \int_s^{\hat{t}_1} \psi_x(t)\, f_{tx}(t, s, x^0(s), u^0(s))\, dt,$$

$$\dot{\psi}_t(s) = \psi_x(s)\, f_s(s, s, x^0(s), u^0(s)) + \int_s^{\hat{t}_1} \psi_x(t)\, f_{ts}(t, s, x^0(s), u^0(s))\, dt - \dot{\psi}_x(s)\, F(s),$$

where $F(s) = \displaystyle\int_{t_0}^{s} f_t(s, z, x^0(z), u^0(z))\, dz$, and the terminal transversality conditions

$$\psi_x(\hat{t}_1) = -l_{x_1}, \qquad \psi_t(\hat{t}_1) = -l_{t_1} + \psi_x(\hat{t}_1)\, F(\hat{t}_1),$$

where $l(t_0, x_0, t_1, x_1)$ is the usual endpoint Lagrange function.

The proof is based on reducing the problem to a problem on a fixed time interval and then using the general Lagrange multipliers rule. However, in contrast to problems with ODEs, this reduction leads to an integral equation of a more general form than the standard Volterra-type equation, and this is the cause of appearance of the new additional terms.

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Optimal Synthesis in the Simplified Goddard Problem on a Constrained Time Interval

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We consider the following optimal control problem (a simplification of the classical Goddard problem on maximizing the height of vertical flight of a "meteorological rocket" [1, 2]) on a constrained time interval:

$$\begin{cases} \dot{s} = x, & s(0) = 0, & s(T) \to \max, \\ \dot{x} = u - \varphi(x) - g, & x(0) = 0, & x(T) \ \text{ is free}, \\ \dot{m} = -u, & m(0) = m_0\,, & m(T) \geqslant m_T\,, \\ 0 \leqslant u \leqslant 1, & T \leqslant T_0\,. & \end{cases} \tag{1}$$

Here, $s(t)$ and $x(t)$ are one-dimensional position and velocity of a vehicle, $m(t)$ describes the total mass of vehicle's body and fuel, $u(t)$ is the rate of fuel expenditure, g is a constant gravity force, and the function $\varphi(x)$ describes the "friction" (media resistance) depending on the velocity. We assume that (see [3, 4]) $\varphi(0) = 0$, $\varphi'(x) \geqslant 0$ for all x,$\varphi(x)$ is twice smooth for $x \neq 0$, $\varphi''(x) < 0$ for all $x < 0$ and $\varphi''(x) > 0$ for all $x > 0$, which, in particular, implies that $\varphi(x)$ works on decreasing the absolute value of the speed $|x|$.

Using the Pontryagin maximum principle, we observe that an optimal trajectory in (1) is either bang-bang or bang-singular-bang, and construct optimal synthesis in the coordinate plane (x, T) w.r.t. the parameter $\gamma = m_T - m_0$ (treshold values γ_*, γ_0, and γ^* are defined from the optimality conditions):

1. $\gamma \in (0, \gamma_*]$ — the optimal trajectory is bang-bang with $x(T) = 0$, $T < T_0$,

2. $\gamma \in (\gamma_*, \gamma_0]$ — the optimal trajectory is bang-singular with $x(T) = 0$, $T \leqslant T_0$,

3. $\gamma \in (\gamma_0, \gamma^*]$ — the optimal trajectory bang-singular with $x(T_0) > 0$,

4. $\gamma \in (\gamma^*, T_0]$ — the optimal trajectory bang-bang with $x(T_0) > 0$.

We also investigate evolution of the obtained synthesis w.r.t. the parameter $g \to 0$.

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New Formulas for the Maslov Canonical Operator in the Neighborhood of the Lagrangian Singularities and Their Applications in the Linear Water Theory

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The Maslov canonical operator [1] (a close object is the Fourier Integral Operator) allows one to construct the asymptotic solutions of evolution and stationary problems for the wide class of linear differential and pseudodifferential equations. The definition of the Maslov canonical operator is connected with Lagrangian manifolds in the the phase space. In particular, the canonical operator gives the description of the asymptotic solutions in the neighborhood of a caustic and a focal point (Lagrangian singularities). Near these objects, the Maslov formulas have an integral form and are based on partial Fourier transforms. The practical realization of such formulas is quite complicated and includes the two following steps: 1) expression of the coordinates on the appropriate Lagrangian manifold via certain momenta in the phase space (which is not trivial as usual) and 2) the integration in these variables. We suggest [2] a new construction based only on the integration in the coordinates on the Lagrangian manifold. This allows one to simplify crucially the process of calculation of asymptotic solutions in the neighborhood of caustic and focal points, establish the relationship between the canonical operator and many special functions, and to extend the application of the canonical operator. As an example, we consider the problem of Airy-Bessel beams in optics [3] and the behavior of the front edge of a wave in the linear water wave theory [4] .

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Manifolds of Potentials and Eigenfunctions of Periodic Eigenvalue Problems

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We consider the family

$$-y" + p(x)y = \lambda y; y(0) - y(2\pi) = y'(0) - y'(2\pi) = 0 \tag{1}$$

of periodic eigenvalue problems with 2π-periodic real potential $p \in P := \{C^0(2\pi) \ \wedge \int_0^{2\pi} p(x)dx = 0\}$ as a functional parameter. For any potential p, the spectrum of the problem consists of isolated real eigenvalues, the multiplicity of each eigenvalue does not exceed two, and the spectrum has the form

$$\lambda_0(p) < \lambda_1^-(p) \leqslant \lambda_1^+(p) < ... < \lambda_k^-(p) \leqslant \lambda_k^+(p) \to \infty.$$

Let $\Delta\lambda \geqslant 0$ be a fixed number. We consider the submanifolds

$$P_k(\Delta\lambda) := \{p \in P |\ \lambda_k^+(p) - \lambda_k^-(p) = \Delta\lambda\}.$$

We provide the description of the analytic and the topological structures of $P_k(\Delta\lambda)$. Also, we consider the manifolds of all eigenfunctions

$$Y := \{y \in C^2(2\pi) |\ \int_0^{2\pi} y^2 dx = 1 \ \wedge \ \exists (p, \lambda) \in P \times \mathbb{R} : (1)\ is\ true\}.$$

Let $Y_k^{\pm} = \{y \in Y |\ \lambda = \lambda_k^{\pm}(p)\}$ and $Y_k^{\pm}(\Delta\lambda) = \{y \in Y_k^{\pm} |\ \lambda_k^+(p) - \lambda_k^-(p) = \Delta\lambda\}$. We claim that if $\Delta\lambda > 0$, then the manifolds $Y_k^{\pm}(\Delta\lambda)$ are diffeomorphic to $P_k(\Delta\lambda)$, and if $\Delta\lambda = 0$, then $Y_k(0)$ is trivially foliated over $P_k(0)$ with a circle as the standard fiber of the bundle.

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The Generalized Concept of Integrability and Dynamics of the Trace Map

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In [1], we extended the definition of integrability which is due to Grigorchuk (see [2] for integrability of polynomial or rational maps), to the case of arbitrary (maybe, even discontinuous) maps in the plane $\mathbb{R}^2$.

Trying to adapt our definition of integrability to the study of the trace map, we generalize here our definition of integrability to an upper semicontinuous two-valued map defined in a convex unbounded domain of the plane.

A criterion is proved for integrability of the above multivalued maps.

The obtained results are applied to a special upper semicontinuous two-valued map connected with the trace map

$$F(x,\, y) = (xy,\, (x-2)^2).$$

Studying the properties of this two-valued map makes it possible to describe the structure of the nonwandering set of the above trace map in some unbounded subset of the plane.

The methods are based on the use of geometric results obtained in [1] for the above trace map.

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A Fractional Dynamical System Related to Tumor Cancer Evolution

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In this article, we present a general fractional dynamical system related to cancer tumor. The considered model describes tumor–immune cell interactions using a system of fractional order differential equations. The conditions for global stability of cancer free state are studied.

In order to stabilize or completely eliminate the cancer, we suggest suitable choices of functions and parameters in our fractional model. The suitable mathematical models of fractional dynamical systems explore important problems in biology. This tool is an ever increasing towards shedding light on these nonlinear fractional systems. The considered model incorporates tumor–immune interaction terms of a form that is qualitatively different from those commonly used.

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New Discrete Hardy Inequalities

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Let $0 < p, q < \infty$. In this paper, necessary and sufficient conditions are given for the validity of discrete Hardy's inequalities of the form

$$\left(\sum_{n=1}^{\infty} v(n)\left(\sum_{m=1}^{b(n)} k(n,m)f(m)\right)^{q}\right)^{\frac{1}{q}} \leqslant C\left(\sum_{n=1}^{\infty} f^{p}(n)w(n)\right)^{\frac{1}{p}}$$

for all $f(n) \geqslant 0$, where $v(n), w(n)$ are weight sequences of positive numbers, $b(n)$ is a strictly increasing sequence of natural numbers and $k(n,m) \geqslant 0$ is non-decreasing in n.

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A New Approach to Identification Problems for Degenerate Evolution Equations

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A new approach to handle some identification problems for degenerate differential equations is introduced. Various applications to PDEs are indicated.

On Dubovitskiĭ–Federer–Luzin Properties for Sobolev and Holder Mappings

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The Morse–Sard theorem requires that a mapping $v\colon \mathbb{R}^n \to \mathbb{R}^m$ is of class C^k, $k > \max(n-m, 0)$. In 1957, Dubovitskiĭ generalized this result by proving that almost all level sets for a C^k mapping has $\mathcal{H}^\nu$-negligible intersection with its critical set, where $\nu = n-m-k+1$ and $\mathcal{H}^\nu$ is the Hausdorff measure. Here the critical set, or m-critical set is defined as $Z_{v,m} = \{x \in \mathbb{R}^n : \operatorname{rank} \nabla v(x) < m\}$. Another generalization was obtained independently by Dubovitskiĭ and Federer in 1966. Namely, they proved for C^k mappings $v\colon \mathbb{R}^n \to \mathbb{R}^d$ and integers $m \leqslant d$ that the set of m–critical values $v(Z_{v,m})$ is $\mathcal{H}^{q_\circ}$-negligible for $q_\circ = m-1+\frac{n-m+1}{k}$. They also established the sharpness of these results within the C^k category.

Here we formulate and prove a *bridge theorem* that includes all the above results as particular cases: namely, if a function $v : \mathbb{R}^n \to \mathbb{R}^d$ belongs to the Holder class $C^{k,\alpha}$, $0 \leqslant \alpha \leqslant 1$, then for every $q > m-1$ the identity

$$\mathcal{H}^\mu(Z_{v,m} \cap v^{-1}(y)) = 0$$

holds for $\mathcal{H}^q$-almost all $y \in \mathbb{R}^d$, where

$$\mu = n - m + 1 - (k + \alpha)(q - m + 1).$$

The result is new even for the classical C^k-case (when $\alpha = 0$); a similar result is established for the Sobolev classes of mappings $W_p^k(\mathbb{R}^n, \mathbb{R}^d)$ with minimal integrability assumptions $p = \max(1, n/k)$, i.e., it guarantees in general only the continuity of a mapping. We cover also the case of fractional Sobolev spaces.

As a limiting case in this bridge theorem (for $q = m - 1$), we also establish a new coarea-type formula. Finally, we establish for Sobolev mappings the relative analogs of the Luzin N-property with respect to lower dimensional Hausdorff measure. We found the sharp version of these N-properties, and the corresponding nontrivial counterexample for the limiting cases is demonstrated (based on the classical theory of the lacunary Fourier series).

The proofs of the most results are based on our previous joint papers with J. Bourgain (Princeton) and J. Kristensen (Oxford), see [1, 2].

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Study of m-Convex Hypersurfaces

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We consider a C^2-hypersurface $\Gamma \subset \mathbb{R}^n$ and the set of principal curvatures of Γ. Recall that the sum of principal curvatures is the mean curvature and the product of principal curvatures is the Gauss curvature of the hypersurface.

The pth order elementary symmetric function of principal curvatures is called the p-curvature of Γ and is denoted by k_p, $1 \leqslant p \leqslant n-1$. A hypersurface $\Gamma \in R^n$ is called m-convex at a point $M \in \Gamma$ if $k_p(M) > 0$ for all $p = 1, 2, \ldots, m$.

If a hypersurface is m-convex, then it is p-convex for all $p = 1, 2, \ldots, m-1$. Notice that a 1-convex hypersurface is just a hypersurface of the positive mean curvature and any $(n-1)$-convex hypersurface is strongly convex in the classical sense. Thus the notion of m-convexity is a generalization of the classical convexity. It appeared at the end of the twentieth century as a result of the successful application of Gårding’s cones [1] in the theory of fully nonlinear partial differential equations [2]. The solvability condition of the Dirichlet problem for the m-Hessian equation in a bounded domain $\Omega \subset \mathbb{R}^n$ is expressed in terms of the $(m-1)$-convexity of the boundary hypersurface $\partial\Omega$.

The systematic study of m-convex hypersurfaces is only just at the beginning. The most complete overview of already accumulated facts and methods is available in [3]. Until now, one of the gaps in this theory has been the construction of the simplest examples of m-convex hypersurfaces. This gap is filled in our research [4] where we study m-convexity of multidimensional quadrics, paraboloids, and hyperboloids.

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Fixed Points and Coincidences of Mappings on Ordered Sets

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The well-known fixed point theorems by Knaster–Tarski, Smithson, and Zermelo (see, for example, [1, 2]) are classical results in the fixed point theory of mappings of ordered sets. These results have a large number of various applications. As for the coincidence theory of mappings between ordered sets, the first results in this area were apparently obtained by A. V. Arutyunov, E. S. Zhukovskii, and S. E. Zhukovskii in their joint works in 2013–2016 for the case of two mappings, one of which is a covering, and the other one is an isotone mapping. In our papers with my graduate student D.A. Podoprikhin [3, 4, 5], we have slightly refined some results obtained by A. V. Arutyunov and coauthors and generalized them to the case of families of multivalued mappings of ordered sets. The above-mentioned Zermelo theorem was generalized by Y. Yachymski (see [1]). It should be noted that the results concerning generalizations of the Knaster–Tarski and the Smithson theorems, on the one hand, and the Zermelo theorem, on the other hand, do not follow from each other. So, the question arises: is it possible to obtain fixed point and coincidence theorems implying all the above-mentioned results? A positive answer to this question will be given in the report. Fixed point and coincidence theorems implying all the above-mentioned results will be formulated. The obtained results represent generalizations of the corresponding results of [3, 4, 5]. It should be noted that in these theorems, the involved mappings need not be coverings or isotone mappings. Instead, the existence of special chains, together with their special lower boundaries, is needed. The report is based on the paper [6].

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Parabolic Equations of Normal Type Connected with 3D Helmholtz System and Its Nonlocal Stabilization

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The talk will be devoted to the normal parabolic equation (NPE) connected with the 3D Helmholtz system whose nonlinear term $B(v)$ is the orthogonal projection of that one for the Helmholtz system on a ray generated by the vector v. Interest in this NPE is due to attempts to solve the problem of nonlocal existence of a smooth solution for the 3D Navier–Stokes equations.

As it became clear now, the study of the NPE have opened the way to construct the method of nonlocal stabilization by feedback control for the 3D Helmholtz as well as for the 3D Navier–Stokes equations.

First we describe the structure of a dynamic flow corresponding to this NPE [1]. After that, we formulate the nonlocal stabilization problem for the NPE with control initially supported in an arbitrary fixed subdomain. The main steps in solving this problem will be discussed [2]. Finally, we explane how to apply this result for solution of the nonlocal stabilization problem with impulse control for the 3D Helmholtz system.

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Derivation of Effective Models for Reaction-Diffusion Processes in Multi-Component Media Including Nonlinear Transmission Conditions

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In this talk, we derive effective models via periodic homogenization for a system of microscopic nonlinear reaction–diffusion equations in a multi-component porous medium, where the components are separated by an interface. One component is connected, the other ones are disconnected and consist of periodically distributed inclusions. The differential equations in the different domains are coupled by nonlinear transmission conditions depending on the solutions on both sides of the interface.

For this system, we derive a macroscopic model using the two-scale convergence and the unfolding method for periodic domains and on periodic surfaces. For the convergence of the nonlinear reaction-rates, especially the transmission conditions on the interface between the components, strong two-scale compactness results are developed using an unfolding argument and a Banach-valued compactness theorem of Kolmogorov type.

Nonexistence of Monotone Solutions to Some Coercive Elliptic Inequalities in the Half-Space

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Let $p > 1$, $q > p - 1$. Using a modification of the test function method developed in [1], we establish nonexistence results for the coercive quasilinear elliptic inequality in a half-space

$$\Delta_p u(x) \geqslant u^q(x) \qquad (x \in \mathbb{R}^n_+)$$

and for some generalizations of this inequality.

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Measure Functional Differential Equations with Infinite Time-dependent Delay

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Measure functional differential equations (**MFDEs** in short) with finite delay of the type

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s)dg(s), \quad t \in [t_0, t_0 + \sigma], \tag{1}$$

have been introduced by Ferderson, Mesquita, and Slavik in [1]. Here y and f are functions with values in $\mathbb{R}^n$, the integral on the right-hand side of (1) is the Kurzweil–Henstock integral with respect to a nondecreasing function g and as is usual in the theory of functional differential equations, y_s represents the "history" of y at s. They showed that functional dynamic equations on time scales represent a special case of **MFDEs**, and they obtained results on the existence and uniqueness of solutions using the theory of generalized ordinary differential equations introduced by J. Kurzweil in 1957 [4]. The case of equation (1) considered with infinite delay was later studied by A. Slavik in [5]. He described axiomatically a suitable phase space similar to classical functional differential equations with infinite delay (see, e.g., [2, 3]), and he obtained results of existence and uniqueness.

We focus our attention on equation (1) with infinite time-dependent delay, i.e. we study the equation

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_{r(s)}, s)dg(s), \quad t \in [t_0, t_0 + \sigma], \tag{2}$$

where r is a nondecreasing function such that $r(s) \leqslant s$ for all $s \in Dom(r)$.

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Time Scales in the Double-deck Structures of the Boundary Layer

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In [1, 2, 3], we considered an incompressible viscous fluid flow past a plate with small irregularities (of periodic and localized types) on the surface for a large Reynolds number $\mathbf{Re}$. In [4], we considered the similar problem for a flow in a "wavy-wall" pipe (and a 2D channel) with the Poiseuille flow inside.

These problems have similar asymptotic solutions with the double-deck structure of the boundary layer containing the following different-scale regions: a thin boundary layer (near-wall region) and the classical Prandtl (or "thick") boundary layer, and the external region with the unperturbed flow. In the thin boundary layer, the flow is described by the Prandtl boundary layer equations with self-induced pressure. In the classical boundary layer, the flow is described by a Rayleigh-type equation (see Eq. (1)) in the plate case and by the Laplace equation in the pipe (and channel) case.

However, in addition to the space scales, we showed the appearance of the following time scales in the non-stationary problems (where $\varepsilon = \mathbf{Re}^{-1/2}$ is a small parameter of the problem):

	Plate case	Pipe (channel) case
Thin boundary layer	$t \sim 1$	$t \sim 1$
Classical boundary layer	$t \sim \varepsilon^{-1/3}$	$t \sim \varepsilon^{2/5}$
Total time (in Navier–Stokes eq.)	$t \sim \varepsilon^{2/3}$	$t \sim \varepsilon^{-2/5}$

The main interest is the study of the Rayleigh-type equation, which, for periodic irregularities, has the form

$$\varepsilon^{1/3}\partial_t \int \Delta_{\xi,\tau} v \, d\xi + f'(\tau/\sqrt{x})\Delta_{\xi,\tau} v - f'''(\tau/\sqrt{x})v/x = 0, \tag{1}$$

$$v\big|_{\tau=0} = V(\xi,t),\ v\big|_{\tau\to\infty} \to 0,\ v\big|_{\xi} = v\big|_{\xi+2\pi},\ v\big|_{t=0} = V_0(\xi,\tau),$$

where $f(\gamma)$ is the Blasius function.

The solution of Eq. (1) depends on the time t (due to the boundary condition at $\tau = 0$) and on the "large" time $t' = \varepsilon^{-1/3}t$. In [2, 5], we proved that a quasistationary solution (i.e., depending only on time t) exists and is unique, and it is stable.

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Blow-up of Solutions of Semilinear Parabolic Equations with Absorption and Nonlinear Nonlocal Neumann Boundary Conditions

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We consider the initial boundary value problem for the semilinear parabolic equation

$$u_t = \Delta u - c(x,t)u^p,\ x \in \Omega,\ t > 0, \tag{1}$$

$$\frac{\partial u(x,t)}{\partial \nu} = \int_{\Omega} k(x,y,t)u^l(y,t)\,dy,\ x \in \partial\Omega,\ t > 0, \tag{2}$$

$$u(x,0) = u_0(x),\ x \in \Omega, \tag{3}$$

where $p > 0$, $l > 0$, Ω is a bounded domain in $\mathbb{R}^n$ for $n \geqslant 1$ with smooth boundary $\partial\Omega$, ν is the unit outward normal on $\partial\Omega$. The functions $c(x,t)$, $k(x,y,t)$ and $u_0(x)$ are nonnegative and satisfy some regularity conditions.

We prove some global existence results. Criteria which determine whether the solutions blow up in finite time for large or for all nontrivial initial data are also given. Our global existence and blow-up results depend on the behavior of $c(x,t)$ and $k(x,y,t)$ as $t \to \infty$.

In particular, we prove the following blow-up result. Let $\psi(x)$ be a positive solution of the problem

$$\Delta\psi = 1,\ x \in \Omega;\ \frac{\partial\psi(x)}{\partial\nu} = \frac{|\Omega|}{|\partial\Omega|},\ x \in \partial\Omega,$$

and $m_0 = \inf\{\sup_\Omega \psi(x)\}$. We assume that

$$c(x,t) \leqslant c_1(t),\ c_1(t) \in C^1([t_0,\infty)),\ c_1(t) > 0 \text{ for } t \geqslant t_0, \tag{4}$$

where t_0 is some positive constant,

$$\liminf_{t\to\infty} \frac{c_1'(t)}{c_1(t)} > -\frac{p-1}{m_0} \tag{5}$$

and

$$\lim_{t\to\infty} \left\{ [c_1(t)]^{(1-l)/(p-1)} \inf_{\partial\Omega\times\Omega} k(x,y,t) \right\} = \infty. \tag{6}$$

Theorem 1. *Let $l > p > 1$ and* (4)–(6) *hold. Then any nontrivial solution of* (1)–(3) *blows up in finite time.*

We prove a certain optimality of Theorem 1. We show under some conditions that the blow-up occurs only on the boundary.

Problem (1)–(3) with the Dirichlet boundary condition was considered in [1, 2].

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On Some Solutions of Heat-and-Mass Transfer Equation in Multilayer Media

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Let us consider a one-dimensional heat flow normal to a multilayer plate. The points of $x_i,\ i=\overline{1,n+1}$, are the boundaries of the layers. There are ideal contact conditions on the inner boundaries of the layers. Functions $a_1^{(i)}(x)$, $a_2^{(i)}(x)$ take the physical and geometric properties of the layers into account. The heat flow is $J^{(i)} = -a_1^{(i)}(x)\frac{\partial T^{(i)}}{\partial x}$ and the temperature field is determined by

$$a_2^{(i)}(x)\frac{\partial}{\partial x}\left(a_1^{(i)}(x)\frac{\partial T^{(i)}}{\partial x}\right)-\frac{\partial T^{(i)}}{\partial t}=0,\ i=\overline{1,n}. \tag{1}$$

To solve (1), we apply the Fourier method: $T^{(i)}(x,t) = u^{(i)}(x)e^{-x^2t}$, $J^{(i)} = J_0^{(i)}(x)e^{-\lambda^2t} = -a_1^{(i)}(x)\frac{\partial T^{(i)}}{\partial x}e^{-\lambda^2 t}$.

Then we obtain the following system of ordinary differential equations and fitting conditions:

$$a_2^{(i)}(x)\frac{\partial}{\partial x}\left(a_1^{(i)}(x)\frac{du^{(i)}}{dx}\right)+\lambda^2u^{(i)}=0,$$

$$u^{(i)}(x_{i+1})=u^{(i+1)}(x_{i+1}),\ J_0^{(i)}(x_{i+1})=J_0(x_{i+1}). \tag{2}$$

Let us write a solution to the system in the matrix form. Using the Bers formalism [1] and applying (2), we can find solutions to any layer in the form

$$\begin{gathered}
V^{(i)}(x)=\begin{pmatrix} u^{(i)}(x)\\ J_0^{(i)}(x)\end{pmatrix}=\\
=\begin{pmatrix}\cos\lambda X_i(x,x_i) & -\frac{1}{\lambda}\sin\lambda X_i(x,x_i)\\ \lambda\sin\lambda\widetilde{X}_i(x,x_i) & \cos\lambda\widetilde{X}_i(x,x_i)\end{pmatrix}\begin{pmatrix}u^{(i)}(x_i)\\ J_0^{(i)}(x_i)\end{pmatrix}=\\
=K^{(i)}(x,x_i)V^{(i)}(x_i)=K^{(i)}(x,x_i)K^{(i-1)}(x_i,x_{i-1})\dots K^{(1)}(x_2,x_1)V^{(1)}(x_1)=\\
=K^{(i,i-1,\dots,1)}(x,x_1)V^{(1)}(x_1).
\end{gathered} \tag{3}$$

Then the condition for the eigenvalues can be written as $k_{1,2}^{(n,n-1,\dots,1)}(\lambda,x_{n+1})=0$.

To normalize the basis functions, we find

$$N_k^2 = (J_0^{(1)}(x_1))^2 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \frac{1}{a_2^{(i)}(\xi)} \left(k_{1,2}^{(i,i-1,\dots,1)}(\lambda_k,\xi)\right)^2 d\xi, \; k=1,2,3,\dots$$

Consequently, the quantity $J^{(1)}(x_1)$ is found from condition (3). After the normalization, the coefficients in the Fourier expansion are determined from the scalar product. Thus, the desired solution is written as

$$T^{(i)}(x,t) = \sum_{k=1}^{\infty} c_k J_{0_k}^{(i)}(x_1) K_k^{(i,i-1,\dots,1)}(x,x_1) e^{-\lambda_k^2 t}.$$

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Cauchy Weight Problem for Euler–Poisson–Darboux Equation

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Let A be a closed operator in a Banach space E with domain $D(A)$ dense in E. Assuming $k<0$, consider the Cauchy weight problem for the Euler–Poisson–Darboux equation

$$u''(t) + \frac{k}{t} u'(t) = Au(t), \quad t>0, \tag{1}$$

$$u(0)=0, \quad \lim_{t\to 0} t^k u'(t) = u_1. \tag{2}$$

In [1], we investigate solvability of the Cauchy problem with the conditions

$$u(0)=u_0, \quad u'(0)=0 \tag{3}$$

for equation (1) as $k>0$. A criterion of uniform correctness of this problem is proved in terms of the norm estimate of the fractional degree of the resolvent $R(\lambda)$ (of the operator A) and its derivatives. The set of operators A for which Cauchy problem (1), (3) is uniformly well-posed for $k>0$ is denoted by G_k, and the corresponding resolving operator called the Bessel operator function, is denoted by $Y_k(t)$.

Definition 1. Problem (1), (2) is said to be *uniformly correct* if there exist an operator function $Z_k(t)$ commuting with A, and numbers $M \geqslant 1$ and $\omega \geqslant 0$ such that for any $u_1 \in D(A)$, the function $Z_k(t)u_1$ is its unique solution with

$$\|Z_k(t)\| \leqslant M\ t^{1-k} \exp(\omega t),$$

$$\|Z_k'(t)u_1\| \leqslant M\ t^{-k} \exp(\omega t) \left(\|u_1\| + t\|Au_1\|\right).$$

The function $Z_k(t)$ for $k < 0$ is called the *Bessel operator function* with negative index of problem (1), (2), and the set of operators for which problem (1), (2) is uniformly correct, is denoted by H_k.

Theorem 1. *Let the operator A be the generator of an analytic C_0-semigroup. Problem (1), (2) is uniformly correct if and only if there exist constants $M \geqslant 1$ and $\omega \geqslant 0$ such that the numbers λ^2 with $Re\ \lambda > \omega$ belong to the resolvent set of the operator A and the estimates*

$$\left\| \frac{d^n}{d\lambda^n} \left(\lambda R^{2-k/2} \left(\lambda^2\right) \right) \right\| \leqslant \frac{M\ \Gamma(n-k+3)}{(Re\ \lambda - \omega)^{n-k+3}}, \quad n = 0,\ 1,\ 2, \ldots$$

hold for the fractional power of the resolvent of the operator A.

Theorem 2. *Let the conditions of Theorem 1 be fulfilled, then for $k \leqslant 0$ the equality $H_k = G_{2-k}$ holds, and in this case $Z_k(t) = \dfrac{1}{1-k} t^{1-k} Y_{2-k}(t)$.*

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Pulse Packets in a System of Two Nonlinearly Coupled Equations with Delay

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Let us consider the system of nonlinear differential-difference equations (see [1])

$$\begin{aligned} \dot{u}_1 &= [\lambda f(u_1(t-1)) + b\, g(u_2(t-h)) \ln(u_*/u_1)] u_1, \\ \dot{u}_2 &= [\lambda f(u_2(t-1)) + b\, g(u_1(t-h)) \ln(u_*/u_2)] u_2, \end{aligned} \tag{1}$$

modelling the association of two neurons with a synaptic connection. Here $h > 0$ characterizes the delay in the connection chain, $u_1(t), u_2(t) > 0$ are the normalized membrane neurons potentials, the parameter $\lambda \gg 1$ characterizes the electrical processes rate in the system, $b = \text{const} > 0$, $u_* = \exp(c\lambda)$ is the threshold value for control interaction, $c = \text{const} \in R$, and the terms $bg\big(u_{j-1}\big) \ln(u_*/u_j) u_j$ model synaptic interaction. The functions $f(u)$, $g(u)$ belong to $C^2(R_+)$, where $R_+ = \{u \in R : u \geqslant 0\}$, and satisfy the following conditions:

$$\begin{gathered} f(0) = 1;\ f(u) + a, uf'(u), u^2 f''(u) = O\big(u^{-1}\big) \text{ as } u \to +\infty,\ a = \text{const} > 0; \\ \forall u > 0\ g(u) > 0,\ g(0) = 0;\ g(u) - 1, ug'(u), u^2 g''(u) = O\big(u^{-1}\big) \text{ as } u \to +\infty. \end{gathered} \tag{2}$$

For any natural n, we find a periodic solution of system (1), containing n asymptotically high bursts on the period.

Let us show that the system has a solution of the form $u_1(t) = \exp(\lambda x(t))$, $u_2(t) = \exp(\lambda x(t+\Delta))$, where $\Delta \geqslant 0$ and $x(t)$ is a periodic solution of the equation

$$\dot{x} = f(\exp(x(t-1)/\varepsilon)) + b\,(c-x)g(\exp(x(t+\Delta-h)/\varepsilon)), \varepsilon), \tag{3}$$

$\varepsilon \ll 1$. Using (2), we come to the limit equation

$$\dot{x} = R(x(t-1)) + b\,(c-x)H(x(t+\Delta-h)), \tag{4}$$

where $R(x) = 1$ and $H(x) = 0$ as $x < 0$, $R(x) = -a$ and $H(x) = 1$ as $x > 0$.

Denote the solution of relay equation (4) by $x_*(t)$, and its period by T_*.

Theorem. *There exist parameters a, b, c, h, and Δ such that for any natural n and $\varepsilon > 0$ small enough, equation (3) has an orbital exponential stable cycle $x_*(t,\varepsilon)$ with period $T_*(\varepsilon)$. Moreover,*

$$\lim_{\varepsilon\to 0} \max_{0\leqslant t\leqslant T_*(\varepsilon)} |x_*(t,\varepsilon) - x_*(t)| = 0, \quad \lim_{\varepsilon\to 0} T_*(\varepsilon) = T_*,$$

and there are n segments within the period where the function $x_*(t,\varepsilon)$ is positive.

The theorem allows us to justify the existence and stability of a periodic solution with n asymptotically high bursts on the period for system (1). If $\Delta = 0$, then system (1) has a homogeneous solution $u_1 \equiv u_2$. And if the constant Δ satisfies the matching condition $\Delta = T_*/2$, then (1) has an anti-phase solution. It is shown thereby that the appearance of bursting in the system of coupled oscillators is a consequence of the delay in the connection chain between them.

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Relaxation Periodic Motion in Circular Hopfield Networks with Delay

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We consider the system of differential-difference equations

$$\dot{u}_j = -\mu_j u_j + \sum_{i=1}^{m} a_{ij} f_i(u_i(t - s_{ij})) + I_j, \;\; j = 1, \ldots, m, \tag{1}$$

modelling Hopfield networks with delay (see, [1]). Here $\mu_j = \mathrm{const} \geqslant 0$, $s_{ij} = \mathrm{const} \geqslant 0$, $a_{ij} = \mathrm{const} \in \mathbb{R}$, $I_j = \mathrm{const} \in \mathbb{R}$, and smooth functions $f_j(u), u \in \mathbb{R}$, are such that $\lim\limits_{u\to-\infty} f_j(u) = 0$, $\lim\limits_{u\to+\infty} f_j(u) = 1$.

The following two specific problems were investigated for Hopfield networks.

1. A mathematical model of a single Hopfield neuron, obtained from (1) for $m = 1$. After appropriate normalizations of the variables t and u, as well as of the parameter, it can be written in the form

$$\dot u = -\mu\, u + \lambda[1 - (a+1) f(u(t-1))]. \tag{2}$$

Here $\mu = \mathrm{const} \geqslant 0$, $a = \mathrm{const} > 0$, and $\lambda \gg 1$. We assume that the nonlinearity $f(u) \in C^\infty(\mathbb{R})$ satisfies the asymptotic equations $f(u) = \sum\limits_{k=1}^{\infty} c_k^- u^{-k}$ as $u \to -\infty$, $f(u) = 1 + \sum\limits_{k=1}^{\infty} c_k^+ u^{-k}$ as $u \to +\infty$. Under the above assumptions, we study problems of existence, asymptotic behaviour, and stability of the relaxation periodic motion for equation (2).

2. A chain of $m > 3$ Hopfield neurons connected unidirectionally into a circle, as described by the system of equations

$$\dot u_j = -\mu\, u_j + \lambda[1 - (a+1) f(u_j(t-1)) - b\, g(u_{j-1})], \;\; j = 1, \ldots, m. \tag{3}$$

Here $u_0 = u_m$, the parameters μ, a, λ and the function $f(u)$ are as in (2), $b = \mathrm{const} > 0$, and the function $g(u) \in C^\infty(\mathbb{R})$ satisfies asymptotic representations similar to that one for $f(u)$: $g(u) = \sum\limits_{k=1}^{\infty} d_k^- u^{-k}$ as $u \to -\infty$, $g(u) = 1 + \sum\limits_{k=1}^{\infty} d_k^+ u^{-k}$ as $u \to +\infty$.

It is proved that under appropriate choice of the parameters μ, a, and b, and for arbitrary $\lambda \gg 1$, system (3) can exhibit arbitrarily prescribed finite number of coexisting stable relaxation cycles. In other words, the buffer phenomenon is realized in this system.

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On the Mean-Field and Semiclassical Limits of the N-Particle Schrödinger Equation

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Consider the motion of N identical particles interacting through a 2-body potential V. If V is a long range potential and the particle system is dilute, then the effect on each particle of the interaction with the cloud of all the other particles can be approximately described by the average of the potential with respect to the particle density in the large N limit. In other words, the motion of the N-particle system can be described in terms of the motion of the *typical* particle subject to the self-consistent interaction potential.

In classical mechanics, the mathematical formulation of this result is expressed in terms of the limit of the system of Newton equations of motion for the N-particle system leading to the Vlasov equation (see Neunzert–Wick [6], Braun–Hepp [1], Dobrushin [2]).

In quantum mechanics, the analogous result is expressed in terms of the limit of the N-particle Schrödinger equation leading to the Hartree equation (Spohn [7]). The purpose of this talk is to discuss the uniformity of the mean-field (large N) limit as the typical particle action is large compared to the Planck constant $\hbar$. In other words, do the mean-field and semiclassical limits of quantum mechanics commute?

Our discussion of this problem is based on a quantum analogue of the quadratic Monge–Kantorovich–Vasershtein distance used in optimal transport, following the proof by Dobrushin [2] of the mean-field limit in classical mechanics. This talk is based on joint work with C. Mouhot, T. Paul and M. Pulvirenti [3, 4, 5].

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On Topology of Manifolds Admitting Morse–Smale Systems without Heteroclinic Intersections

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We show that if a closed orientable manifold M^n ($n \geqslant 3$) admits a Morse–Smale system F without intersections of codimension one separatrices, and if, in addition, in the case where F is a flow, F has no periodic trajectories, then M^n is either the n-sphere S^n or the connected sum of copies of $S^{n-1} \times S^1$ and of special manifolds which admit polar Morse–Smale systems.

We present two applications of the result above.

The first one concerns an existence of heteroclinic intersections of codimension one separatrices that form codimension two submanifolds. In the particular case $n = 3$, a connected component of intersection of two-dimensional separatrices is called heteroclinic curve. A heteroclinic curve is the mathematical model of the so-called separators considered in Solar Magnetohydrodynamics. From the modern point of view, reconnections of Solar magnetic field along separators are responsible for Solar flairs. The second one is a topological sufficient condition of existence of a closed trajectory of a Morse–Smale flow.

For a Morse–Smale diffeoomorphism f (a flow f^t), denote by μ the number of sinks and source periodic points (equilibrium states), and let ν be the number of codimension one saddle periodic points (equilibrium states). Set

$$g = \frac{\nu - \mu + 2}{2}$$

and denote

$$\Gamma_g = \mathbb{Z} * \cdots * \mathbb{Z}$$

the free product of g copies of the group of integers $\mathbb{Z}$.

1. Let $f : M^n \to M^n$ be a preserving orientation Morse–Smale diffeomorphism of a closed orientable n-manifold M^n, $n \geqslant 3$. If $g \geqslant 1$ and the fundamental group $\pi_1(M^n)$ does not contain a subgroup isomorphic to Γ_g, then there exist saddle periodic points p, $q \in NW(f)$ such that the Morse index of the point p equals 1, the Morse index of the point q is equal to $n-1$, and $W^s(p) \cap W^u(q) \neq \varnothing$.

2. Let f^t be a Morse–Smale flow without heteroclinic intersections on a closed orientable manifold M^n of dimension $n \geqslant 3$, Then, if $g \geqslant 1$ and the fundamental group $\pi_1(M^n)$ does not contain a subgroup isomorphic to Γ_g, then the flow f^t has at least one periodic orbit.

The above results have been obtained in collaboration with E. Ya. Gurevich, V. S. Medvedev, O. V. Pochinka, and E. V. Zhuzhoma (see [1], [2]). The author thanks the Russian Science Foundation (project 17-11-01041) for financial support.

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Singular Solitons and Indefinite Metrics

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In addition to regular solutions, the soliton equations admit important classes of singular solutions. First nontrivial example of such type was found in [1]: by choosing the Cauchy data for the Korteweg-de Vries (KdV) equation in the form $u(x,t) = 6\wp(x)$, one obtains $u(x,t) = \sum_{k=1}^{3} 2\wp(x - x_k(t))$. Singular solitons with pole-type singularities were studied by many authors.

These solutions could not be interpreted in the framework of the spectral transform for the auxiliary linear operators using the standard Hilbert spaces. In particular, in the case of the KdV equation, the evolution is well-defined on the functional class $H^{-1}(T,R)$ (see [2]) but the meromorphic solutions do not belong to this space.

The authors have shown recently (see [3] and references therein) that the spectral transform for the Schrödinger operators with potentials corresponding to meromorphic KdV solutions is well-defined in terms of Hilbert spaces with **indefinite** scalar products. The number of negative squares provides a new conservation law. Let us point out that the scattering transform for meromorphic KdV solutions was discussed in [4] but the corresponding Hilbert spaces were not discussed in this paper.

In our talk we discuss the extensions of the results of [3] to higher order operators.

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The Periodic Cauchy Problem for Self-Focusing NLS Equation Near Constant Solutions

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The focusing Nonlinear Schrödinger (NLS) equation is the simplest universal model describing the modulation instability (MI) of quasi monochromatic waves in weakly nonlinear media, considered the main physical mechanism for the appearance of rogue (anomalous) waves (RWs) in Nature.

We study, using the finite gap method, the NLS Cauchy problem for periodic initial perturbations of the unstable background solution of NLS. We show that the finite gap method applied to this problem provides the solution at the leading order in terms of different elementary functions in different time intervals.

In the case of one unstable mode only, such solution describes an exact deterministic alternate recurrence of linear and nonlinear stages of MI, and the nonlinear RW stages are described by the 1-breather Akhmediev solution, whose parameters, different at each RW appearance, are always given in terms of the initial data through elementary functions.

In the case of one unstable mode, the exact recurrence actually reduces to periodicity in time, up to an overall shift in the x-direction, and up to a multiplicative phase factor. We also write down, in the case of a finite number of unstable modes, the elementary analytic formulae describing the first nonlinear stage of MI, given again by the 1-breather Akhmediev solution, when the initial perturbation excites only one of the unstable modes.

Since the solution of the Cauchy problem is given in terms of different elementary functions in different time intervals, obviously matching in the corresponding overlapping regions, an alternative approach, based on matched asymptotic expansions, is suggested and presented in a separate paper in which the RW recurrence, in the case of a finite number of unstable modes and of a generic initial perturbation exciting democratically all of them, is again described in term of elementary functions.

On Stability Properties of Extremum Seeking Systems with Oscillating Controls

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In many practical applications, it is necessary to steer a control system to the point of minimum (or maximum) of a partially or completely unknown and potentially time-varying cost or performance function [1]. Namely, assume that the values of the cost function $J(x) \in C^2(\mathbb{R}^n;\mathbb{R})$ can be measured for each $x=(x_1,\ldots,x_n)^T\in\mathbb{R}^n$, and x^* is a point of minimum of J. The purpose is to construct a control system whose trajectories $x(t)$ tend asymptotically to an arbitrary small neighborhood of x^*, assuming that only the values of the cost function $J(x(t))$ are available for control design. In this talk, we present a novel class of extremum seeking algorithms which solve the above problem based on the Lie brackets approximation idea [2]. In particular, we introduce the following system:

$$\dot{x} = \sum_{i=1}^{n} \Big(F_{1i}(J(x))u_{1i}(t) + F_{2i}(J(x))u_{2i}(t) \Big) e_i, \tag{1}$$

where $u_{1i}, u_{2i} : [0, +\infty) \to \mathbb{R}$ are periodic inputs, $F_{1i}, F_{2i} : \mathbb{R} \to \mathbb{R}$. In (1), e_i denotes ith unit vector in $\mathbb{R}^n$. We will describe the whole class of functions F_{1i}, F_{2i} such that the trajectories of system (1) approximate the gradient flow corresponding to the cost function J. As was shown in [4], this result unifies some known extremum seeking algorithms and also allows us to propose a new one. It will be shown that, under certain additional conditions, the point x^* is practically exponentially stable for system (1). Furthermore, we will also provide asymptotic stability conditions in the sense of Lyapunov. Besides, we will consider the case of a dynamical cost function $J = J(x,t)$ for which the conditions for the asymptotic stability in a neighborhood of the time-varying point of extremum $x^*(t)$ will be presented based on the results of [3]. The obtained results will be illustrated by several examples.

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Rattling in Spatially Discrete Diffusion Equations with Hysteresis

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We discuss reaction-diffusion equations with hysteretic nonlinearities on one- and two-dimensional lattices. Such equations arise as a result of the spatial discretization of continuous hysteretic and slow-fast reaction-diffusion models. We will show that the solutions typically form a propagating microstructure, which we call rattling. We analyze this microstructure and determine its propagation speed [1, 2].

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A Necessary and Sufficient Condition for Existence of Measurable Flow of a Bounded Borel Vector Field

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Let $b\colon I \times \mathbb{R}^d \to \mathbb{R}^d$ be a bounded Borel vector field, where $I = [0,T]$, $T > 0$. Consider the standard Cauchy problem

$$\begin{cases} \partial_t \gamma(t) = b(t, \gamma(t)), & t \in (0,T) \\ \gamma(0) = x, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^d$ and $\gamma\colon I \to \mathbb{R}^d$.

The classical sufficient conditions for existence of solutions of (1), such as Peano's or Caratheodory's theorems, require continuity of $b(t,x)$ with respect to x. In [1] it was proved that under such assumptions it is possible to choose for each $x \in \mathbb{R}^d$ a solution $\gamma_x\colon I \to \mathbb{R}^d$ of (1) in such a way that the map $F\colon x \mapsto \gamma_x$ is Borel measurable.

Existence of the Borel measurable *flow* F allows one to construct measure-valued solutions μ_t of the continuity equation

$$\begin{cases}\partial_t\mu_t + \operatorname{div}(b\mu_t) = 0,\\ \mu_t|_{t=0} = \bar{\mu}\end{cases} \tag{2}$$

for any given Radon measure $\bar{\mu}$ on $\mathbb{R}^d$.

Let us denote $\Gamma := C(I;\mathbb{R}^d)$ (endowed with the uniform metric) and let $\bar{\mu}$ be a non-negative Radon measure on $\mathbb{R}^d$. A Borel function $F\colon \mathbb{R}^d \to \Gamma$ is called a $\bar{\mu}$-measurable flow of b if for $\bar{\mu}$-a.e. $x \in \mathbb{R}^d$ the function $\gamma = F(x)$ solves (1).

We prove that a $\bar{\mu}$-measurable flow of b exists *if and only if* (2) has a non-negative measure-valued solution with the initial condition $\bar{\mu}$.

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Analysis of Maxwell's Equations in Metamaterials

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Metamaterial is an artificially structured composite material which has exotic properties with revolutionary applications in many areas. Since the first successful construction of the double negative metamaterial in 2000, it has attracted great interest of researchers from various areas. In this talk, I will present some modeling equations in simulating wave propagation in metamaterials.

Well-posedness of the PDEs for backward wave propagation, invisibility cloaks will be discussed. Some numerical methods and simulation results will be presented also.

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Solution of the Dirichlét Problem for a Degenerating *B*-elliptic Equation

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Let $\mathbb{E}_p^{++}$ be the part of the p-dimensional Euclidean space where $x_{p-1} > 0$ and $x_p > 0$. In $\mathbb{E}_p^{++}$, we consider the partial differential equation

$$x_p^m \left(\sum_{l=1}^{p-2} \frac{\partial^2 u}{\partial x_l^2} + B_{x_{p-1}} u \right) + \frac{\partial^2 u}{\partial x_p^2} + \lambda^2 x_p^m u = 0, \tag{1}$$

which is a B-elliptic equation with a positive parameter degenerating on the plane $x_p = 0$. Here $B_{x_{p-1}} = \dfrac{\partial^2}{\partial x_{p-1}^2} + \dfrac{k}{x_{p-1}} \dfrac{\partial}{\partial x_{p-1}}$ is the Bessel operator, $m > 0$, $k > 0$, $p \geqslant 3$, $\lambda \in \mathbb{R}$.

Denote by D the finite domain in $\mathbb{E}_p^{++}$ bounded by the surface Γ and the parts Γ_0 and Γ_1 of the planes $x_{p-1} = 0$ and $x_p = 0$, respectively, and by $C_{B^l}^k(D)$ the set of k times continuously differentiable functions in D satisfying the condition $\frac{\partial u}{\partial x_l} = \mathrm{o}(1)$ as $x_l \to 0$.

The aim of the study is to prove the existence of a unique solution of the following boundary value problems:

Dirichlét internal boundary value problem. To find a solution of equation (1) in the domain D, satisfying the conditions

$$u(x) \in C_{B^{p-1}}^2(D) \cap C_{B^p}^2(D) \cap C(\overline{D}) \cap C^1(D \cup \Gamma_1 \cup \Gamma_0),$$
$$u(x) = \mathrm{o}(1) \quad \text{as} \quad x_p \to 0,$$
$$u\big|_{\Gamma} = f(\xi), \quad f(\xi) \in C(\Gamma).$$

Dirichlét external boundary value problem. To find a solution of equation (1) in the domain D_e, satisfying the conditions

$$u(x) \in C_{B^{p-1}}^2(D_e) \cap C_{B^p}^2(D_e) \cap C(\overline{D_e}) \cap C^1(D_e \cup \Gamma_1 \cup \Gamma_0),$$
$$u(x) = \mathrm{o}(1) \quad \text{as} \quad x_p \to 0,$$
$$u\big|_{\Gamma} = f(\xi), \quad f(\xi) \in C(\Gamma),$$

and such that

$$\int_{S_R^+} |u|^2 \, x_p^k \, dS_R = \mathrm{O}(1), \quad \int_{S_R^+} \left| \frac{\partial u}{\partial r} - i\lambda u \right|^2 x_p^k \, dS_R = \mathrm{o}(1)$$

as $R \to \infty$, where $A[u] = \xi_p^m \sum\limits_{l=1}^{p-1} \cos(n, \xi_l) \frac{\partial u}{\partial \xi_l} + \cos(n, \xi_p) \frac{\partial u}{\partial \xi_p}$ is the conormal derivative.

The following theorem is proved:

Theorem. *If Γ is a Lyapunov surface and forms right angles with the planes $x_{p-1} = 0$ and $x_p = 0$, then the Dirichlét internal and external boundary value problems have a unique solution for any boundary data $f(\xi) \in C(\Gamma)$ and the solution of each of them can be represented as the potential of a double layer.*

A New Approach to the Nonlinear Generalization of the Rayleigh Quotient

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The main purpose of this talk is to present a new approach to the generalization of the Rayleigh quotient. Using the fibering map, we introduce the so-called Nonlinear Generalized (NG-) Rayleigh quotient. It turns out that the critical values of the NG–Rayleigh quotient contain an important information on the nonlinear differential equations [1]. In particular, we show that the applicability of the Nehari manifold method can be studied by means of the critical values of a corresponding NG–Rayleigh quotient. Theoretical results are illustrated by several examples of nonlinear boundary value problems. Furthermore, we demonstrate that the introduced tool of the nonlinear generalized Rayleigh quotient can also be applied to prove new results on the existence of multiple solutions for nonlinear elliptic equations [2].

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The Vanishing Discount Problem for Fully Nonlinear Degenerate Elliptic PDEs

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I explain an approach, based on generalized Mather measures, to the vanishing discount problem for fully nonlinear degenerate elliptic partial differential equations. Under mild assumptions, we introduce Mather viscosity measures for such PDEs, which are natural extensions of Mather measures originally due to J. Mather. Using the Mather viscosity measures, one can show that the whole family of solutions v^λ of the discount problem, with the discount factor λ, converges to a solution of the ergodic problem as λ goes to 0. This is based on joint work with Hiroyoshi Mitake (Hiroshima University) and Hung V. Tran (University of Wisconsin, Madison).

Doubly Asymptotic Trajectories near the Adiabatic Limit for Lagrangian Systems with Turning Points

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Let $\mathcal{M}$ be a compact Riemannian manifold. We consider a natural Lagrangian system with the Lagrangian

$$L(q, \dot{q}, t, \varepsilon) = \frac{1}{2}|\dot{q}|^2 - f(\varepsilon t)V(q), \tag{1}$$

where the potential $V(q)$ belongs to $C^2(\mathcal{M})$ and f is a periodic function of period T. The parameter ε is assumed to be small. It is also assumed that system (1) has N turning points, i.e.

(A_1) there exist N different solutions $\tau_l \in [0, T), l = 1, \ldots, N$, of the equation $f(\tau) = 0$;

(A_2) for each $l = 1, \ldots, N$, there exists a neighborhood of τ_l where the function f can be represented as $f(\tau) = (\tau - \tau_l)^{\varkappa_l} g_l(\tau)$ with $\varkappa_l \in \mathbb{N}$ and some C^1-function g_l such that $g_l(\tau_l) \neq 0$.

Since $\mathcal{M}$ is compact, the function V attains its maximum and minimum on $\mathcal{M}$. Let $X_{\max} = V^{-1}(V_{\max})$ and $X_{\min} = V^{-1}(V_{\min})$, where $V_{\max}$ ($V_{\min}$) stands for the maximum (minimum) value of V. Define a set X_c as follows:

$$X_c = \begin{cases} X_{\max}, & \mathcal{T}_- = \varnothing, \\ X_{\max} \cup X_{\min}, & \mathcal{T}_\pm \neq \varnothing, \\ X_{\min}, & \mathcal{T}_+ = \varnothing, \end{cases}$$

$$\mathcal{T}_+ = \{\tau \in \mathbb{R} : f(\tau) > 0\}, \qquad \mathcal{T}_- = \{\tau \in \mathbb{R} : f(\tau) < 0\}.$$

We assume that

(A_3) X_c consists of isolated nondegenerate critical points of V.

Theorem 1. *For any $x_\pm \in X_c$ there exist $\varepsilon_0 > 0$ and a subset $\mathcal{E}_h \subset (0, \varepsilon_0)$ such that*

1. *for any $\varepsilon_1 < \varepsilon_0$, one has* $\operatorname{leb}((0,\varepsilon_1) \setminus \mathcal{E}_h) = O\left(\mathrm{e}^{-c/\varepsilon_1}\right)$ *with some positive constant c, where* leb *is the Lebesgue measure;*
2. *for any $\varepsilon \in \mathcal{E}_h$, $x_\pm$ are hyperbolic equilibria of system* (1)*;*
3. *for any $\varepsilon \in \mathcal{E}_h$ there exist infinitely many doubly asymptoic trajectories of system* (1) *emanating from x_1 and terminating at x_2.*

The proof is based on the observation that in a small neighborhood of a turning point system (1) can be approximated by a model system with the Lagrangian

$$L(q, q', \zeta) = \frac{1}{2}|q'|^2 - \zeta^{\varkappa} V(q), \quad \varkappa \in \mathbb{N}, \quad q' = \frac{\mathrm{d}q}{\mathrm{d}\zeta}.$$

The existence of doubly asymptotic trajectories for the model system was earlier established in [1] by variational methods. Using the contraction principle in an appropriate functional space, one can prove the existence of doubly asymptotic trajectories for system (1).

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Boundary Value Problems for Differential-Difference Equations with Incommensurable Shifts of Arguments

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Consider the boundary value problem

$$-\sum_{i,j=1}^{n} (R_{ijQ} u_{x_j})_{x_i} = f(x) \quad (x \in Q), \tag{1}$$

$$u(x) = 0 \quad (x \notin Q). \tag{2}$$

Here Q is a bounded domain in $\mathbb{R}^n$ with piecewise-smooth boundary and $f \in L_2(Q)$. Difference operators $R_{ij} = R^a_{ij} + R^b_{ij}$, $R^a_{ij}, R^b_{ij} : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ look as follows

$$R^a_{ij} u(x) = \sum_{h \in M_1} a_{ijh} u(x+h) \quad (\, a_{ijh} \in \mathbb{R}),$$

$$R^b_{ij} u(x) = \sum_{p \in M_2} b_{ijp} u(x+p) \quad (\, b_{pij} \in \mathbb{R}).$$

Here M_1 is a finite set of vectors with commensurable coordinates, M_2 is also a finite set of vectors with commensurable coordinates, meanwhile the coordinates of the vectors h are not commensurable with the coordinates of the vectors p.

We are interested in the unique solvability of the problem in the Sobolev space $\mathring{H}^1(Q)$ (by a solution, we mean a generalized solution understood in the standard way). Exact solvability conditions in terms of coefficients were earlier found [1] for a particular form of the operator. We obtain strong ellipticity conditions taking the shape and the size of the domain Q into account and stable with respect to small perturbations of argument's shifts.

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Joint Optimization of the Trajectory and the Main Design Parameters of Electric Propulsion System

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The problem of joint optimization for the main design parameters of the electric propulsion system and the trajectory of the spacecraft is considered. We consider the following two problems: to minimize thrust and to maximize the useful mass of the spacecraft with the optimum characteristics of the propulsion system (for example, thrust, exhaust velocity, and power). The first problem is related to the existence of optimal trajectories with finite thrust. If there is no a priori information on the existence of solutions, then it is difficult to construct stable and efficient methods of numerical optimization. Indeed, if a numerical method does not converge, then it can not be determined whether the method fails or the optimal control problem have no solutions at all. Therefore, it is important to find the region of existence of the solution for the numerical simulation. The latter problem is directly related to the optimization of the characteristics of the propulsion system within the domain of existence of the solution.

The main goal of joint optimization of parameters and trajectory of spacecraft is the maximization of the useful mass of the spacecraft. This problem has been considered by many authors. A typical approach is related to the separation of the problem into dynamic and parametric subproblems. The aim of the dynamic subproblem is to optimize the trajectory for fixed values of the main design parameters (thrust, exhaust velocity, power), and the goal of the parametric subproblem is to optimize the main design parameters. Unfortunately, the complete separation of the joint optimization problem into dynamic and parametric subproblems was performed only for a limited power problem. A realistic mathematical model for electric propulsion with a constant exhaust velocity requires recalculation of the optimal trajectory for each new value of any design parameter. Therefore, joint optimization requires iterations over a set of main design parameters and optimization of the trajectory at each iteration. Direct methods of minimization allow one to optimize the trajectory and main design parameters jointly by means of non-linear programming. But the direct approach has a low degree of convergence, low or medium accuracy, and an inaccurate completion criterion. To overcome these shortcomings, we consider the problem of joint optimization using an indirect approach based on the necessary optimality conditions.

For optimization, we use a simple design model of a spacecraft with an electric propulsion system and the Pontryagin maximum principle. We give numerical examples of the solution of the joint optimization problem applied to the interplanetary space missions.

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On the Two Symmetries in the Theory of m-Hessian Operators

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Let Ω be a bounded domain in $\mathbb{R}^n$, $u \in C^2(\bar{\Omega})$, u_{xx} be the Hessian matrix for u. The operators $F[u] := F(u_{xx})$ are called Hessian operators. The simplest examples of those are the p-Hessian operators $T_p[u] = T_p(u_{xx})$, $1 \leqslant p \leqslant n$, where $T_p(S)$ is the p-trace of a symmetric matrix S. The first kind of symmetry to be discussed is the orthogonal invariance of the operators F:

$$F(u_{xx}) = F(Bu_{xx}B^T), \quad B \in O(n). \tag{1}$$

The second one is the skew symmetry of the operator F, which means by our definition that $F(u_{xx})$ is a linear combination of minors of $\det u_{xx}$. We plan to show the interaction of these symmetries in the theory of p-Hessian operators.

It is well known that symmetry (1) is necessary for well-posedness of the Dirichlet problem for Hessian equations. For instance, the following proposition holds due to (1).

Theorem 1. *Let $n \geqslant 2$, $\Omega \subset \mathbb{R}^n$, $\partial\Omega \in C^4$, and $f \in C^2(\bar{\Omega})$. Assume that the Gauss curvature $\mathbf{k}_n[\partial\Omega]$ is positive and $f \geqslant \nu > 0$. Then there exists in $C^2(\bar{\Omega})$ a unique solution u to the problem*

$$T_{m;l}[u] = \frac{T_m}{T_l}[u] = f, \quad u|_{\partial\Omega} = 0, \quad 0 \leqslant l < m \leqslant n, \tag{2}$$

if $(m-l)$ is odd, and there are exactly two solutions, u and $-u$, otherwise.

The skew symmetry of m-Hessian operators brings out the qualitative results. For instance, the following imbedding-type theorem is true.

Theorem 2. *Let $u \in \overset{\circ}{W}{}^2_1(\Omega)$, $\partial\Omega$ satisfy the condition of Theorem 1, and v^m be a solution to problem (2) with $l = 0$. Then the inequalities*

$$\left(\int_\Omega u dx\right)^2 \leqslant \frac{1}{m}\int_\Omega -v^m dx \int_\Omega T^{ij}_m[v^m]u_i u_j dx, \quad m = 1, \ldots, n, \tag{3}$$

are valid. Here $T^{ij}_m(S) := \frac{\partial T_m(S)}{\partial s_{ij}}$, $u_i = \frac{\partial u}{\partial x^i}$, $i, j = 1, \ldots, n$.

Notice that all inequalities (3) are sharp.

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Mathematical Modelling and Analysis of Processes in Poroelastic Media — Applications in Medicine

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Modelling reactive flows, diffusion, transport and mechanical interactions in media consisting of multiple phases, e.g. of a fluid and a solid phase in a porous medium, is giving rise to many open problems for multi-scale analysis and simulation. The following processes are taken into account in this lecture:

- diffusion, transport, and reaction of substances in the fluid and the solid phase,
- mechanical interactions of the fluid and solid phase,
- change of the mechanical properties of the solid phase by chemical reactions,
- growth of the materials.

Using a scale limit, an effective model is derived, coupling the filtration flow, the mechanical deformation and the chemical reactions on the macroscopic level. A Biot-law is replacing the Darcy-law, which is used for non-deformable media. Processes in biological tissues are discussed as applications and the following examples are presented:

- the swelling of cells caused by hypoxia, modelled and simulated, using a Biot-law for the flow inside cells,
- the biochemical and biophysical processes in the arterial wall relevant for its mechanical behaviour, for inflammation and disorders of the cardiovascular systems,
- the movement of a tooth in the alveolar bone under external forces,
- the mechanical behaviour of a ligament, a tissue connecting teeth with bone, modelled by a Biot-law.

The lecture is based on results obtained in co-operation with Andro Mikelic, Maria Neuss-Radu, Martina Kihn, Valeria Malieva.

Regularization of the Continuation Problem for Solutions from Data Given on the Part of the Boundary

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We consider several numerical approaches for parameters identification of the medium. One of them is based on the continuation problems of physical fields with the data on the part of the boundary [1,2,3], which arise in geophysics, tomography, and in the problem of protection of nuclear reactors. The second one is a method based on the conservation laws and the solution of the coefficient inverse problems.

Continuation problems are ill-posed and we formulate this problems in the form of the operator equation $Aq = f$, for which the minimization of the objective functional and the method of singular value decomposition [2,3] are applied. We study the properties of the operator A and the algorithm of minimization of the functional $J(q) = \|Aq - f\|^2$ by the conjugate gradient method. Series of numerical experiments show that this allows us to recover the boundary conditions on the inaccessible part of the boundary, as well as to obtain information about inhomogeneities (the number, location, approximate volume) located in the region of inaccessibility.

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Direct Scheme Method of Constructing Asymptotic Solutions for Three-Tempo Optimal Control Problems

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The most part of publications devoted to singular perturbations deals with problems with one slow and one fast variables. On the other hand, a lot of mathematical models is described with the help of multi-tempo systems (see [1]). We consider the following three-tempo optimal control problem

$$J_\varepsilon(u) = \int_0^T F(v,t,\varepsilon)\,dt \to \min,$$

$$\mathbb{E}(\varepsilon)dw/dt = \Phi(v,t,\varepsilon),\ w(0,\varepsilon) = w^0,$$

where $\varepsilon>0$ is a small parameter, T is fixed, $t \in [0,T]$, $w(t,\varepsilon)=(x(t,\varepsilon)', y(t,\varepsilon)', z(t,\varepsilon)')'$, $v(t,\varepsilon) = (w(t,\varepsilon)', u(t,\varepsilon)')'$, $x(t,\varepsilon) \in \mathbb{R}^{n_1}$, $y(t,\varepsilon) \in \mathbb{R}^{n_2}$, $z(t,\varepsilon) \in \mathbb{R}^{n_3}$, $u(t,\varepsilon) \in \mathbb{R}^m$, the prime denotes the transposition, $\mathbb{E}(\varepsilon) = diag(I_{n_1}, \varepsilon I_{n_2}, \varepsilon^2 I_{n_3})$, I_{n_i} are identity operators in spaces $\mathbb{R}^{n_i}$, $i = \overline{1,3}$. Following [2], we construct an arbitrary order asymptotic solution by the direct scheme method. The last one consists of an immediate substitution of the asymptotic expansion $v(t,\varepsilon) = \overline{v}(t,\varepsilon)+\sum_{i=0}^{1}(\Pi_i v(\tau_i,\varepsilon)+Q_i v(\sigma_i,\varepsilon))$ for $v(t,\varepsilon)$ in the problem, and construction of a series of optimal control problems to find the terms of the asymptotic expansion. Here $\tau_i = t/\varepsilon^{i+1}$, $\sigma_i = (t-T)/\varepsilon^{i+1}, i = 0,1$, $\overline{v}(t,\varepsilon) = \sum_{j\geqslant 0}\varepsilon^j \overline{v}_j(t)$, $\Pi_i v(\tau_i,\varepsilon) = \sum_{j\geqslant 0}\varepsilon^j \Pi_{ij} v(\tau_i)$, $Q_i v(\sigma_i,\varepsilon) = \sum_{j\geqslant 0}\varepsilon^j Q_{ij} v(\sigma_i)$; $\overline{v}_j(t)$ are regular functions, $\Pi_{ij} v(\tau_i)$ and $Q_{ij} v(\sigma_i)$ are boundary functions of the exponential type in neighborhoods of $t = 0$ and $t = T$, respectively. Some problems for finding the zero order asymptotic solution were given in [1]. We construct linear-quadratic optimal control problems for finding higher order asymptotic solutions.

Linear-quadratic optimal control problems under cheap control with two different costs are reduced by change of variables to three-tempo linear-quadratic optimal control problems in the singular case [3]. Using the direct scheme method, we construct asymptotic solutions of arbitrary orders. Note that the first order asymptotic solution for this problem was obtained in [3].

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Higher Derivatives of Solutions for Elliptic Systems with Discontinuous Coefficients

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Consider an elliptic equation or a system of the form

$$\mathrm{div}^m(A(x)D^m u) = f(x), \quad x \in \mathbb{R}^n,$$

with the matrix coefficient $A \in L_\infty$ satisfying the standard structure conditions and the natural energy space W_2^m. Obviously, if the derivative DA lies in L_n, then there exists $D^{m+1}u \in L_2$. But the typical discontinuity $x/|x|$ just fails to be in W_n^1, and $x'/|x'|$ is far beyond W_n^1 (here x' is a subvector of coordinates).

We treat general discontinuities of such type (including dense discontinuities like in Souček and John–Malý–Stará examples of solutions with dense set of discontinuities) by means of the dual Morrey spaces. We will discuss estimates for higher derivatives $D^{m+\alpha}u$ of solution as well as for lower derivatives $D^{m-\alpha}u$, and also the solvability in $W_2^{m-\alpha}$. The results are new even for second-order equations.

Bitsadze–Samarsky Boundary Condition for Elliptic-Parabolic Volume Potential

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In [1], A. V. Bitsadze and A. A. Samarsky considered for the first time a non-classical boundary value problem for the Laplace equation where the usual boundary conditions were only set on one part of the boundary, while the values of the unknown function on the remaining part were related to the values on an inner surface. In view of the great theoretical and practical importance, the study of this class of non-classical problems has developed rapidly [2]. Problems of this type are called Bitsadze–Samarsky problems.

In this paper, we obtain an integral representation for the elliptic-parabolic volume potential. Further, using the boundary condition of the Newtonian and the heat potentials, we find the Bitsadze–Samarsky type boundary condition for this potential.

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Resonance Capture in the System of Two Oscillators

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We consider a system of differential equations that describes the interaction of two weakly connected nonlinear oscillators. The initial data are such that, in the absence of connection, the first oscillator is far from equilibrium and the second oscillator is near equilibrium; the eigenfrequencies of the oscillators are close to each other. The capture into resonance is investigated when the frequencies of the connected oscillators remain close and the amplitudes of their oscillations undergo significant time variations; in particular, the second oscillator moves far from the equilibrium. We find that the initial stage of the resonance capture is described by a solution of the second Painleve equation. The description is obtained under an asymptotic approximation with respect to a small parameter corresponding to the connection factor.

A simplest model of interacting oscillators is given by the system of differential equations

$$\frac{d^2x}{dt^2} + x - 2x^3 = \varepsilon[f_0\xi + f_1\dot{\xi}], \quad \frac{d^2\xi}{dt^2} + \omega^2\xi - w\xi^3 = \varepsilon[g_0x + g_1\dot{x}], \quad t > 0,\ 0 < \varepsilon \ll 1.$$

Here ε is a small parameter, $\omega, w, f_1, f_2, g_1, g_2 = \mathrm{const}$.

This system is reduced to three differential equations by averaging method [1]. The averaged system is similar to one considered in [2], and it is reduced to the Painleve-2 equation [3]:

$$\frac{d^2z}{d\tau^2} = z(J + \nu\tau - \frac{1}{2}z^2) + \mu.$$

Here $\tau = \varepsilon^{2/3}t$ is a slow time, $J, \nu, \mu = \mathrm{const}$.

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On Oscillations of a String System with the Hysteresis-Type Condition[1]

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In this talk, we consider the initial-boundary value problem describing the oscillation process with a condition of the hysteresis type. This kind of problem occurs in modeling of the string oscillation where the movement is restricted by a sleeve concentrated at one point. We investigate the case where the sleeve is located at the node of the graph-star. An analogue of the d'Alambert formula is obtained.

Consider a mechanical system of n strings, which are segments OA_1, OA_2,... OA_n in equilibrium, where O, A_1, A_2,... A_n belong to a plane π. The ends of the strings are interconnected at the point O. There is a vertical borehole inside the sleeve passing through the point O. Introduce a coordinate system to describe string deformations. The axis Ox_i for ith string ($i = 1, 2, ...n$) contains the segment OA_i and is directed from A_i to O. The axis OY is perpendicular to the plane π. Let $u_i(x, t)$ be the deviation of ith string from the equilibrium position at time t. We assume that the length of all the strings equals l, i.e. $0 \leqslant x \leqslant l$.

Thus, the oscillation of each string of the system can be described by the wave equation $\dfrac{\partial^2 u_i}{\partial x^2} = \dfrac{\partial^2 u_i}{\partial t^2}$. The connection between the strings at the node is expressed by $u_1(l, t) = u_i(l, t)$ $(i = 1, 2, ..n)$. There is the sleeve at the point $x = l$, whose motion perpendicular to the plane π is given by

$$C(t) = [-h, h] + \xi(t).$$

The mathematical model of such problem is described by

$$\begin{cases} \dfrac{\partial^2 u_i}{\partial x^2} = \dfrac{\partial^2 u_i}{\partial t^2}, \quad 0 < x < l,\, 0 < t < T \quad (i = 1, 2, ...n), \\ u_i(x, 0) = \varphi_i(x), \\ \dfrac{\partial u_i}{\partial t}(x, 0) = 0, \\ -\sum\limits_{i=1}^{n} \dfrac{\partial u_i}{\partial x}(l - 0, t) \in N_{C(t)}(u(l, t)), \\ u(l, t) = u_1(l, t) = u_2(l, t) = \ldots = u_n(l, t), \\ u(l, t) \in C(t), \\ u_i(0, t) = 0, \end{cases} \tag{1}$$

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where the set $N_C(a)$ is the outward normal cone to C at a defined by

$$N_C(a) = \{\xi \in R^1 : \xi \cdot (c - a) \leqslant 0 \quad \forall c \in C\}.$$

Theorem. *Assume that the function $\xi(t)$ satisfies the Lipschitz condition and $\varphi_i \in W_2^1[0,l]$. Then the solution of problem (1) can be represented as*

$$u_i(x,t) = \frac{\Phi_i(x-t) + \Phi_i(x+t)}{2}.$$

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Relaxation Periodic Solutions in a System with Delay and Finite Nonlinearity

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Consider the system of two delay differential equations

$$\begin{cases} \dot{u_1} + u_1 = \lambda F(u_1(t-T)) + \gamma(\ln\lambda)^{-1}\lambda^{-\alpha}(u_2 - u_1), \\ \dot{u_2} + u_2 = \lambda F(u_2(t-T)) + \gamma(\ln\lambda)^{-1}\lambda^{-\alpha}(u_1 - u_2), \end{cases} \tag{1}$$

which simulates two coupled oscillators with nonlinear feedback. Here $T > 0$ is the delay time, $\gamma > 0$ is a parameter, and α belongs to the interval $(0, \frac{1}{2}]$. The feedback function $F(u)$ is compactly supported ($F(u) = 0$ if $|u| \geqslant p$, where p is some fixed positive constant), piecewise smooth, $F(u) < a_1 < 0$ if $-p < u < 0$, $F(0) = 0$, and $F(u) > a_2 > 0$ if $0 < u < p$.

The goal is to study dynamics of system (1) under the assumption that the positive parameter λ is sufficiently large ($\lambda \gg 1$).

Introduce a set of initial conditions for system (1). Let x and β be such that $|x| \geqslant 1$ and $0 < \beta \leqslant \alpha$, a parameter k be 1 or -1, a parameter m be 1 or 2 with $m+1 = 2$ if $m = 1$ and $m+1 = 1$ if $m = 2$. Fix some values x, k, and m. Let $S(x) \subset C[-T, 0](\mathbb{R}^2)$ be a set of functions $u_m(s)$ and $u_{m+1}(s)$ ($s \in [-T, 0]$) such that

$$\begin{cases} |u_m(s)| \geqslant p, & u_m(0) = kp, \\ |u_{m+1}(s)| \geqslant p, & u_{m+1}(0) = xp\lambda^{\beta}. \end{cases} \tag{2}$$

All solutions of system (1) with initial conditions from the set $S(x)$ were studied by a special method of large parameter. It was proved that system (1) has four stable coexisting relaxation periodic solutions satisfying initial conditions (2) with period $t_0 = (2(1-\alpha) + o(1))\ln\lambda$.

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Dynamics of Equation with Large Spatial Control

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Consider a complex parabolic equation with the so-called spatial control,

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} + (a - b|u|^2)u + Ke^{i\varphi}\Big(\int\limits_{-\infty}^{\infty} F(s)u(t, x+s)ds - u\Big), \tag{1}$$

$$u(t, x + 2\pi) \equiv u(t, x). \tag{2}$$

Here $Re\, a, Re\, b > 0$, $K > 0$ $(Im\, K = 0)$, $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $d \geqslant 0$,

$$F(x) = \frac{1}{\sqrt{\mu\pi}} \exp\left(-\frac{(x+h)^2}{\mu}\right), \qquad 0 < \mu \ll 1.$$

Our main assumption is that K is large enough.

Let $K = \varepsilon^{-1}$, $0 < \varepsilon \ll 1$ and there exist $\varkappa > 0$ such that $\mu = \varkappa\varepsilon$.

The goal is to study behaviour of solutions to (1), (2) for sufficiently small ε in some neighbourhood of zero in the phase space $C^2_{[0,2\pi]}(\mathbb{C})$.

After division (1) by K and time renormalization, we come to the equivalent problem

$$\frac{\partial u}{\partial \tau} = \varepsilon d\frac{\partial^2 u}{\partial x^2} + \varepsilon[a - b|u|^2]u + e^{i\varphi}\Big[\int_{-\infty}^{\infty} F(s)u(\tau, x+s)ds - u\Big],$$

$$u(\tau, x + 2\pi) \equiv u(\tau, x).$$

The characteristic equation for the linearized problem is

$$\lambda_m = \varepsilon a + e^{i\varphi}[\exp[-ihm - \varkappa\varepsilon m^2] - 1] - \varepsilon d m^2 \qquad (m \in \mathbb{Z}). \tag{3}$$

The so-called critical cases (when (3) has no roots with positive real parts and there exists λ_m whose real part tends to zero as $\varepsilon \to 0$) are considered. In fact, the number of such λ_m is infinite, so all critical cases have the infinite dimension. The theory of local invariant integral manifolds and methods of normal forms are not applicable directly.

In order to study the local dynamics for infinite-critical cases, a special method of the so-called quasi-normal forms is used in [1, 2]. The main idea of this method is to construct a special substitution by means of which the initial equation is reduced to a problem that does not contain small parameters (or depends on them regularly). Unlike the initial equation, this problem (the quasinormal form), can easily be investigated numerically. In all situations, quasi-normal forms are complex parabolic equations. Solutions of quasi-normal forms give the main part of asymptotics of solutions of the principal problem.

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Vector mKdV Equation and Inverse Scattering Transform

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We propose the vector modified Korteweg–de Vries equation (v-mKdV) of the form

$$\partial_t U + 6[U \times \partial_x^2 U] + 6(U\partial_x |U|^2 - |U|^2 \partial_x U) + \partial_x^3 U = 0, \quad (x,t) \in \mathbb{R}^2,$$

where $U(x,t)$ is a three-dimensional vector-function, and the sign "$\times$" means the vector product in $\mathbb{R}^3$. This equation is a new completely integrable evolutionary equation. We develop the Inverse Scattering Transform for it. By use of this method, we construct an infinite sequence of the integrals of motion for the v-mKdV equation. Our main results consist in obtaining explicit formulas for the vector solutions of the multi-soliton, doublet, and the breather types.

Limit Theorems for Interval Exchange Transformations

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Consider a translation flow h_t^+ on a generic flat surface (M,ω) from the given stratum in the moduli space. Let φ be any weakly Lipschitz function on M. Then its ergodic integral

$$S_T(\varphi, x) = \int_0^T \varphi(h_t^+ x)\, dt$$

can be considered as a random variable if x is chosen randomly according to the Lebesgue measure on M, which is h_t^+-invariant.

A. Bufetov studied asymptotics of *distributions* of (normalized) random variables $S_T(\varphi)$ as $T \to \infty$. Namely, put $\hat{S}_T(\varphi) = [S_T(\varphi) - \mathbb{E}S_T(\varphi)]/\sqrt{\operatorname{Var} S_T(\varphi)}$.

Theorem 1 (A. Bufetov, [1]). *The function* $\mathrm{Law}(\hat{S}_T(\varphi))$ *does not converge but has only partial limits as* $T \to \infty$. *These partial limits correspond to subsequences* (T_n) *such that the image* $g^{\log T_n}(M)$ *of* M *under the Teichmüller flow approaches a fixed flat surface* M_0 *belonging to some full-measure set in the moduli space. The function* φ *can be chosen arbitrarily from the complement to a finite-codimension space (which depends on* M_0*).*

We will discuss a similar result for interval exchange transformations with

$$S_n(\varphi, x) = \sum_{k=0}^{n-1} \varphi(T^n x).$$

Here the following difficulty occurs. For a translation flow, there is a flow along another foliation, hence, one can use duality between the corresponding cocycles. But for interval exchange transformations there are no such flows. Another aspect of this difficulty is that the Teichmüller flow rescales time by a fixed factor in ergodic integral, while the Rauzy induction for IET rescaling depends on the point.

Theorem 2. *The function* $\mathrm{Law}(\hat{S}_n(\varphi))$ *has partial limits as* $n \to \infty$. *These limit distributions are the same as for translation flows.*

The key idea in the proof is to approximate ergodic sums for IET by ergodic integrals for an appropriate translation flow.

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Dissipative Evolution Problems in Metric Spaces

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This talk is concerned with the evolution problems in complete metric spaces, which are described by the mutational equations introduced by J. P. Aubin [1].

The results in [3], on the existence, uniqueness and continuous dependence on data of solutions to the initial value problems for ordinary differential equations in Banach spaces, are extended here to the case of mutational equations in complete metric spaces. For this purpose, the dissipativity condition on mutational equations with respect to metric-like functionals on metric spaces is proposed together with the subtangential condition combined with growth conditions. Our results are applied to the initial value problems for the quasilinear evolution equations investigated by Hughes et al. [2].

The details of the results are given in [4]. We refer to the bibliographies in Lorenz's book [5] for more information on mutational equations in metric spaces.

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On Global Attractors of Hamilton Nonlinear PDEs

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The theory of global attractors of Hamilton nonlinear PDEs was initiated by the author in 1990, and developed since 1995 in collaboration with H. Spohn (Muenchen TU) and M. Kunze (Essen Univ.), and since 2004 in collaboration with V. S. Buslaev (St. Petersburg and Toronto Univ.), P. Joly (INRIA, Rocquencourt), A. A. Komech (A&M Texas Univ.), E. Kopylova (IITP RAS), D. Stuart (Cambridge Univ.), A. Vinnichenko (MCCME) and others. The investigation is inspired by mathematical problems of quantum physics: Bohr's transitions between quantum stationary states and wave-particle duality, see the details in [1]. A survey of the results can be found in [2]. Main results mean the following global attraction:

Each finite energy solution converges to a finite-dimensional attractor A in the Hilbert phase space as $t \to \pm\infty$.

The structure of the global attractor crucially depends on the symmetry group of the equation:

I. For a generic equation, the attractor A is the set of all stationary states $s(x)$.

II. For generic $U(1)$-invariant equations, the attractor A is the set of all solitary waves $e^{-i\omega t}\psi(x)$.

III. For generic translation-invariant equations, the attractor A is the set of all soliton solitons $\psi(x-vt)$.

Our general (open) conjecture is the following. Let G be a Lie group (for example, $SU(2)$ or $U(3)$).

For generic G-invariant nonlinear Hamilton PDE each finite energy solution converges as $t \to \pm\infty$ to a finite-dimensional attractor A which is the set of solutions $U(t)\psi$ where $U(t)$ is a one-parametric subgroup $U(t)$ of G.

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On Stabilization Conditions for Solutions of Nonlinear Parabolic Equations

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We study solutions of the equations

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,u) \frac{\partial^2 u}{\partial x_i \partial x_j} + f(x,u,Du) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1}$$

where $D = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ is the gradient operator and $\|a_{ij}(x,\zeta)\|$ is a positive definite matrix for all $x \in \mathbb{R}^n$ and $\zeta \in \mathbb{R} \setminus \{0\}$. Also, let there exist locally bounded measurable functions $g : (0,\infty) \to (0,\infty)$, $h : (0,\infty) \to (0,\infty)$, and $p : \mathbb{R}^n \to [0,\infty)$ such that

$$\inf_K g > 0 \quad \text{and} \quad \inf_K h > 0$$

for any compact set $K \subset (0,\infty)$ and, moreover,

$$f(x,\zeta,\lambda x) \operatorname{sign} \zeta \geqslant p(x) g(|\zeta|) \left(1 + \sum_{i,j=1}^{n} |a_{ij}(x,\zeta)| \right)$$

for all $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R} \setminus \{0\}$, and $0 \leqslant \lambda \leqslant p(x) h(|\zeta|)$.

By a solution of (1) we mean a function u having two continuous derivatives with respect to x and one continuous derivative with respect to t and satisfying equation (1) in the classical sense [1]. Denote

$$q(r) = \inf_{B_r} p, \quad r \in (0,\infty),$$

where $B_r = \{x \in \mathbb{R}^n : |x| < r\}$. For any function $\varphi : (0,\infty) \to \mathbb{R}$ and a real number $\theta > 1$, we put

$$\varphi_\theta(\zeta) = \inf_{(\zeta/\theta, \theta\zeta)} \varphi, \quad \zeta \in (0,\infty).$$

Theorem 1. *Let*

$$\int_1^\infty rq(r)\,dr = \infty$$

and, moreover,

$$\int_1^\infty (g_\theta(\zeta)\zeta)^{-1/2}\,d\zeta < \infty \quad \textit{and} \quad \int_1^\infty \frac{d\zeta}{h_\theta(\zeta)} < \infty$$

for some real number $\theta > 1$. *Then any solution of* (1) *stabilizes to zero uniformly on an arbitrary compact set* $K \subset \mathbb{R}^n$ *as* $t \to \infty$, *i.e.*

$$\lim_{t\to\infty} \sup_{x\in K} |u(x,t)| = 0.$$

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On Oscillations of Joined Bodies with Fluid-Filled Cavities

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We consider the problem of small oscillations of two pendulums joined one to another by a spherical hinge. Each pendulum has a cavity partially filled with homogeneous fluid.

For the case of ideal fluids, we formulate and study the initial boundary value problem for movements of bodies and fluids in cavities. This problem is transformed to the Cauchy problem for a first order differential-operator equation in some Hilbert space. It is shown that the main operator matrix of this equation is the generator of a contracting semigroup. Based on this fact, we prove the theorem on correct solvability of the considered problem.

If there is no friction in the hinge (conservative dynamic system), then we consider the corresponding eigenvalue problem for frequencies of oscillations. We obtain the variational principle for this problem and propose the Ritz approach for calculations.

For the case where fluids in cavities are viscous, we also formulate and study the problem of small movements of the hydromechanical system. In the same way as in the previous case, we transform the original problem to the Cauchy problem for a first order differential-operator equation in a Hilbert space with the main operator matrix generating an analytic semigroup. Thus we prove the theorem on correct solvability of the problem for viscous fluids.

Further, we study the so-called normal oscillations of the system, coming to investigation of the known nonselfadjoint S. Krein operator pencil.

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Depression of Euler Equations for Motion of Compressible Medium Flow

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1. *The Euler Equations.* These equations describe motion of ideal (inviscid) fluid and gas medium flow, the case of compressible medium being most popular in practice. Equations of that type are of mathematical interest and have a lot of applications in engineering, aviation and rocket-space engineering, and pipeline transportation. The general version for the $3D$ case is a system of four nonlinear equations with five unknowns [1]. These unknown are the components u, v, and w of the velocity vector, pressure p, and density ρ. One needs to determine these unknown as functions of spatial coordinates x, y, z and time t. To date, theoretical study of these equations lags behind requirements of practice. The main disadvantage is the lack of a constructive method of solution while preserving the nonlinear terms. The first step in this direction is depression of the original equations.

2. *Approach to problem solution.* The essence of the proposed approach is to reduce the basic problem of solving the initial equations to a set of more simple tasks. If we proceed from the purely mathematical properties of the original equations, then we can state that each of them can be represented in the free divergent form

$$\frac{\partial P_i}{\partial x}+\frac{\partial Q_i}{\partial y}+\frac{\partial R_i}{\partial z}+\frac{\partial S_i}{\partial t}=0. \tag{1}$$

Here P_i, Q_i, R_i, S_i, are some combinations of the major unknowns u, v, w, p, ρ. Each equation of form (1) allows integration in general. So the general solution has the form

$$P_i=\frac{\partial\Psi_{2,i}}{\partial y}-\frac{\partial\Psi_{4,i}}{\partial z}-\frac{\partial\Psi_{6,i}}{\partial t},\quad Q_i=-\frac{\partial\Psi_{2,i}}{\partial x}+\frac{\partial\Psi_{5,i}}{\partial z}-\frac{\partial\Psi_{3,i}}{\partial t},$$
$$R_i=\frac{\partial\Psi_{4,i}}{\partial x}-\frac{\partial\Psi_{5,i}}{\partial y}+\frac{\partial\Psi_{1,i}}{\partial t},\quad S_i=-\frac{\partial\Psi_{6,i}}{\partial x}+\frac{\partial\Psi_{3,i}}{\partial y}-\frac{\partial\Psi_{1,i}}{\partial t}. \tag{2}$$

Here $\Psi_{k,i}$, $k=1,2,...,6$ are some twice differentiable functions of four variables called the associated unknowns [2, 3].

Equality of form (2) for each initial equation can be converted so as to exclude any nonlinear and non-divergent terms. As a result, we arrive at ten equations linking the major unknowns u, v, w, p, ρ and the associated ones. They do not contain derivatives of the five major unknowns. With respect to the major unknowns, these ten equations represent algebraic relations. Considered together, they provide the first integral of the Euler equations for the case of compressible medium flow. Thus in mathematical terms, solving these equations is a more simple task.

In the case of incompressible medium, the similar approach was effectively used by the author to solve some specific problems [4, 5].

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On Global Attraction to Solitary Waves for Klein–Gordon Equation with Concentrated Nonlinearities

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The long-time asymptotics is analyzed for solutions to 3D Klein–Gordon equation with concentrated nonlinearities. The nonlinear generator coincides with the free Klein–Gordon operator when restricted to sufficiently regular functions.

Our main result is the global attraction of each "finite energy solution" to the set of all solitary waves as $t \to \pm\infty$. This attraction is caused by the nonlinear energy transfer from lower harmonics to the continuous spectrum and subsequent dispersion radiation.

We justify this mechanism by the following strategy based on inflation of spectrum by the nonlinearity. We show that any omega-limit trajectory has the time-spectrum in the spectral gap $[-m, m]$ and satisfies the original equation. The corresponding equation implies the key spectral inclusion for spectrum of the nonlinear term. Then the application of the Titchmarsh Convolution Theorem reduces the spectrum of each omega-limit trajectory to a single harmonic $\omega \in [-m, m]$.

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Random Guiding Functions and Asymptotic Behavior of Trajectories for Random Differential Inclusions

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The present talk is devoted to the study of some qualitative properties of solutions for random differential inclusions. The asymptotic behavior of a solution $x(\cdot)$ of an inclusion could be characterized, for example, by the existence of finite limits at $+\infty$ or $-\infty$.

This type of behavior is closely related to the existence of heteroclinic and homoclinic solutions. Indeed, in the case of an autonomous system $x' = f(x)$, each solution $x(\cdot)$ for which there exists $x(\pm\infty)$ is a heteroclinic solution and a solution $x(\cdot)$ for which $x(+\infty) = x(-\infty) = 0$ is a homoclinic solution.

We define the notion of the random guiding function and by applying the random topological degree we investigate the behavior of solutions on $\mathbb{R}$ and provide upper bounds for the norms of trajectories for random differential inclusions.

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Magnetic Schrödinger Operators on Periodic Discrete Graphs

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We consider magnetic Schrödinger operators with periodic magnetic and electric potentials on periodic discrete graphs. The spectrum of the operators consists of an absolutely continuous part (the union of a finite number of non-degenerate bands) and a finite number of flat bands, i.e., eigenvalues of infinite multiplicity. We estimate the Lebesgue measure of the spectrum in terms of the Betti numbers and show that these estimates become identities for specific graphs. We estimate a variation of the spectrum of the Schrödinger operators under a perturbation caused by a magnetic field in terms of magnetic fluxes. The proof is based on the Floquet theory and a precise representation of fiber magnetic Schrödinger operators.

Classical Solutions for One Dimensional Biwave Equation

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The biwave equation has been studied in some models related to the mathematical elasticity theory. This is an issue of mathematical formulation for the displacement equation of a homogeneous isotropic elastic body. In later studies, the symmetry analysis of the biwave equation was done and exact solutions obtained by Fushchych, Roman, and Zhdanov [1]; the existence of a unique solution to the initial value problem and the boundary value problem were given by Korzyuk [3, 4, 5], the finite element methods for approximations of the biwave equation were developed by Feng and Neilan [2]. In our present work, we mainly focuse on classical solutions to some mixed problems for the one dimensional biwave equation studied by the method of characteristics. The uniqueness of the solution is proved and the matching conditions are indicated.

In the closed domain $\overline{Q} = [0, \infty) \times [0, l]$ of two independent variables $(t, x) \in \overline{Q} \subset \mathbb{R}^2$, consider the biwave equation

$$\left(\partial_t^2 - a^2\partial_x^2\right)\left(\partial_t^2 - b^2\partial_x^2\right) u\,(t,x) = f\,(t,x)\,, \quad (t,x) \in \overline{Q}, \tag{1}$$

where a,b and l are positive real numbers.

Theorem 1. *The general solution of Eq.* (1) *is given by the sum*

$$
\begin{aligned}
u(t,x) = g_1(x-at) + g_2(x+at) + g_3(x-bt) + g_4(x+bt) + \\
+\frac{1}{2a^3-2ab^2}\int\limits_0^t \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} \int\limits_0^y \int\limits_0^\chi f(\tau,\xi)\,d\xi d\chi dy d\tau - \\
-\frac{1}{2a^2b-2b^3}\int\limits_0^t \int\limits_{x-b(t-\tau)}^{x+b(t-\tau)} \int\limits_0^y \int\limits_0^\chi f(\tau,\xi)\,d\xi d\chi dy d\tau.
\end{aligned}
\tag{2}
$$

Theorem 2. *Solution* (2) *of Eq.* (1) *belongs to the class* $C^4\left(\overline{Q}\right)$ *if and only if*

$$g_1, g_3 \in C^4(-\infty, l], g_2, g_4 \in C^4[0,\infty), \tag{3}$$

$$f \in C\left(\overline{Q}\right), \int\limits_0^t f(\tau, x \pm h(t-\tau))d\tau \in C^1\left(\overline{Q}\right), h = a, b. \tag{4}$$

We consider some mixed boundary value problems for biwave equation (1) with the Cauchy conditions and different types of boundary conditions (among them are the Dirichlet conditions, the Robin conditions, the periodic boundary conditions, and the integral conditions).

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Variational Problems with Regular and Irregular Bilateral Constraints in Variable Domains

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Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geqslant 2)$, $\{\Omega_s\}$ a sequence of domains in $\mathbb{R}^n$ contained in Ω, and $p > 1$. Let $c_1, c_2 > 0$ and suppose that $\mu_s \in L^1(\Omega_s)$, $\mu_s \geqslant 0$ in Ω_s, for every $s \in \mathbb{N}$. We assume that the sequence of the norms $\|\mu_s\|_{L^1(\Omega_s)}$ is bounded. Let, for every $s \in \mathbb{N}$, $f_s : \Omega_s \times \mathbb{R}^n \to \mathbb{R}$ be a function satisfying the following conditions: for every $\xi \in \mathbb{R}^n$, the function $f_s(\cdot, \xi)$ is measurable on Ω_s; for almost every $x \in \Omega_s$, the function $f_s(x, \cdot)$ is convex on $\mathbb{R}^n$; for almost every $x \in \Omega_s$ and for every $\xi \in \mathbb{R}^n$, we have $c_1|\xi|^p - \mu_s(x) \leqslant f_s(x,\xi) \leqslant c_2|\xi|^p + \mu_s(x)$.

Let, for every $s \in \mathbb{N}$, $F_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ be a functional defined by

$$F_s(v) = \int_{\Omega_s} f_s(x, \nabla v)dx, \quad v \in W^{1,p}(\Omega_s).$$

Let $c_3, c_4 > 0$ and $G_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ be a weakly continuous functional for all $s \in \mathbb{N}$. Assume that $G_s(v) \geqslant c_3\|v\|^p_{L^p(\Omega_s)} - c_4$ for every $s \in \mathbb{N}$ and $v \in W^{1,p}(\Omega_s)$.

Now, suppose $\varphi, \psi \in W^{1,p}(\Omega)$ and $\varphi \leqslant \psi$ a.e. in Ω. We define

$$V(\varphi,\psi) = \{v \in W^{1,p}(\Omega) : \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega\}$$

and put $V_s(\varphi,\psi) = \{v \in W^{1,p}(\Omega_s) : \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega_s\}$ for every $s \in \mathbb{N}$.

Theorem 1. *Assume that the embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact, the sequence of the spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$, and, for every sequence of measurable sets $H_s \subset \Omega_s$ such that $\operatorname{meas} H_s \to 0$, we have $\int_{H_s} \mu_s\, dx \to 0$. Assume in addition that the sequence $\{F_s\}$ Γ-converges to a functional $F : W^{1,p}(\Omega) \to \mathbb{R}$ and there exists a functional $G : W^{1,p}(\Omega) \to \mathbb{R}$ such that, for every function $v \in W^{1,p}(\Omega)$ and for every sequence $v_s \in W^{1,p}(\Omega_s)$ with the property $\|v_s - v\|_{L^p(\Omega_s)} \to 0$, we have $G_s(v_s) \to G(v)$. Moreover, assume that $\psi - \varphi > 0$ a.e. in Ω. Finally, let, for every $s \in \mathbb{N}$, u_s be a function in $V_s(\varphi,\psi)$ minimizing the functional F_s+G_s on the set $V_s(\varphi,\psi)$. Then there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $u \in V(\varphi,\psi)$ such that u minimizes the functional $F+G$ on the set $V(\varphi,\psi)$, $\|u_{s_j} - u\|_{L^p(\Omega_{s_j})} \to 0$, and $(F_{s_j}+G_{s_j})(u_{s_j}) \to (F+G)(u)$.*

For the above mentioned notions of strong connectedness of spaces and Γ-convergence of functionals, and for the proof of Theorem 1, see [1]. A similar result was obtained in [2] for the case where the exhaustion condition of the domain Ω by the domains Ω_s is satisfied, $\|\mu_s\|_{L^1(\Omega_s)} \to 0$, $\varphi \equiv 0$, and $\psi : \Omega \to \overline{\mathbb{R}}$ is an arbitrary nonnegative measurable function.

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Entropy Solutions of Anisotropic Elliptic Equations with Variable Nonlinearity in Unbounded Domains

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For elliptic equations with power-like nonlinearities and with L_1-data on the right-hand side, the notion of an entropy solution of the Dirichlet problem was suggested and its existence and uniqueness were proved in [1]. In the present paper, the above result is generalized to some class of elliptic equations with variable nonlinearities

$$\operatorname{div} \mathrm{a}(\mathrm{x}, \nabla u) = |u|^{p_0(\mathrm{x})-2}u + a(\mathrm{x}, u), \quad u(\mathrm{x})|_{\partial\Omega} = 0. \tag{1}$$

Here Ω is an arbitrary domain, $\Omega \subsetneq \mathbb{R}^n = \{\mathrm{x} = (x_1, x_2, \dots, x_n)\}$, $n \geqslant 2$.

Put $\overrightarrow{\mathbf{p}}(\mathrm{x}) = (p_0(\mathrm{x}), p_1(\mathrm{x}), ..., p_n(\mathrm{x})) \in (C^+(\overline{\Omega}))^{n+1}$. We set $p_+(\mathrm{x}) = \max\limits_{i=\overline{1,n}} p_i(\mathrm{x})$,

$$\overline{p}(\mathrm{x}) = n\left(\sum_{i=1}^{n} 1/p_i(\mathrm{x})\right)^{-1}, \quad p_*(\mathrm{x}) = \begin{cases} \frac{n\overline{p}(\mathrm{x})}{n-\overline{p}(\mathrm{x})}, & \overline{p}(\mathrm{x}) > n, \\ +\infty, & \overline{p}(\mathrm{x}) \leqslant n \end{cases}.$$

We assume that

$$p_+(\mathrm{x}) \leqslant p_0(\mathrm{x}) < p_*(\mathrm{x}), \quad \mathrm{x} \in \Omega. \tag{2}$$

It is assumed that the functions $\mathrm{a}(\mathrm{x}, \mathrm{s}) = (a_1(\mathrm{x}, \mathrm{s}), \dots, a_n(\mathrm{x}, \mathrm{s}))$, $a(\mathrm{x}, s_0)$, $\mathrm{x} \in \Omega$, $s_0 \in \mathbb{R}$, $\mathrm{s} = (s_1, \dots, s_n) \in \mathbb{R}^n$, satisfy the Caratheodory conditions, $a(\mathrm{x}, s_0)$ is nondecreasing in $s_0 \in \mathbb{R}$, there exist numbers $\widehat{A}, \overline{a} > 0$ and measurable functions $\Phi_i(\mathrm{x}) \geqslant 0$ such that for a. a. $\mathrm{x} \in \Omega$ and any $\mathrm{s}, \mathrm{t} \in \mathbb{R}^n$, the following inequalities hold:

$$|a_i(\mathrm{x}, \mathrm{s})| \leqslant \widehat{A}\mathrm{P}(\mathrm{s})^{1/p_i'(\mathrm{x})} + \Phi_i(\mathrm{x}), \quad i = 1, \dots, n; \tag{3}$$

$$(\mathrm{a}(\mathrm{x}, \mathrm{s}) - \mathrm{a}(\mathrm{x}, \mathrm{t})) \cdot (\mathrm{s} - \mathrm{t}) > 0, \ \mathrm{s} \neq \mathrm{t}; \quad \mathrm{a}(\mathrm{x}, \mathrm{s}) \cdot \mathrm{s} \geqslant \overline{a}\mathrm{P}(\mathrm{s}); \quad \mathrm{P}(\mathrm{s}) = \sum_{i=1}^{n} |s_i|^{p_i(\mathrm{x})}. \tag{4}$$

Here $\mathrm{s} \cdot \mathrm{t}$ is the scalar product. We set $T_k(r) = r$ as $|r| \leqslant k$, $T_k(r) = k \operatorname{sign} r$ as $|r| > k$; $\langle u \rangle = \int\limits_{\Omega} u d\mathrm{x}$.

Definition 1. By an entropy solution of problem (1) we understand a measurable function $u : \Omega \to \mathbb{R}$ such that $A(\mathrm{x}) = a(\mathrm{x}, u) \in L_1(\Omega)$, $T_k(u) \in \mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$ for all $k > 0$, and the inequality

$$\langle (a(\mathrm{x}, u) + |u|^{p_0(\mathrm{x})-2}u) T_k(u - \xi) \rangle + \langle \mathrm{a}(\mathrm{x}, \nabla u) \cdot \nabla T_k(u - \xi) \rangle \leqslant 0$$

holds for all $k > 0$ and $\xi(\mathrm{x}) \in C_0^1(\Omega)$.

Theorem 1. *Let conditions* (2)–(4) *hold. Then the entropy solution to problem* (1) *is unique.*

We formulate additional conditions that are used in the existence theorem. We set $a(\mathrm{x}, s_0) = a(\mathrm{x}, 0) + b(\mathrm{x}, s_0)$ and assume that

$$a(\mathrm{x}, 0) \in L_1(\Omega). \tag{5}$$

Theorem 2. *Let conditions* (2)–(5) *hold. Then there exists the entropy solution to problem* (1).

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Closing and Connecting Lemmas for Conservative Flows in Euclidean Spaces

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Let V be a bounded Lipschitz continuous vector field in $\mathbb{R}^d$. Suppose that V is divergence-free and satisfies the so-called small mean drift condition

$$\lim_{L\to\infty} \sup_{x\in\mathbb{R}^d} \left\| \frac{1}{L^d} \int_{[0,L]^d} V(x+y)\,dy \right\| = 0. \tag{1}$$

This set of conditions was first introduced by Burago, Ivanov, and Novikov [2]. There, the authors demonstrate that the system

$$\dot{x} = V(x) \tag{2}$$

is controllable. Namely, for any $\sigma > 0$ there exist positive values C_1 and C_2 with the following property: for any two points $x_0, y_0 \in \mathbb{R}^d$, there is a value $\tau(x_0, y_0) \in (0, C_1|x-y| + C_2)$ and a continuous function $\alpha : [0, \tau(x_0, y_0)] \to \mathbb{R}^d$ such that $\varphi_\alpha(\tau(x_0, y_0), 0, x_0) = y_0$. Here $\varphi_\alpha(\tau(t_2, t_1, \xi)$ is the solution of the system $\dot{x} = V(x) + \alpha(t)$. In other words, system (2) is chain transitive. However, points of the Euclidean space may be wandering.

In our talk, we discuss possible generalizations of this result on controlability (we are mostly interested in possible analogs of Pugh's closing lemma [3]). First of all, it follows from results of [1] that if, in addition to supposed above, system (2) does not have non-hyperbolic periodic solutions, then for any $\varepsilon > 0$ and any points $x_0, y_0 \in \mathbb{R}^d$

there exists an ε-small (in the C^1-norm) vector field $W(x)$ such that the point y_0 belongs to the positive semi-trajectory of the point x_0 with respect to the perturbed system

$$\dot{x} = V(x) + W(x). \tag{3}$$

We demonstrate that the assumption on hyperbolicity of periodic orbits can be omitted, at least for an analog of the closing lemma.

Theorem 1. *Let the vector field V be bounded, uniformly Lipschitz continuous, divergence-free and satisfy condition* (1). *Then for any $\varepsilon > 0$ and any $x_0 \in \mathbb{R}^d$ there exists a perturbation $W(x)$ such that $\|W\|_{C^1} < \varepsilon$ and the point x_0 is periodic with respect to system* (3).

In the proof, we use an original techniques of constructing for system (3) probability invariant measures distributed everywhere, and applying Poincaré's Recurrence Theorem and Pugh's lemma.

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Bifurcation Analysis of Periodic Solutions of the Ikeda Equation

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We consider the equation

$$\dot{x}(t) + x(t) = \mu \sin(x(t-\tau) - c), \tag{1}$$

where the unknown $x(t)$ means the phase shift of the electric field in the nonlinear medium of a ring resonator, τ is the light propagation time in the ring resonator, $0 \leqslant c < 2\pi$ is a constant phase shift, $\mu > 0$ is the coefficient characterizing the intensity of the laser emission. This equation was proposed by K. Ikeda to describe the dynamics of a passive optical resonator [1, 2]. We study bifurcation from equilibrium states of periodic solutions of the Ikeda equation.

The equation, written in a characteristic time scale, has the following form

$$\varepsilon_1 \dot{x}(t) + x(t) = \mu \sin(x(t-1) - c). \tag{2}$$

Here $\varepsilon_1 = \tau^{-1} \ll 1$. This equation contains a small parameter at the first derivative, which makes it singular. We apply the uniform normalization method to study periodic solutions. The normal form of equation (2) contains "fast" and "slow" variables, where equilibrium states for “slow” variables determine periodic solutions. Analysis of equilibrium states allows us to study the bifurcation of periodic solutions depending on the parameters. We show the possibility of simultaneous bifurcation of large number of stable periodic solutions. This phenomenon is called multistability. It is shown that chaotic attractors arise through a series of the period-doubling bifurcations from periodic solutions, which leads to the appearance of the chaotic multistability in the behavior of solutions.

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Spectral Properties of Fractional Differentiation Operators

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We assume that Ω is a convex domain in the n-dimensional Euclidean space, P is a fixed point of the boundary $\partial\Omega$, and $Q(r,\vec{\mathbf{e}})$ is an arbitrary point of Ω; we denote by $\vec{\mathbf{e}}$ the unit vector directed from P to Q, where r is the Euclidean distance between points P and Q. We consider classes $L_p(\Omega)$ of Lebesgue complex-valued functions. Consider the operator of fractional differentiation in the sense of Kipriyanov [1]

$$\mathfrak{D}^\alpha : \overset{0}{W}{}_p^l(\Omega) \to L_q(\Omega),$$

$$lp \leqslant n,\ 0 < \alpha < l - \frac{n}{p} + \frac{n}{q},\ \left(p < q < \frac{np}{n-lp}\right),$$

$$(\mathfrak{D}^\alpha f)(Q) = \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_0^r \frac{[f(Q) - f(P+\vec{\mathbf{e}}t)]}{(r-t)^{\alpha+1}} \left(\frac{t}{r}\right)^{n-1} dt + \frac{(n-1)!}{\Gamma(n-\alpha)} f(Q) r^{-\alpha}.$$

Under the assumption $n \geqslant 2$, consider a uniformly elliptic operator L containing the fractional derivative of order $0 < \alpha < 1$ in lower terms, defined by the expression

$$Lu := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + p(x)\mathfrak{D}^\alpha u,\ u \in \overset{0}{W}{}_2^2(\Omega),$$

where the coefficients of the operator L are real-valued functions

$$a_{ij}(x) \in C^1(\bar{\Omega}),\ p(x) \in \operatorname{Lip}\lambda,\ (\alpha < \lambda \leqslant 1),\ p(x) > 0.$$

The following theorem establishes some spectral properties for the closure $\tilde{L}$ of the operator L.

Theorem 1. *The operator $\tilde{L}$ is strongly accretive and its numerical range belongs to the sector*

$$\mathfrak{S} := \{\zeta \in \mathbb{C} : \ |\arg(\zeta - \gamma)| \leqslant \theta\}$$

with θ and γ defined by the coefficients of the operator L. For all $\zeta \in \mathbb{C} \setminus \mathfrak{S}$, the operator $\tilde{L} - \zeta$ has a closed range, moreover,

$$\operatorname{nul}(\tilde{L} - \zeta) = 0, \ \operatorname{def}(\tilde{L} - \zeta) = \mu, \ \mu = \text{const}.$$

In the case $\mu = 0$, we have the estimate

$$\|(\tilde{L} - \zeta)^{-1}\| \leqslant 1/\operatorname{dist}(\zeta, \mathfrak{S}), \ \zeta \in \mathbb{C} \setminus \mathfrak{S}.$$

The proof is based on the property of strong accretivity for the fractional differentiation operator in the sense of Kipriyanov. The analogous results can be obtained for the operators of fractional differentiation in the sense of Marchaud and Riemann-Liouville on the segment. Note that applied to sufficiently nice functions, the Kipriyanov operator in the one-dimensional case is reduced to the operator in the sense of Marchaud which coincides a.e. with the Riemann–Liouville operator.

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Local Bifurcations in the Ginzburg–Landau Equation

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The equation

$$u_t = u - (1 + ic)u|u|^2 - id\Delta u, \quad c \in \mathbb{R}, \, d > 0, \tag{1}$$

which can be interpreted as a “weakly-dissipative” form of the Ginzburg–Landau equation for a complex-valued function $u(t, x)$, $t \geqslant 0$, $x \in \mathbb{R}^n$ $(x = (x_1, x_2, \ldots, x_n))$ [1-3] was considered in [1-3]. We study (1) with the following periodic boundary conditions:

$$u(t, x + 2\pi e_j) = u(t, x), \, e_j = (e_{1j}, \ldots, e_{nj}), e_{kj} = \delta_{kj}, \tag{2}$$

where δ_{kj} is the Kronecker delta. Boundary value problem (1), (2) has the countable family of solutions $u_k(t, x) = \exp(i\sigma_k t + i(k, x)), k = (k_1, \ldots, k_n), k_j \in Z, x \in \mathbb{R}$, where $(*, **)$ is the scalar product and $\sigma_k = d(k, k) - c$.

The solutions $u_k(t, x)$ are stable if $d > 2c$, unstable if $d < 2c$, and a critical case in the problem of stability for the traveling plane waves is realized with $d = 2c$.

Let $a = 4c^2(1 - c^2)$, $b = (30c^4 - 9c^2 + 1)/6$,

$$\Theta_m = ((54 - 24m)c^4 + (24m - 33)c^2 + 1)/6, \quad m = 1, 2, \ldots, n,$$

and $d = 2c - \gamma\varepsilon, \varepsilon \in (0, \varepsilon_0), \gamma = \pm 1$.

The following assertion was proved in [4].

Theorem 1. *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $a \neq b$, and $\gamma = 1$, boundary value problem (1), (2) has invariant tori T_{m+1} ($dim\, T_{m+1}(\varepsilon) = m+1$) lying in any neighborhood of the solution $u_k(t,x)$ if $\Theta_m > 0$.*

The tori $T_2(\varepsilon)$ are stable if $b < a$, and unstable if $b > a$. The torus $T_{n+1}(\varepsilon)$ is stable if $b > a$, and unstable if $b < a$. The tori $T_p(\varepsilon)$ with $p = 3, \ldots, n$ are always saddle.

All of the existing tori are unstable if $\gamma = -1$.

Asymptotic formulas for the solutions belonging to the invariant tori are given.

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A Class of Integral Operators on $\mathbb{R}^c$ with Continuously Varying Kernels

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Let X and Y be Banach spaces. We denote by $\mathbf{B}(X,Y)$ the space of all bounded linear operators acting from X to Y. If $X = Y$, then we use the brief notation $\mathbf{B}(X)$. We denote by $\mathbf{1} \in \mathbf{B}(X)$ the identity operator.

Let $\mathbb{E}$ be a fixed finite-dimensional Banach space with norm $|\cdot|$. We consider the spaces $L_p = L_p(\mathbb{R}^c, \mathbb{E})$, $1 \leqslant p \leqslant \infty$.

We denote by $\mathbf{N}_1 = \mathbf{N}_1(\mathbb{R}^c, \mathbb{E})$ the set of all measurable functions $n : \mathbb{R}^c \times \mathbb{R}^c \to \mathbf{B}(\mathbb{E})$ satisfying the following property: there exists a function $\beta \in L_1(\mathbb{R}^c, \mathbb{R})$ such that

$$\|n(x,y)\| \leqslant \beta(y)$$

for almost all $(x,y) \in \mathbb{R}^c \times \mathbb{R}^c$.

We denote by $\mathbf{N}_1 = \mathbf{N}_1(L_p)$ the set of all operators $N \in \mathbf{B}(L_p)$, $1 \leqslant p \leqslant \infty$, of the form

$$\big(Nu\big)(x) = \int_{\mathbb{R}^c} n(x, x-y)\, u(y)\, dy, \tag{1}$$

where $n \in \mathbf{N}_1(\mathbb{R}^c, \mathbb{E})$.

For any $n \in \mathbf{N}_1(\mathbb{R}^c, \mathbb{E})$, we denote by $\bar{n}$ the function that assigns to each $x \in \mathbb{R}^c$ the function $\bar{n}(x) : \mathbb{R}^c \to \mathbf{B}(\mathbb{E})$ defined by the rule

$$\bar{n}(x)(y) = n(x, x - y).$$

We denote by $\mathbf{CN}_1 = \mathbf{CN}_1(\mathbb{R}^c, \mathbb{E})$ the class of all kernels $n \in \mathbf{N}_1$ such that the corresponding function $x \mapsto \bar{n}(x)$ is continuous in the norm of $L_1\bigl(\mathbb{R}^c, \mathbf{B}(\mathbb{E})\bigr)$.

Theorem 1. *Let $n \in \mathbf{CN}_1$ and the operator $N \in \mathbf{B}(L_p)$, $1 \leqslant p \leqslant \infty$, be defined by formula* (1). *If the operator $\mathbf{1} + N$ is invertible, then $(\mathbf{1} + N)^{-1} = \mathbf{1} + M$, where*

$$\bigl(Mu\bigr)(x) = \int_{\mathbb{R}^c} m(x, x - y)\, u(y)\, dy$$

with $m \in \mathbf{CN}_1$.

The proof is published in [1].

An application to differential-difference operators is discussed.

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Periodic Solutions for N-Dimensional Cyclic Non-Monotone Systems with Negative Delayed Feedback[1]

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We present a framework for the proof of existence of periodic solutions for a class of cyclically coupled systems. This framework includes the results known so far, which were restricted to the dimensions $N = 1, 2, 3$, and covers large classes of equation for general n. Our approach exhibits the common structure of the methods used so far, and generalizes it to arbitrary dimension. Main tools are a return map defined by oscillation of solutions, and application of fixed point theory to this map. The analysis of the characteristic equation coming from the linearization at zero, and of the associated eigenprojections, is a part which gets increasingly difficult with higher dimension. Our approach shows where “classical” methods known from the cases $N = 1, 2, 3$ may fail here, and under which conditions these methods are still applicable for general N.

[1] This is a joint work with A. F. Ivanov.

Ultimate Poisson Boundedness of Solutions of Systems of Differential Equations

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We consider an arbitrary system of differential equations for n unknowns,

$$\frac{dx}{dt} = F(t,x), \quad F(t,x) = (F_1(t,x), \dots, F_n(t,x)), \tag{1}$$

where the functions on the right-hand sides are defined and continuous in $\mathbb{R}^+ \times \mathbb{R}^n$, $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geqslant 0\}$.

For each $t_0 \in \mathbb{R}^+$ we denote by $\mathbb{R}^+(t_0)$ the set $\{t \in \mathbb{R} \mid t \geqslant t_0\}$. A non-negative increasing numerical sequence $\tau = \{\tau_i\}_{i \geqslant 1}$ with $\lim\limits_{i\to\infty} \tau_i = +\infty$ is called $\mathcal{P}$-sequence. For each $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\geqslant 1}$, let $M(\tau)$ denote the set $\bigcup\limits_{i=1}^{\infty} [\tau_{2i-1}; \tau_{2i}]$.

Definition 1. The solutions of system (1) are said to be uniform-ultimately Poisson bounded if there exist a number $B > 0$ and a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\geqslant 1}$ with the following properties: for any $\alpha > 0$, there exists a number $T \geqslant 0$ such that any solution $x = x(t, t_0, x_0)$ of system (1) with $t_0 \in M(\tau)$ and $\|x_0\| \leqslant \alpha$ satisfies the condition $\|x(t,t_0,x_0)\| < B$ for all $t \in \mathbb{R}^+(t_0+T) \bigcap M(\tau)$.

It is clear that if solutions are uniform-ultimately bounded (see [1]), then they are uniform-ultimately Poisson bounded as well.

Theorem 1. *Suppose that there exist for system* (1) *a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\geqslant 1}$ and a function $V(t,x) \geqslant 0$ defined in $\mathbb{R}^+(\tau_1) \times \mathbb{R}^n$ and satisfying the following conditions:*

1. *$b(\|x\|) \leqslant V(t,x) \leqslant a(\|x\|)$ for all $(t,x) \in M(\tau) \times \mathbb{R}^n$, where $a(r) \geqslant 0$ is an increasing function, $b(r) \geqslant 0$ is an nondecreasing function, and $b(r) \to \infty$ as $r \to \infty$;*
2. *$V'^{+}_{F(t,x)}(t,x) \leqslant -c(\|x\|)$ for all $(t,x) \in \mathbb{R}^+(\tau_1) \times \mathbb{R}^n$, where $V'^{+}_{F(t,x)}(t,x)$ is the upper Dini derivative of a function $V(t,x)$ due to system* (1) *and $c(r)$ is a continuous function positive for $r > 0$.*

Then the solutions of system (1) *are uniform-ultimately Poisson bounded.*

If we require in Definition 1 that T depend not only on α but also on t_0, then we have the definition of equiultimate Poisson boundedness of solutions of system (1). If in Theorem 1 we require that the function $a(r)$ depend not only on r but also on t, then we obtain sufficient conditions for the equiultimate Poisson boundedness of the solutions of system (1).

Furthermore, we introduce the notions of partial uniform-ultimate Poisson boundedness of solutions, partial uniform-ultimate Poisson boundedness of solutions with partially controlled initial conditions, partial equiultimate Poisson boundedness of solutions, and partial equiultimate Poisson boundedness of solutions with partially controlled initial conditions. We obtain sufficient conditions for these types of Poisson boundedness of solutions.

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Instability in Shear Flow[1]

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Molecules of liquid crystals in shear flow can respond to this flow in many different kinds. Some solutions are called log-rolloing, tumbling, and kayaking. The state space of a molecule can be viewed as the set of symmetric traceless 3×3 matrices . There is a natural action of the group SO(3) on this space. We explore this symmetry to make predictions on the existence of such kayaking solutions.

Mathematical Simulation of Plasma Physics: Elaboration, Test Solutions, and Results of Computational Experiments

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In ICMMG SB RAS, we work on mathematical modeling of plasma confinement, electromagnetic radiation of plasma turbulence, space plasma, production of gamma-rays in the collision of beams of charged particles, and resistance to the effects of plasma. Simulation results are in demand at the Institute of Nuclear Physics of the SB RAS for understanding problems of plasma physics and controlled thermonuclear fusion. Conducting both laboratory and computer experiments is necessary to study these problems. Computing experiments help to reduce the number of expensive laboratory tests. Mathematical modeling is based on the solution of systems of equations describing the dynamics of plasma, liquid, and gas, and the elastic-plastic deformations.

The Vlasov equation is implemented by the PIC method for equations of the elasticity theory, and the gas dynamics equations are solved by the method of Godunov. There are some specific tests for problems of this type.

Nowadays, the tokamak is the most developed concept of implementation of controlled thermonuclear fusion to generate energy. The main method of accumulation and heating of high-temperature plasma held in a magnetic field is the injection of powerful beams of neutral atoms, while the atoms in plasma lose their weakly bound

[1]Joint work with David Chillingworth (Southampton, UK), Greg Forest (North Carolina, USA) and Claudia Wulff (Surrey, UK)

electrons and become ions of high-temperature plasma. The creation of sources of such beams is an important part of the problem of controlled thermonuclear fusion.

Mathematical modeling of the dynamics of plasma in the trap is performed, and evaluation of the distribution and effectiveness of plasma confinement is obtained. The results are compared with laboratory experiments.

One of the key problems of the tokamak based fusion reactor is the resistance of the materials of the first wall to the impact of plasma. Improved containment of hot plasma in tokamaks is associated with high regular and periodic pulse flows of plasma on the divertor plate. Creating plasma resistant materials requires understanding of the processes occurring during pulse loads. Therefore, the experiments need to measure not only the final result of the impact but also the dynamics of fast processes, and to conduct modeling. Mathematical modeling of the dynamics of cracking, melting, and splashing of tungsten under pulsed heat loads, typical for conditions in prospective fusion reactor and simulated with an electron beam, will provide a basis for the development of theoretical models.

On Solutions of Schlesinger Equation for Third Order Upper-Triangular Matrices

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For the Schlesinger equation

$$\mathrm{d}\, B_i(a) = -\sum_{j=1,\, j\neq i}^{n} [B_i(a),\, B_j(a)]\,\frac{\mathrm{d}(a_i - a_j)}{a_i - a_j}, \tag{1}$$

we seek for triangular solutions of the form

$$B_i(a) = \begin{pmatrix} \lambda_i^1 & u_i(a) & w_i(a) \\ 0 & \lambda_i^2 & v_i(a) \\ 0 & 0 & \lambda_i^3 \end{pmatrix}, \qquad i = 1, \dots, n.$$

Then nonlinear equation (1) for functions $u_i(a)$ and $v_i(a)$, $i = 1, 2, \dots n$, is reduced to the linear systems

$$\frac{\partial u_i}{\partial a_j} = \frac{(\lambda_j^1 - \lambda_j^2)u_i - (\lambda_i^1 - \lambda_i^2)u_j}{a_i - a_j},$$

$$i \neq j = 1, \dots n,$$

$$\frac{\partial v_i}{\partial a_j} = \frac{(\lambda_j^2 - \lambda_j^3)v_i - (\lambda_i^2 - \lambda_i^3)v_j}{a_i - a_j},$$

$$i \neq j = 1, \dots n,$$

with the conditions $\sum_i^n u_i = \sum_i^n v_i = 0$. For functions $w_i(a), i = 1, 2, \dots, n$, we obtain the linear nonhomogeneous system

$$\frac{\partial w_i}{\partial a_j} = \frac{(\lambda_j^1 - \lambda_j^3)w_i - (\lambda_i^1 - \lambda_i^3)w_j}{a_i - a_j} + \frac{(u_i(a)v_j(a) - u_j(a)v_i(a))}{a_i - a_j},$$

$$i \neq j = 1, \dots n,$$

and the condition $\sum_i^n w_i = 0$.

Theorem 1. *If for each $i = 1, 2, \dots, n$ the eigenvalues λ_i^1, λ_i^2, λ_i^3 of the triangular solution $B_i(a)$ to system (1) form an arithmetic progression with a fixed base $d = \pm\frac{1}{2}$, then the entries $u_i(a)$ and $v_i(a)$ of B_i may be obtained as periods of meromorphic differentials on hyperelliptic curves in $\mathbb{C}^2$, and the entries $w_i(a)$ as polynomials of $a = (a_1, \dots, a_n)$.*

See [1] for matrices $B_i(a)$ of order $p = 2$ with hyperelliptic entries. For the case of $p \geqslant 3$ and hypergeometric entries, see [2].

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Analytical and Numerical Methods for the Study of Attractors: Homoclinic Bifurcations, Localization, and Dimension Characteristics

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This lecture is devoted to recent results of the study of attractors in dynamical systems. Effective analytical and numerical methods for the study of transition to chaos via homoclinic bifurcations, and the localization and dimension characteristics of chaotic attractors are discussed.

The homoclinic orbits play an important role in the bifurcation theory and in scenarios of the transition to chaos. In the case of dissipative systems, the proof of the existence of homoclinic orbits is a challenging task. Recently, an effective method called *the Fishing principle* was developed, which allows one to obtain necessary and sufficient conditions of the existence of homoclinic orbits in various well-known dynamical systems [1, 2, 3, 4]. The advantage of the Fishing principle in comparison with other methods of proving the existence of homoclinic and heteroclinic orbits is that there is no need to consider a mutual disposition of stable and unstable manifolds of saddle equilibria and to introduce a small parameter.

Recently, it was suggested to classify the attractors as being *hidden* either *self-excited* [5, 6]. Namely, an attractor is called a *self-excited attractor* if its basin of attraction intersects with any vicinity of an equilibrium, otherwise it is called a *hidden attractor*. This served the purpose of connecting the notions of transient processes (engineering), visualization (numerical analysis), and basins of attraction and stability (dynamical systems). The classification not only demonstrated the difficulties of fundamental problems (e.g., the second part of Hilbert's 16th problem on the number and the mutual disposition of limit cycles, Aizerman's and Kalman's conjecture on the monostability of nonlinear systems) and applied systems analysis, but also triggered the discovery of new hidden attractors in the well-known physical and engineering models [7].

The concept of the Lyapunov dimension was suggested by Kaplan and Yorke for estimating the Hausdorff dimension of attractors. Along with widely used numerical methods for estimating and computing the Lyapunov dimension, an effective analytical was also developed based on the direct Lyapunov method with special Lyapunov-like functions. The advantage of the method is that it allows one, in many cases, to estimate the Lyapunov dimension of an invariant set without localization of the set in the phase space, to prove Eden's conjecture for the self-excited attractors and get exact Lyapunov dimension formula for attractors of various well-known dynamical systems (e.g., such as the Henon, Lorenz, Shimizu–Morioka, and Glukhovsky–Dolzhansky systems) [8, 9, 10, 11].

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On the Riemann-Hilbert Problem for Difference and q-Difference Systems

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The talk is devoted to an analogue of the classical Riemann-Hilbert problem, stated for the classes of difference and q-difference systems.

Construct a difference (q-difference) linear system of the form

$$Y(z+1) = A(z)Y(z) \quad \textit{or} \quad Y(qz) = A(z)Y(z)$$

with a given monodromy matrix, a prescribed set of characteristic constants, and a condition for the coefficient matrix $A(z)$ to be a polynomial of fixed power.

We generalize Birkhoff's existence theorem (see [1]). The result obtained by Birkhoff (see [1]) could be insufficient in some cases since his theorem sometimes leads to systems with shifted characteristic constants. There might be integer additions to the characteristic constants corresponding to power asymptotics of the solutions of the systems. In the talk, we consider further research on this problem and propose a solution which shows that there exist systems with the correct monodromy data and characteristic constants. As a result, the roots of the determinant of the coefficient matrix could be shifted by an integer, but those are not fixed in the monodromy data. Complete statement and proof of the theorem can be found in [2].

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On Subordination of One Minimal Differential Operator to the Tensor Product of Two Others

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Let Ω be a domain in $\mathbb{R}^n$, $p \in [1, \infty]$, and let $P(D)$ be a differential polynomial, $D = (D_1, \dots, D_n)$, $D_j = -i\partial/\partial x^j$. We consider the problem of description of the linear space $L_{p,\Omega}(P)$ of all minimal differential operators $Q(D)$ subordinate to $P(D)$ in the $L^p(\Omega)$-norm, i.e., the space of the operators $Q(D)$ obeying the a priori estimate

$$\|Q(D)f\|_{L^p(\Omega)} \leqslant C_1\|P(D)f\|_{L^p(\Omega)} + C_2\|f\|_{L^p(\Omega)}, \qquad f \in C_0^\infty(\Omega),$$

with constants $C_1, C_2 > 0$ independent of f.

In this talk, we consider the case of $n = 2$, $\Omega = \mathbb{R}^2$ and the operator $P(D)$ being the tensor product of two ordinary differential operators, i.e., $P(D) = p_1(D_1)\otimes p_2(D_2)$. In [1], some general results were established concerning the structure of the space $L_{\infty,\mathbb{R}^n}(p_1\otimes p_2)$, where $p_i(D)$ were elliptic operators in an arbitrary number of variables. Further, in [2], the space $L_{\infty,\mathbb{R}^2}(p_1 \otimes p_2)$ was completely described for the case where all the zeros of the symbol $p_1(\xi_1)$ were real and simple.

Here we continue the investigations of [2], considering several cases where the zeros of $p_i(\xi_i)$ are not necessarily real or simple. In particular, the description of the space $L_{\infty,\mathbb{R}^2}(p_1 \otimes p_2)$ where the both cofactors $p_i(\xi_i)$ $(i = 1, 2)$ have no multiple real zeros is obtained, and the dependence of $\dim L_{\infty,\mathbb{R}^2}(p_1 \otimes p_2)$ on the structure of the cofactors' zeros is studied. Proofs are based both on the harmonic analysis technique and on the Newton polygon method.

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Mathematical Models of Gene Regulation: New Mathematics from Biology (again)

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Simple bacterial gene regulatory motifs can be viewed from the perspective of dynamical systems theory, and mathematical models of these have existed almost since the statement of the operon concept.

In this talk I review the three basic types of these regulatory mechanisms and the underlying dynamical systems concepts that apply in each case. In the latter part of the talk I discuss the exciting mathematical challenges that arise when trying to make honest mathematical models of the underlying biology. These include transcriptional and translational delays, the fact that these delays may be state dependent, as well as the interesting and often unsolved problems of characterizing the noise inherent in bacterial dynamics (which is not of the usual type).

All of the mathematical details may be found in the recently published [1].

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Accuracy Estimate with Respect to State of Finite-Dimensional Approximations for Optimization Problems for Nonlinear Elliptic Equations with Mixed Derivatives and Unbounded Nonlinearity

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Let $\Omega = \big\{x = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_\alpha < l_\alpha, \alpha = 1, 2\big\}$ be a rectangle in $\mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$. Suppose that a controlled physical process is described by the first boundary value problem for a second-order nonlinear differential equation: to find a function $u = u(x)$, $x \in \overline{\Omega}$, satisfying the conditions

$$\begin{aligned} &-\sum_{\alpha=1}^{2} k_{\alpha\alpha}(u)\frac{\partial^2 u}{\partial x_\alpha^2} - 2k_{12}(u)\frac{\partial^2 u}{\partial x_1 \partial x_2} + q(u)u = f(u), \quad x \in \Omega, \\ &k_{12}(u) = k_{21}(u); \end{aligned} \tag{1}$$

$$u(x) = 0, \qquad x \in \partial\Omega = \Gamma, \tag{2}$$

where $q(\eta)$, $f(\eta)$ are given functions of η.

We introduce the set of admissible controls

$$\begin{gathered} g = \big(k_{11}, k_{22}, k_{12},\big) \in U = \Bigg\{k_{\alpha\beta} \equiv g_{\alpha\beta} \in W_\infty^1(\Omega), \alpha, \beta = 1, 2 : \\ \nu \sum_{\alpha=1}^{2} \xi_\alpha^2 \leqslant \sum_{\alpha,\beta=1}^{2} k_{\alpha\beta}(\eta)\xi_\alpha\xi_\beta \leqslant \mu \sum_{\alpha=1}^{2} \xi_\alpha^2, \quad k_{\alpha\beta}(\eta) = k_{\beta\alpha}(\eta), \quad \alpha, \beta = 1, 2, \\ \forall x \in \Omega, \xi \neq 0, \xi \in \mathbb{R}^2, \left|\frac{\partial k_{\alpha\beta}}{\partial x_1}\right| \leqslant R_1, \left|\frac{\partial k_{\alpha\beta}}{\partial x_2}\right| \leqslant R_2, \alpha, \beta = 1, 2\Bigg\}. \end{gathered} \tag{3}$$

We define a cost functional $J : U \to \mathbb{R}^1$ of the form

$$g \to J(g) = \int\limits_{\Omega_1} |u(r_1, r_2; g) - u_0^{(1)}(r)|^2 d\Omega_1, \tag{4}$$

where $u_0^{(1)} \in W_2^1(\Omega_1)$ is a given function.

The optimal control problem is to find a control $g_* \in U$ minimizing the functional $g \to J(g)$ on the set U. More precisely, we need to minimize functional (4) on the solutions $u(g)$ to problem (1)-(2) obeying all admissible controls $g \in U$, see (3).

It is a priori assumed that problem (1)-(2) is uniquely solvable in the class $W_{2,0}^m(\Omega) = W_2^m(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$, $3 < m \leqslant 4$. We denote by

$$M_u = \{M_1 \leqslant u(x) \leqslant M_2, x \in \Omega\}$$

the range of the exact solution to problem (1)-(2) (which is a bounded set by the assumption that the solution to the original problem is smooth). We define a neighborhood D_u (δ-neighborhood) of the exact solution as

$$D_u = \big\{\bar{u} : \bar{M}_1 = M_1 - \delta \leqslant \bar{u}(x) \leqslant M_2 + \delta = \bar{M}_2, \quad x \in K \subseteq \bar{\Omega}, \delta > 0\big\},$$

where $\delta > 0$ is an arbitrary constant and can be quite small.

It is supposed that

$$0 \leqslant q_0 \leqslant q(\eta) \leqslant \bar{q}_0, \qquad |f(\eta)| \leqslant f_0, \quad \forall \eta \in D_u;$$
$$|q(\eta_1) - q(\eta_2)| \leqslant L_q|\eta_1 - \eta_2|, \quad |f(\eta_1) - f(\eta_2)| \leqslant L_f|\eta_1 - \eta_2|, \forall \eta_1, \eta_2 \in D_u;$$
$$\frac{2\mu(\max l_\alpha)^2}{\nu^2 - \mu\bar{q}_0(\max l_\alpha)^2}\left\{L_f + \frac{\mu f_0[2(2+\sqrt{2})L + L_q(\max l_\alpha)^2]}{\nu^2 - \mu\bar{q}_0(\max l_\alpha)^2}\right\} = q_0^*,$$
$$\nu^2 - \mu\bar{q}_0(\max l_\alpha)^2 > 0, \qquad q_1^* = \frac{q_0^*}{2} < 1.$$

Here, ν, μ, R_α, $\alpha = 1, 2$, $q_0, \bar{q}_0$, L_q, L_f, f_0, are given constants. Note that the conditions imposed on the coefficients of the state equation are satisfied only in the vicinity of the exact solution values, which indicates the presence of nonlinearities of the unbounded growth.

When the solution of the optimization problem is smooth enough, a fairly complete study of convergence was performed and the estimates of accuracy were obtained in the theory of finite difference schemes (see, for example, [1],[2], and the literature cited therein). However, real physical processes, as a rule, take place in heterogeneous environments when various solution domains have various physical characteristics. Nowadays, the study of computational methods for optimization problems described by nonlinear PDFs with unbounded nonlinearities is actual and in demand, and is a rather complex technical problem.

In the present work we construct difference approximations for extremum problems and obtain estimates for accuracy of approximation with respect to the state.

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On the Domain of Fractional Powers of Operators in Banach Spaces and Generalized Dirichlet-to-Neumann Maps

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Let $A : \mathcal{D}(A) \to X$ be a non-negative linear operator in a Banach space X and $\alpha \in \mathbb{C}$, $0 < \operatorname{Re}\alpha < 1$. We study the generalized Dirichlet-to-Neumann map for the Bessel-type differential equation

$$u''(t) + \frac{1-2\alpha}{t}u'(t) = Au(t), \quad u(0) = x.$$

As it turns out, for "nice" x, the generalized Neumann data $\lim\limits_{t\to 0+} -t^{1-2\alpha}u'(t)$ equals $A^\alpha x$ up to a constant $c_\alpha \in \mathbb{C}$.

The results presented here are based on [1].

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Free Boundaries in Rock Mechanics

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In the present talk we deal with some physical processes in rock mechanics which are described by free-boundary problems. Some of them are well known (the Muskat problems), some of them are completely new (in-situ leaching and dynamics of cracks in underground rocks).

The Muskat problem describes a motion of two immiscible incompressible viscous liquids separated by a free (unknown) boundary and it is still unsolved up to now. We suggest a new formulation for this physical problem, which uses microstructure and homogenization and discuss some theoretical results for a joint motion of two immiscible incompressible viscous liquids separated by a free (unknown) boundary in a single capillary. Among them is heap and in-situ leaching, important technological processes to extract uranium, precious metals, nickel, copper and other compound, and biological tissue growth in vitro.

To understand the complex systems and their mathematical description, we develop a general mathematical approach to treat it on the mesoscale (~ 1000 nm). The main point here is new conditions at the free (unknown) boundary between liquid and solid phases ("pore space – solid skeleton"), which express usual mass conservation laws, and the derivation of mathematical model, describing the process at the macroscopic level. As a result, the proposed method allows one to study how the dynamics of the free boundary depends on the heterogeneous solution rate and external parameters (the temperature, pressure and reagent concentration). In turn, the overall rate of the process can be controlled either by the rate of chemical reaction on the free boundary or by the rate at which dissolved substances are transported from the solid. We outline how to surpass from the microscopic description to a macroscopic model as compared to the commonly used techniques.

The last problem is dynamics of cracks (with unknown boundaries) in rocks under frequently repeated heat impulses coming from the Earth's core. Following K. Kasahara, these geological faults in rocks are earthquakes centers, and any earthquake is a consequence of a collapse in a deep-seated geological fault, and it begins when the pressure inside the fault achieves some limiting value (for a given fault). We suggest the simplest mathematical model of a deep-seated geological fault's collapse based on the poroelasticity theory.

On Doubly Nonlinear Evolution Equations with Dynamic Relation between the State Variables

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Doubly nonlinear evolution equations have the abstract form $\frac{d}{dt}v + Au = f$ with a static relation $v = Bu$ between the state variables u, v for nonlinear operators A, B. A particular case are systems of doubly nonlinear reaction-diffusion equations

$$\frac{\partial v}{\partial t} - \operatorname{div}(a(\nabla u)) = f\,, \tag{1}$$

where u is vector-valued and the operator $Au = -\operatorname{div}(a(\nabla u))$ may be degenerate or singular. After a short review of the case where $Bu = b(u)$ is non-potential, we concentrate in this talk on the case of an additional dynamic equation like

$$\frac{\partial u}{\partial t} - \Delta u = \frac{1}{\varepsilon}(v - b(u)) \tag{2}$$

with a relaxation time $\varepsilon > 0$. Combining (1) and (2) leads to the abstract second order equation

$$\varepsilon\frac{d^2u}{dt^2} + \frac{d}{dt}(\varepsilon J_Z u + Bu) + Au = f\,, \tag{3}$$

with a duality mapping J_Z on a Hilbert space Z given above by $J_Z u = -\Delta u$ on $Z = W_0^{1,2}(\Omega)$. We show how to prove existence of solutions of (3) by a Rothe method, which works even in the case where A is a multivalued and unbounded potential operator [1].

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On the Dynamics of Nonlinear Interaction of the String with an Oscillator (the Nonlinear Lamb System)

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The Lamb system describes small transverse oscillations of an infinite string interacting with a particle of mass m locally at the origin. The interaction force causing the string to oscillate, is, in general, nonlinear and conservative, such that the system is Hamiltonian. Mathematically, the Lamb system is the following Cauchy problem:

$$\left.\begin{array}{l} \ddot{u}(x,t) = u''(x,t), \qquad x \in \mathbb{R}\setminus 0 \\[2ex] m\ddot{y}(t) = F(y(t)) + u'(0+,t) - u'(0-,t), \qquad y(t) = u(0,t) \end{array}\right| \quad t \in \mathbb{R}$$

$$u|_{t=0} = u_0(x), \quad \dot{u}|_{t=0} = v_0(x) \quad \dot{y}(t)|_{t=0} = p_0.$$

Here the solution $u(x,t)$ takes values in $\mathbb{R}^n$. The system was originally introduced by H. Lamb [4] in the linear case where $F(y) = -\omega^2 y$ and $n = 1$. The Lamb system with general nonlinear function $F(y)$ was studied in [1], where the global attraction to stationary states was established for the first time. In the talk we formulate the main facts related to the dynamics of the system, such as the existence and uniqueness of a solution, energy conservation, system stabilization to stationary states, and asymptotic behavior of the dynamics. We establish the *asymptotic completeness* in the system with a positive mass for hyperbolic stationary states (for a massless particle, it was proved in [2]). For the proof, with the help of the inverse function theorem in a Banach space, we construct a trajectory of the reduced equation (which is a nonautonomous nonlinear ordinary differential equation) converging to a hyperbolic stationary point. We give a counterexample showing nonexistence of such trajectories for non-hyperbolic stationary points.

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Homogenization of Periodic Hyperbolic Systems with the Correction Term Taken into Account

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In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we consider the selfadjoint second order differential operator $A_\varepsilon = b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$, $\varepsilon > 0$. The matrix-valued function $g(\mathbf{x})$ is assumed to be periodic, bounded and uniformly positive definite. Next, $b(\mathbf{D}) = \sum_{j=1}^d b_j D_j$ is a first order differential operator with constant coefficients. The symbol $b(\boldsymbol{\xi})$ is subject to some condition which ensures strong ellipticity of the operator A_ε. We study the behavior of the operator $A_\varepsilon^{-1/2} \sin(\tau A_\varepsilon^{1/2})$, $\tau \in \mathbb{R}$, in the small period limit.

In [1], M. Sh. Birman and T. A. Suslina proved that

$$\| \cos(\tau A_\varepsilon^{1/2}) - \cos(\tau (A^0)^{1/2}) \|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant C\varepsilon(1 + |\tau|), \quad \tau \in \mathbb{R}, \quad \varepsilon > 0. \quad (1)$$

Here $A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ is the effective operator with the constant positive effective matrix g^0. The results of this type are called operator error estimates in homogenization theory.

Later M. A. Dorodnyi and T. A. Suslina [2] showed that estimate (1) is sharp with respect to the type of the operator norm. On the other hand, in [2], under some additional assumptions on the operator, the result (1) was improved with respect to the type of the norm. In [1, 2], by virtue of the identity $A_\varepsilon^{-1/2}\sin(\tau A_\varepsilon^{1/2}) = \int_0^\tau \cos(\widetilde{\tau} A_\varepsilon^{1/2})\,d\widetilde{\tau}$ and the similar identity for the effective operator, the estimate

$$\|A_\varepsilon^{-1/2}\sin(\tau A_\varepsilon^{1/2}) - (A^0)^{-1/2}\sin(\tau (A^0)^{1/2})\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C\varepsilon(1+|\tau|)^2 \qquad (2)$$

was deduced from (1) as a (rough) consequence.

Our first result is improvement of estimate (2). Our second result is approximation for the operator $A_\varepsilon^{-1/2}\sin(\tau A_\varepsilon^{1/2})$ in the $(H^2 \to H^1)$-norm with the corrector taken into account.

Theorem (see [3]). *For $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|A_\varepsilon^{-1/2}\sin(\tau A_\varepsilon^{1/2}) - (A^0)^{-1/2}\sin(\tau (A^0)^{1/2})\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C_1\varepsilon(1+|\tau|),$$
$$\left\|A_\varepsilon^{-1/2}\sin(\tau A_\varepsilon^{1/2}) - (A^0)^{-1/2}\sin(\tau (A^0)^{1/2}) - \varepsilon K(\varepsilon,\tau)\right\|_{H^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant C_2\varepsilon(1+|\tau|).$$

Here $K(\varepsilon,\tau)$ is the so-called corrector. The constants C_1 and C_2 are controlled explicitly in terms of the problem data.

The results are applied to homogenization for the solutions of the nonhomogeneous hyperbolic equation $\partial_\tau^2 \mathbf{u}_\varepsilon(\mathbf{x},\tau) = -A_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x},\tau) + \mathbf{F}(\mathbf{x},\tau)$, $\mathbf{F} \in L_{1,\mathrm{loc}}\big(\mathbb{R}; L_2(\mathbb{R}^d;\mathbb{C}^n)\big)$.

The study was supported by project of RSF No. 17-11-01069.

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On the Uniqueness of Solutions of Boundary Value Problems for Mixed-type Equations

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Consider an analogue of the classical Gellerstedt problem for the Lavrentyev–Bitsadze equation

$$u_{xx} + (\operatorname{sgn} y)u_{yy} = 0 \qquad (1)$$

in the domain $D = D^+ \cup D_1 \cup D_2$, where D is a semicircle in the upper half-plane, $D = \{(r,\theta) : 0 < r < 1,\, 0 < \theta < \pi\}$, D_1 is the triangle in the lower half-plane bounded by the axis OX and by the segments $\gamma_1 = \{(x,y) : y = -x-1, -1 \leqslant x \leqslant -\frac{1}{2}\}$, $\gamma_2 = \{(x,y) : y = x, -\frac{1}{2} \leqslant x \leqslant 0\}$, and D_2 is the triangle in the lower half-plane bounded by the axis OX and by the segments $\gamma_3 = \{(x,y) : y = -x, 0 \leqslant x \leqslant \frac{1}{2}\}$, $\gamma_4 = \{(x,y) : y = x-1, \frac{1}{2} \leqslant x \leqslant 1\}$.

The problem is to find a function $u(x,y)$ satisfying equation (1), the inhomogeneous boundary condition at the circular curve

$$u(1,\theta) = f(\theta),\ 0 \leqslant \theta \leqslant \pi, \tag{2}$$

and the following conditions at the parallel characteristics γ_1 and γ_3:

$$u(x,y)|_{\gamma_1} = 0, \tag{3}$$

$$\left(\frac{\partial u(x,y)}{\partial x} - \frac{\partial u(x,y)}{\partial y}\right)\bigg|_{\gamma_3} = 0. \tag{4}$$

In addition, we set one of the following Frankl type gluing conditions along the degeneracy line of the type of the equation:

$$\begin{cases} \dfrac{\partial u(x,+0)}{\partial y} = \pm \dfrac{\partial u(x,-0)}{\partial y},\ x \in (-1,\, 0), \\ \dfrac{\partial u(x,+0)}{\partial y} = \pm \dfrac{\partial u(x,-0)}{\partial y},\ x \in (0,\, 1); \end{cases} \tag{5}$$

The classical Gellerstedt problems were investigated in [1, 2], while the Gellerstedt problems with data given at internal or external characteristics were investigated in [3, 4].

We obtain some results about solvability of these problems, including the case where condition (4) is replaced by the condition $u(x,y)|_{\gamma_3} = 0$.

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Multistability in the Mackey–Glass Equation

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Multistability, i.e., the simultaneous existence of several attractors for a given set of parameters, is one of characteristic features of nonlinear systems. We investigate bifurcation from equilibrium states of periodic solutions of the well-known Mackey–Glass equation [1] with the uniform normalization method. We show that several stable periodic solutions can coexist for given parameters.

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On Maximum Principle for Differential-Difference Equations

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For the case of classical elliptic and parabolic partial differential equations, the maximum principle is derived from the form of the equation itself: assuming that a function satisfies the considered equation, we immediately see that it satisfies a classical sufficient extremum condition for multi-variable functions. Thus, the maximum principle is an a priori estimate.

Once we include nonlocal terms, the above arguing becomes impossible because the equation links the values of derivatives of the equation at different points. However, if we have a Poisson-type integral representation of the solution and criteria of its stabilization (see [1, 2, 3]), then it is possible to establish the maximum principle for such solutions as well; note that it is not an a priori estimate anymore: its validity is guaranteed only for solutions represented by the said Poisson-type expression.

At the moment, the greatest generality achieved in this area is as follows:

(i) the half-space Cauchy problem for the parabolic equation

$$\frac{\partial u}{\partial t} = Lu \equiv \sum_{k,j=1}^{n} a_{kj} \frac{\partial^2 u}{\partial x_k^2} (x + h_{kj} e_j, t),$$

where e_j, $j = \overline{1,n}$, denotes the unit vector of the jth coordinate direction, while the real coefficients a_{kj} and h_{kj}, $k,j = \overline{1,n}$, guarantee the strong ellipticity of the differential-difference operator $-L$;

(ii) the half-plane Dirichlét problem for the elliptic equation

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \sum_{k=1}^{m} a_k \frac{\partial^2 u}{\partial x^2}(x - h_k, y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0$$

under the assumption that there exists a positive constant C such that the inequality $1 + \sum\limits_{k=1}^{m} a_k \cos h_k \xi \geqslant C$ holds on the real line.

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Uniform Asymptotics of Boundary Values of the Solution to the Linear Problem on the Run-Up of Waves on a Shallow Beach

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In a domain $\Omega \subset \mathbb{R}^2$, we consider the Cauchy problem with localized initial data for the wave equation with variable velocity degenerating at the boundary $\partial\Omega$ as the square root of the distance to $\partial\Omega$. In particular, this problem describes the run-up of tsunami waves on a shallow beach in the linear approximation (e.g., see [1]). A method for constructing asymptotic solutions of this problem for small values of the source-to-basin size ratio was developed in [2] (see also references therein) on the base of Maslov's canonical operator [3] modified in [4] to take into account the nonstandard behavior of characteristics (they go to infinity in the momentum variables) near the boundary, which can be viewed as a caustic of a special kind. The modified canonical operator is defined via the Hankel transform, which arises as Fock's quantization [5] of a canonical transformation regularizing these nonstandard characteristics.

In the present talk, we are interested in the boundary values of the asymptotic solutions. It turns out that they can be expressed via the standard canonical operator on a Lagrangian submanifold of the cotangent bundle of the boundary [6]. This simplifies the formulas dramatically; for initial perturbations of a special form, the boundary values of the asymptotic solutions can be expressed via simple algebraic functions. Further, if the initial perturbation shape is specified by a function sufficiently rapidly decaying at infinity, then the boundary values of the asymptotic solution give the asymptotics of the boundary values of the exact solution in the uniform norm [7]. To show this, we, in particular, prove a trace theorem for nonstandard Sobolev type spaces with degeneration at the boundary.

The results were obtained jointly with S. Yu. Dobrokhotov and A. A. Tolchennikov. This work was supported by the Russian Science Foundation under grant 16-11-10282.

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Variational Inequalities for the Fractional Laplacians

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We study the obstacle problems for the Dirichlet and Navier fractional Laplacians of order $s \in (0,1)$ in a bounded domain $\Omega \subset \mathbb{R}^n$, under mild assumptions on the data.

Let Ω be a bounded and Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 1$. Given $s \in (0,1)$, a measurable function ψ on Ω and $f \in \widetilde{H}^s(\Omega)'$, we consider the variational inequalities

$$u \in K^s_\psi\,, \qquad \langle (-\Delta)^s\, u - f, v - u\rangle \geqslant 0 \quad \forall v \in K^s_\psi\,. \tag{Dir}$$

$$u \in K^s_\psi\,, \qquad \langle (-\Delta_\Omega)^s\, u - f, v - u\rangle \geqslant 0 \quad \forall v \in K^s_\psi\,. \tag{Nav}$$

Here

$$\widetilde{H}^s(\Omega) := \{u \in H^s(\mathbb{R}^n) \mid u \equiv 0 \ \text{ on } \ \mathbb{R}^n \setminus \overline{\Omega}\};$$

$$K^s_\psi = \Big\{ v \in \widetilde{H}^s(\Omega) \mid v \geqslant \psi \ \textit{a.e. on } \Omega \ \Big\}\;;$$

$(-\Delta)^s$ is the *restricted* (or *Dirichlet*) fractional Laplacian defined via the Fourier transform by

$$\mathcal{F}[(-\Delta)^s\, u](\xi) = |\xi|^{2s}\mathcal{F}[u](\xi) = \frac{|\xi|^{2s}}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, dx\,;$$

$(-\Delta_\Omega)^s$ is the *spectral* (or *Navier*) fractional Laplacian being sth power of the standard Laplacian in the sense of spectral theory, i.e.

$$(-\Delta_\Omega)^s u = \sum_{j=1}^{\infty} \lambda_j^s \Big(\int\limits_\Omega u\varphi_j \Big) \varphi_j \,.$$

where λ_j and φ_j, $j \geqslant 1$, are the eigenvalues and (L_2-normalized) eigenfunctions of the standard Dirichlet Laplacian, and the series is understood in the sense of distributions.

We obtain some regularity results and compare the solutions of (Dir) and (Nav).

This talk is based on papers [1] and [2].

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On Long-Term Dynamics of Slow-Fast Systems with Passages through Resonances

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Small perturbations imposed on an integrable nonlinear multifrequency oscillatory system cause a slow evolution. During this evolution the system may pass through resonant states. There are important phenomena related to such passages: capture into resonance and scattering on resonance. We will discuss the dynamics on long time intervals on which many passages through resonances occur.

Effects of passages through resonances can be considered as random events. Such effects separated by long time intervals can be treated as statistically independent. In this talk we describe model examples from charged particles dynamics that demonstrate these quasi-random effects. In particular, we present an analog of kinetic equation for description of such kind of dynamics.

The talk is based on papers [1, 2, 3].

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Asymptotic Integration of Functional Differential Equations with Oscillatory Decreasing Coefficients

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We study the asymptotic integration problem for the functional differential system (FDS)

$$\dot{x} = B_0 x_t + G(t, x_t) \tag{1}$$

as $t \to \infty$. Here $x \in \mathbb{C}^m$ and $x_t(\theta) = x(t+\theta)$ $(-\tau \leqslant \theta \leqslant 0)$ denotes the element of C_τ, where $C_\tau \equiv C\big([-\tau, 0], \mathbb{C}^m\big)$ is the set of all continuous functions defined on $[-\tau, 0]$ and valued in $\mathbb{C}^m$. We consider Eq. (1) as a perturbation of the linear autonomous system $\dot{x} = B_0 x_t$, where B_0 is a constant bounded linear functional acting from C_τ to $\mathbb{C}^m$. And the perturbation in the form of a bounded linear functional $G(t, \cdot)$ acting from C_τ to $\mathbb{C}^m$, is assumed to be small in a certain sense. Roughly speaking, we assume that for each infinitely differentiable function $\varphi(\theta) \in C_\tau$, the vector function $G(\cdot, \varphi)$ consists of two terms: the first one is of oscillatory decreasing form as $t \to \infty$ and the second one is a certain function depending on $\varphi(\theta)$ and absolutely integrable on $[t_0, \infty)$. The main assumption concerning the unperturbed equation is the following. The characteristic equation $\det \Delta(\lambda) = 0$, where $\Delta(\lambda) = \lambda I - B_0(e^{\lambda\theta} I)$, has N roots (with account of their multiplicities) $\lambda_1, \ldots, \lambda_N$ with zero real parts, while the remaining roots have all negative real parts. This makes possible to use the ideas of the center manifold theory (see, e.g., [1]) for asymptotic integration of Eq. (1).

The asymptotic integration problem for FDS having form (1) was studied by the author in paper [3]. The essence of the proposed asymptotic integration method is to construct the center-like manifold (also called critical manifold) for Eq. (1) which is positively invariant for trajectories of Eq. (1) for $t \geqslant t_*$ and attracts all the trajectories for sufficiently large t. It turns out that the dynamics of solutions of Eq. (1) lying on this manifold can be described by some N-dimensional linear system of ordinary differential equations. Therefore, in order to obtain the asymptotics for all solutions of Eq. (1), we should construct the asymptotic formulas for the fundamental solutions of a system on the critical manifold. This can be done by using the averaging technique from [2] together with the well-known asymptotic theorems. We illustrate the proposed method by constructing the asymptotic formulas as $t \to \infty$ for solutions of the delay differential equations

$$\dot{x} = -\frac{\pi}{2} x(t-1) + \frac{a \sin \omega t}{t^\rho} x(t-h), \tag{2}$$

$$\dot{x} = -\frac{\pi}{2} x\Big(t - 1 + \frac{a \sin \omega t}{t^\rho}\Big). \tag{3}$$

In Eqs. (2) and (3), we assume that $a, \omega \in \mathbb{R} \backslash \{0\}$, $h \geqslant 0$, and $\rho > 0$.

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Analyticity and Non-analyticity for Bounded Solutions of Functional Differential Equations

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Let $\mathbb{R}$ and $\mathbb{C}$ denote the real numbers and the complex numbers respectively. An old theorem of Nussbaum [1] implies as a very special case that if x is a bounded, continuously differentiable function from $(-\infty, T]$ to $\mathbb{C}$ which satisfies the equation

$$x'(t) = f(x(t), x(t-r)),$$

where r is a positive constant and f is analytic on an open neighborhood of the closure of the range of $t \to (x(t), x(t-r))$ in $\mathbb{C} \times \mathbb{C}$, then $t \to x(t)$ extends to a bounded analytic function defined on a $\delta > 0$ neighborhood of $(-\infty, T]$ in $\mathbb{C}$.

We shall describe work with Professor J. Mallet-Paret in which we consider a non-autonomous FDE with a non-constant time lag:

$$x'(t) = f(t, x(t), x(t-r(t))).$$

Here $r(t)$ is nonnegative and real analytic on $(-\infty, T]$ and f is analytic on an appropriate domain in $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$. Somewhat surprisingly, the situation in this context is much more complicated than in the “classical” case. We shall discuss a class of examples of the above form which possess infinitely differentiable periodic solutions which are analytic on certain nonempty open sets, but which also fail to be analytic on nonempty, Cantor-like sets. The Krein–Rutman theorem plays a role in our arguments.

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On Periodic Problem for Random Functional Differential Inclusions

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We consider the notion of a random (either smooth or non-smooth) integral guiding function for some classes of functional differential inclusions in finite-dimensional spaces. Combined with the random topological degree, this notion is applied to the study of periodic and asymptotic problems. Some applications to random differential complementarity systems are discussed.

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On Integration of Smooth Multidimensional Bol Loop Structure Equations

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We continue to investigate the smooth multidimensional loop with the B_m identity $(x/y)(z\backslash x) = x((zy)\backslash x)$ (the so-called Bol loop or B_m-loop). We find a system of PDEs defining a special class of such loops (denoted by $CB_m^{\triangledown}(1)$), arising on the semi-direct product of two Abelian groups. From the geometric point of view, this system represents the structure equations of some multidimensional three-web, written in the Cartan symbols of external differential forms. We succeeded in integrating this system and finding the equations of the corresponding Bol loop in some local coordinates. The following assertion holds.

Theorem. *For some choice of local coordinates, the r-dimensional Bol loop $CB_m^{\triangledown}(1)$ equations can be reduced to the form*

$$
\begin{aligned}
Z^a &= X^a + e^a_b(X)\left(Y^b - \frac{1}{2}d^b_{uv}(X^uY^v - X^vY^u) - \Lambda^b_{(uw)v}X^uX^wY^v\right) + \\
&\quad + \frac{2}{3}S^a_c(X)a^c_{b(t}\Lambda^b_{uw)v}X^tX^uX^wY^v, \\
Z^u &= X^u + Y^u.
\end{aligned}
$$

Here

- *the indices take the following values: $a, b, c, d, ... = 1, ..., \rho$; $u, v, w, z, ... = \rho + 1, ..., r$;*
- *a^a_{bu} and d^b_{uv} are some constants, the quantities Λ^b_{uvw} are defined by the formulas*
$$\Lambda^a_{wuv} \equiv \frac{1}{2}b^a_{wuv} - a^a_{bw}d^b_{uv},$$
were b^a_{wuv} are also constants;
- *the quantities $e^a_b(X)$ are the components of the matrix $\exp(2C(X))$, where $C(X) = (a^a_{bu}X^u)$;*
- *the quantities $S^a_c(X)$ are the components of the function*
$$S(x) = 3\sum_{k=o}^{\infty}\frac{1}{k!(k+3)}(2C(x))^k.$$

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Numerical Solution of a Parabolic Equation by Means of the Rational Approximation of the Operator Exponential

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Let $\mathbf{H}$ be a Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$. Consider the abstract parabolic equation

$$\dot{x} = Bx + bv(t), \qquad x(0) = 0,$$

with the scalar output function

$$y(t) = \langle x(t), c\rangle.$$

Here $B\colon D(B) \subset \mathbf{H} \to \mathbf{H}$ is a negative semidefinite operator, $v\colon [0, +\infty) \to \mathbb{R}$, and b, $c \in \mathbf{H}$. The function $h_{bc}(t) = \int_{-\infty}^{0} \exp_t(\xi)\, dE_{bc}(\xi)$, where $\exp_t(\xi) = e^{\xi t}$ and E is the spectral decomposition of the operator B, is called a *fundamental solution*.

We suggest a numerical method [1] for the construction of an approximate fundamental solution. A similar approach was earlier used in the finite-dimensional case in [2]. For each $t > 0$, we replace the function $\exp_t$ by a rational function r_t approximating the original function on the interval $(-\infty, 0]$. For the approximate fundamental solution we take the function $\tilde{h}_{bc}(t) = \int_{-\infty}^{0} r_t(\xi) dE_{bc}(\xi)$.

We obtain a priori estimates for the approximation error.

Theorem 1. *Let $t > 0$ be a given number. If the rational function r_t approximates the function $\exp_t$ with an absolute error $\varepsilon(t) \geqslant 0$, i.e.*

$$|r_t(\xi) - e^{\xi t}| \leqslant \varepsilon(t), \qquad \xi \in (-\infty, 0],$$

then the approximate fundamental solution $\tilde{h}_{bc}$ satisfies the estimate

$$|\tilde{h}_{bc}(t) - h_{bc}(t)| \leqslant \varepsilon(t)\|b\|\,\|c\|.$$

Theorem 2. *Let $t > 0$ be a given number. If the rational function r_t approximates the function $\exp_t$ with a relative error $\varepsilon(t) \geqslant 0$, i.e.*

$$|r_t(\xi) - e^{\xi t}| \leqslant \varepsilon(t)e^{\xi t}, \qquad \xi \in (-\infty, 0],$$

then the approximate fundamental solution $\tilde{h}_{bc}$ satisfies the estimate

$$|\tilde{h}_{bc}(t) - h_{bc}(t)| \leqslant \varepsilon(t)\sqrt{h_{bb}(t)h_{cc}(t)}.$$

The suggested method can be used in the remodeling problems for complicated objects and systems.

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Method of Asymptotic Partial Decomposition for Multistructures: the Steady Stokes Equations

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Method of asymptotic partial decomposition of a domain proposed and justified earlier for thin domains (rod structures, tube structures, see [1, 2, 3]) is generalized and justified for the multistructures, i.e. domains consisting of a set of thin cylinders connecting some massive 3D domains. In the present paper, the Dirichlet boundary value problem for the steady state Stokes equations is considered. This problem is reduced to the Stokes equations in the massive domains coupled with the Poiseuille type flows within the thin cylinders at some distance from the bases (the MAPDD approximation problem). The estimates for the difference of the exact solution to the initial problem and the solution to the MAPDD approximation problem is proved.

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Hilbert Inequality on Function Spaces

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Hilbert's inequality reads as

$$\int_0^\infty \int_0^\infty \frac{f(y)g(x)}{x+y} \; dx\; dy < \pi \csc \frac{\pi}{p} \; \|f\|_{L^p(\mathbb{R}^+)}\|g\|_{L^{p'}(\mathbb{R}^+)}, \tag{1}$$

where $1 < p < \infty$, $f \in L^p(0,\infty)$ and $g \in L^{p'}(0,\infty)$. It is well known that the Hardy inequality can be deduced from (1), which in turn plays the key role in studying the imbeddings of Sobolev spaces into Lebesgue spaces.

We shall discuss in this talk that inequality (1) can be obtained from certain convolution inequalities. Next, we shall discuss various generalizations of (1), viz, when the expression $\frac{1}{x+y}$ is replaced by a suitable kernel $k(x,y)$ and Lebesgue spaces are replaced by various other spaces such as grand Lebesgue spaces, fully measurable grand Lebesgue spaces, and grand Bochner Lebesgue spaces. Finally, we shall talk about the determination of the best constant in (1) for the case where the parameters are non-conjugate.

On Approximation of Entropy Solutions to Degenerate Nonlinear Parabolic Equations

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We study the Cauchy problem for the degenerate parabolic equation

$$u_t + \operatorname{div}_x f(u) - \Delta_x g(u) = 0, \quad u(0,x) = u_0(x) \in L^\infty(\mathbb{T}^n), \tag{1}$$

where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the n-dimensional torus.

The flux vector $f(u) = (f_1(u),\dots,f_n(u)) \in C(\mathbb{R},\mathbb{R}^n)$ is supposed to have bounded variation in any segment while the diffusion function $g(u)$ is a continuous nondecreasing function (it is possible that $g(u) \equiv 0$ when equation (1) is reduced to a multidimensional conservation law). By our assumptions, we can represent the flux vector $f(u)$ as the difference $f(u) = f^+(u) - f^-(u)$ of vector functions with continuous nondecreasing components f_i^+, f_i^-, $i = 1,\dots,n$. For a fixed $\varepsilon > 0$, we introduce the nondecreasing functions $h_i^\pm(u) = f_i^\pm(u) + \frac{1}{\varepsilon}g(u)$ and define the approximate solution $u = u(t,x;\varepsilon)$ of (1) as a solution of the ODE

$$\dot{u} = \frac{1}{\varepsilon}\sum_{i=1}^n [h_i^+(u(t,x-\varepsilon e_i)) + h_i^-(u(t,x+\varepsilon e_i)) - h_i^+(u(t,x)) - h_i^-(u(t,x))], \; u(0) = u_0, \tag{2}$$

considered in the Banach space $L^\infty(\mathbb{T}^n)$. Here e_i, $i = 1,\dots,n$, is the standard basis in $\mathbb{R}^n$. Approximation scheme (2) is analogous to that one proposed in [3].

Theorem 1. *There exists a unique solution $u(t,x;\varepsilon) \in C^1(\mathbb{R}_+, L^\infty(\mathbb{T}^n))$ of problem (2). Moreover, $\|u(t,\cdot;\varepsilon)\|_\infty \leqslant \|u_0\|_\infty$, and the map $u_0 \to u(t,\cdot;\varepsilon)$ is monotone and nonexpansive in $L^1(\mathbb{T}^n)$. As $\varepsilon \to 0$, we have $u(t,x;\varepsilon) \to u(t,x)$ in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{T}^n)$, where $u(t,x)$ is a unique entropy solution of problem (1) in the sense of Carrillo [2].*

In the hyperbolic case $g \equiv 0$, the presented results were established in [1].

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Current Trends in Deterministic and Stochastic Dynamical Systems Modelling in Mathematical Ecology, from Lotka–Volterra to Hyper-Network Models

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There has been a remarkable interest in Dynamical Systems Modelling in Mathematical Ecology in the late 20th century. Numerous researches have discovered chaotic phenomena, Hopf bifurcations and other interesting behaviors from within two-or-three-species ODE models. However, nonlinearity presents considerable difficulties in the derivation of analytical solutions, even in relatively simple models.

There are obvious restrictions to these modelling approaches. Additionally to the increased requirements in computational resources, it was gradually realized that the predictive capacity of these deterministic models would remain of limited value without taking spatial interactions and stochasticity into account.

Given these, the aim of our study is to identify the key trends emerging in Mathematical Ecological Modelling currently, after tracing the key stages of its development from the celebrated Verhulst and Lotka–Volterra models until 2017, and to assess their forthcoming future.

This study presents the results of an extensive evaluation work on how some of the shortcomings of traditional ODE models can partly be overcome by introducing new components in low-dimensional models; such as Laplacian matrices, stochasticity (i.e. the Gaussian white noise added to each population's dynamics), and others.

Differential equation models have consequently become even more “dynamic” now, by allowing for the study of hitherto unscathed research fields. Furthermore, even the dynamics of “networks of networks” (such as multiplex, multiplayer, hyper-networks) can now be examined analytically also.

Application of Fractional Calculus to the Theory of Hereditary Dynamical Systems

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For the hereditary dynamical system, the following Cauchy problem is posed:

$$\partial_{0t}^{\beta} x(\eta) + \lambda \partial_{0t}^{\gamma} x(\eta) = f(x(t), t), x(0) = \alpha_1, \dot{x}(0) = \alpha_2, 1 < \beta < 2, 0 < \gamma < 1. \quad (1)$$

Here $\partial_{0t}^{\beta} x(\eta) = \frac{1}{\Gamma(2-\beta)} \int\limits_0^t \frac{\ddot{x}(\eta)d\eta}{(t-\eta)^{\beta-1}}$ and $\partial_{0t}^{\gamma} x(\eta) = \frac{1}{\Gamma(1-\gamma)} \int\limits_0^t \frac{\dot{x}(\eta)d\eta}{(t-\eta)^{\gamma}}$ are the operators of Gerasimov–Kaputo fractional orders β and γ, α_1, α_2 are constants, $t \in [0, T]$ is the time variable, $\dot{x}(t) = dx/dt, \ddot{x}(t) = d^2x/dt^2$, $f(x(t), t)$ is a nonlinear function satisfying the Lipschitz condition with respect to the argument $x(t)$, i.e.

$$|f(x_1(t), t) - f(x_2(t), t)| \leqslant L |x_1(t) - x_2(t)|,$$

and $x(t) \in C^2[0, T]$ is a solution.

Using the methodology of paper [1], we come to the discrete Cauchy problem

$$A \sum_{j=0}^{k-1} a_j (x_{k-j+1} - 2x_{k-j} + x_{k-j-1}) + B \sum_{j=0}^{k-1} b_j (x_{k-j+1} - x_{k-j-1}) = f_k \quad (2)$$

with $A = \frac{\tau^{-\beta}}{\Gamma(3-\beta)}, B = \frac{\lambda \tau^{-\gamma}}{2\Gamma(2-\gamma)}, \tau = \frac{T}{N}$, $a_j = (j+1)^{2-\beta} - j^{2-\beta}$, $b_j = (j+1)^{1-\gamma} - j^{1-\gamma}$, where N is the number of nodes and $x(t_k) = x_k$ is the grid function. Equation (3) can be rewritten in a more convenient form of an explicit-finite difference scheme,

$$x_{k+1} = \frac{1}{A+B} \left(2Ax_k - (A-B)x_{k-1} - A \sum_{j=1}^{k-1} a_j (x_{k-j+1} - 2x_{k-j} + x_{k-j-1}) \right) \quad (3)$$

$$- \frac{1}{A+B} \left(B \sum_{j=1}^{k-1} b_j (x_{k-j+1} - x_{k-j-1}) + f_k \right).$$

In the paper, the following key theorems are proved.

Theorem 1 (see [1]). *Explicit finite-difference scheme (3) converges with* $|\bar{x}_k - x_k| = O(\tau)$ *under the condition*

$$\tau \leqslant \tau_0 = \min \left(1, \left(\frac{2\Gamma(2-\gamma)}{\lambda \Gamma(3-\beta)} \right)^{\frac{1}{\beta-\gamma}} \right). \quad (4)$$

Theorem 2 (see [2]). *Explicit finite-difference scheme (3) is conditionally stable if condition (4) is satisfied and the estimate* $|Y_k - X_k| \leqslant C\,|Y_0 - X_0|$ *holds for any* k *with a constant* $C > 0$ *independent of* τ.

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Homogenization Estimates in L^p-norms

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Consider a semigroup e^{-tA_ε} in $\mathbb{R}^d_x$ ($d \geqslant 2$) with the generator

$$A_\varepsilon = -\operatorname{div}(\mathrm{a}^\varepsilon \nabla), \quad \mathrm{a}^\varepsilon(\mathrm{x}) = \mathrm{a}(\mathrm{x}/\varepsilon),$$

where $\varepsilon \in (0,1]$, $a(x)$ is a measurable symmetric matrix 1-periodic in each of the variables $x_1, \ldots, x_d$ and satisfying the ellipticity and boundedness conditions

$$\nu\xi^2 \leqslant a(x)\xi \cdot \xi \leqslant \nu^{-1}\xi^2 \quad \forall \xi \in \mathbb{R}^d, \quad \nu > 0. \tag{1}$$

Our goal is to investigate the asymptotic behaviour as $t \to +\infty$ of the semigroup e^{-tA_ε} for a fixed parameter ε, and also the asymptotic behaviour as $\varepsilon \to 0$ of this semigroup for a finite time $t > 0$.

It turns out that in both cases (for large times or for small ε) the behaviour of the semigroup is governed by a constant homogenized matrix a^0 of class (1) defined via certain procedure usual in homogenization. So, one can construct the homogenized operator $A_0 = -\operatorname{div}(\mathrm{a}^0\nabla)$ of the same type as the initial operator A_ε but having constant coefficients.

Theorem 1. *The estimate*

$$\|e^{-tA_\varepsilon} - e^{tA_0}\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \leqslant c_0\frac{\varepsilon}{\sqrt{t}}, \quad t > 0, \quad 1 \leqslant p \leqslant \infty, \tag{2}$$

holds with a singe constant c_0 *depending on the dimension* d *and the ellipticity constant* ν *only.*

As a corollary of (2), we deduce the similar estimate for the difference of resolvents corresponding to the operators A_ε and A_0.

Theorem 2. *The estimate*

$$\|(A_\varepsilon + 1)^{-1} - (A_0 + 1)^{-1}\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \leqslant c_1\varepsilon, \quad 1 \leqslant p \leqslant \infty, \tag{3}$$

holds with a single constant c_1 *only depending on the dimension* d *and the ellipticity constant* ν.

To obtain (3) from (2), one should use the well-known representation of the resolvent in terms of the semigroup:

$$(A+1)^{-1} = \int_0^\infty e^{-t} e^{-tA} dt.$$

The proof of (2) is based on the use of the asymptotics for large times of the fundamental solution $K(x,y,t)$ that is the kernel of the integral operator e^{-tA}, $A = A_\varepsilon|_{\varepsilon=1}$.

These results are obtained jointly with Professor V.V. Zhikov. They are published in [1].

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The Dirichlét Problem and Disappearance of the Imaginary Part of the Laplace Transform on the Imaginary Axis in Connection with the Fourier–Laplace Operators

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With the help of Theorem 1, we obtain a special solution to the Dirichlét problem in $Re\, p \geqslant 0$ with the zero imaginary part on the imaginary axis (see the integrals of Poisson and Schwartz [1], p. 209). A similar situation was considered in [2] (an additional part, Sec. 2, p. 103) with the difference that the real part of the solution was zero on the imaginary axis.

By definition, $L_\pm S(x)(\cdot)(r) = \int\limits_0^\infty e^{\pm rx} S(x)dx$, $r \in [0,\infty), L_- \equiv L$,$F_\pm S(x)(\cdot)(v) = \int\limits_{-\infty}^\infty e^{\pm vxi} S(x)dx$, $v \in (-\infty, \infty)$.

Theorem 1.

$$Im\, LF_- u(x)(\cdot)(is) = -\int\limits_{-\infty}^\infty u(r)dr/(r+s) \equiv 0, s \in (-\infty, \infty),$$

if $u(0)=0$, *and* $|u(x)||x|^{2+\delta} \to 0, |x| \to \pm\infty, \delta > 0$; $\delta = const, x \in (-\infty,\infty)$, *where the function* $d^2u(x)/dx^2$ *is continuous for all* $x \in (-\infty, \infty)$.

Proof. It follows from the definition that $L(s) = L_+F_-u(x)(\cdot)(is) = \overline{LF_+u(x)(\cdot)(is)}$, $Im\, u(s) \equiv 0, s \in (-\infty,\infty)$, and we obtain $Re\, L(s) = Re\, l(s)$, $l(s) = LF_+u(x)(\cdot)(is)$, $s \in (-\infty, \infty)$. Then we use the fractionally-linear representation $S \to G_+$ (see [1], p. 127)

$$p = s = ih(az+d)/(bz+c) = f(z), a=1, b=-1, d=c=e^{i\alpha}, z \in \{z : |z| \leqslant 1\} = S,$$

$p \in \{p : Im\, p \geqslant 0\} = G_+$, $\alpha, h \in (0, +\infty)$, with $f(w) = s \in (-\infty, \infty), |w| = 1, C_1 \to (-\infty, \infty)$, and $L(f(w)) = L(s), w \in C_1 = \{w : |w| = 1\}, s \in (-\infty, \infty)$.

Note that the real parts of the functions $l(f(z))$ and $L(f(z))$ are the same on the circle $|z| = 1$: $\pi u(s) = Re\, L(s) = Re\, l(s), s \in (-\infty, \infty), s = f(w), |w| = 1$, and it follows from the solution to the Dirichlét problem in $|z| \leqslant 1$ that the the imaginary parts of the functions $L(f(z))$ and $l(f(z))$ (the two integrals of Schwartz, see [1], p. 209) are the same, $Im\, L(f(z)) = Im\, l(f(z)) + c$, $c = 0$. We use the relations $L(p) \to 0$, $l(p) \to 0$ for all p in the definitions of the functions. Moreover, both $L(f(z))$ and $l(f(z))$ are regular in $|z| \leqslant 1$ and continuous on $|z| = 1$ from the inside (see also [1], p. 127).

Thus we obtain $Im\, l(s) = -Im\, l(s) = 0, s \in (-\infty, \infty)$.

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To the Question of Robust Controllability of Differential-Algebraic Equations

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We consider the system of first order ordinary differential equations

$$A(t)x'(t) + B(t)x(t) + U(t)u(t) = 0, \quad t \in I = [0, +\infty), \tag{1}$$

$$\det A(t) \equiv 0,$$

where $A(t)$, $B(t)$ are given $(n \times n)$-matrices, $U(t)$ is a given $(n \times l)$-matrix, $x(t)$ is a desired n-dimensional state function, $u(t)$ is an l-dimensional control function. Such systems are called differential-algebraic equations (DAEs). The insolvability measure with respect to the derivatives for a DAE is an integer called the index of the DAE.

We investigate robust controllability of system (1). Let (1) be R-controllable (i.e., controllable in the reachable set) on an interval T. Robust controllability problem is the search for conditions on matrices ΔB and ΔU such that the system

$$A(t)x'(t) + (B(t) + \Delta B)x(t) + (U(t) + \Delta U)u(t) = 0$$

remains R-controllable on T.

The analysis is carried out under the assumptions that ensure the existence of a global structural form that separates "algebraic" and "differential" subsystems,

$$x_1'(t) + J_1(t)x_1(t) + \mathcal{H}(t)\overline{u}(t) = 0, \tag{2}$$

$$x_2(t) + J_2(t)x_1(t) + \mathcal{G}(t)\overline{u}(t) = 0, \quad t \in I, \tag{3}$$

having the same solutions as original system (1), see [1]. Form (2), (3) is obtained with the help of the operator

$$\mathcal{R} = R_0(t) + R_1(t)\frac{d}{dt} + \ldots + R_r(t)\left(\frac{d}{dt}\right)^r, \tag{4}$$

which has a left inverse on I. Herein colon $(x_1(t), x_2(t)) = Qx(t)$, Q is the corresponding permutation matrix, $\overline{u}(t) = \mathrm{colon}(u(t), u'(t), \ldots, u^{(r)}(t))$; $J_1(t), J_2(t), \mathcal{H}(t), \mathcal{G}(t)$ are determined by the matrices $A(t), B(t), U(t)$, their derivatives up to rth order inclusively, and the coefficients of operator (4).

Necessary and sufficient conditions for robust R-controllability for DAE (1) are obtained.

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The Order of Convergence of Difference Schemes for Fractional Equations of Order $0 < \alpha < 1$

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In this talk, we continue our investigations on convergence of difference schemes for fractional differential equations in Banach spaces. Using an implicit difference scheme and an explicit difference scheme, we deal with full discretization of solutions of fractional differential equations in time variables and get the order of convergence.

Lots of works were devoted to discretization of C_0-semigroups in a traditional way, see [1] and references therein. Recently, Li and Piskarev considered the discrete approximation of integrated semigroups [2, 3, 4], where the order of convergence was obtained using ill-posed problems theory. In this talk, we continue our research [5, 6, 7] on discretization of fractional differential equations in Banach spaces and get the order of convergence $O(\tau_n^\alpha)$, where α is the order of fractional derivative.

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2D Boundary Value Problem for Navier–Stokes–Fourier Equations

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We consider the 2D boundary value problem for the Navier–Stokes–Fourier equations under the assumption that the pressure function satisfies the Mendeleev constitutive law. It is assumed that the compressible fluid occupies a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary. The state of the fluid is characterized by the macroscopic quantities: the density $\varrho(x,t)$, the velocity $\mathbf{u}(x,t)$, and the temperature $\vartheta(x,t)$. We need to find ϱ, $\mathbf{u}$, and ϑ from the equations

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \quad \text{in } \Omega \times (0,T), \tag{1}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \varrho \mathbf{f} + \operatorname{div} \mathbb{S}(\mathbf{u}) \quad \text{in } \Omega \times (0,T), \tag{2}$$

$$\partial_t E + \operatorname{div}((E+p)\mathbf{u}) = \operatorname{div}(\mathbb{S}(\mathbf{u})\mathbf{u}) + \div(\varkappa \nabla \vartheta) + (\varrho \mathbf{f}) \cdot \mathbf{u} \quad \text{in } \Omega \times (0,T). \tag{3}$$

$$\begin{gathered} \mathbf{u}) = 0, \quad \partial_n \vartheta = 0 \text{ on } \partial\Omega \times (0,T), \\ \varrho = \varrho_0, \quad \mathbf{u} = \mathbf{u}_0, \vartheta = \vartheta_0 \ \text{ in } \ \Omega. \end{gathered} \tag{4}$$

Here, $\mathbf{f}$ is a bounded vector field denoting the density of the external mass force, $\varkappa = 1 + \vartheta^n$ is the heat conduction coefficient, the viscous stress tensor $\mathbb{S}$ has the form

$$\mathbb{S}(\mathbf{u}) = \nu_1 \Big(\nabla \mathbf{u} + \nabla \mathbf{u}^\top - \frac{2}{3} \operatorname{div} \mathbf{u} \mathbb{I} \Big) + \nu_2 \operatorname{div} \mathbf{u} \mathbb{I},$$

E is given by

$$E = \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta),$$

where e is the density of the internal energy; $\varrho_0 \geqslant 0$, $\mathbf{u}_0$, $\vartheta_0 \geqslant 0$ are given bounded functions. It is assumed that the pressure function satisfies the Mendeleev law $p = c_1 \varrho + c_2 \varrho \vartheta$. We prove that for a suitable value of the exponent n, problem (1)–(4) has a weak variational solution satisfying the energy inequality.

On Spaces of Bounded q-Variation in Dimension N

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Motivated by the formula

$$\lim_{\varepsilon \to 0^+} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^q}{|x-y|^q} \rho_\varepsilon(x-y)\, dx\, dy = K_{q,N} \|\nabla u\|_{L^q}^q$$

due to Bourgain, Brezis and Mironescu, which characterizes the functions in L^q that belong to $W^{1,q}$ (for $q > 1$) and BV (for $q = 1$), respectively, we study what happens when one replaces the denominator in the expression above by $|x - y|$. It turns out that for $q > 1$ the resulting expression gives rise to a new space that we denote by $BV^q(\Omega)$. We show, among other things, that $BV^q(\Omega)$ contains both the spaces $BV(\Omega) \cap L^\infty(\Omega)$ and $W^{1/q,q}(\Omega)$. We also present applications of this space to the study of singular perturbation problems of Aviles–Giga type.

Spectral Analysis of Even Order Differential Operator

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We consider operators $L_{bc} : D(L_{bc}) \subset \mathrm{L}_2[0,\omega] \to \mathrm{L}_2[0,\omega]$ defined by the differential expression

$$l(y) = (-1)^k y^{(2k)} - qy \quad \text{with} \quad k > 1 \quad \text{and} \quad q \in \mathrm{L}_2[0,\omega]$$

with boundary conditions bc of the following types:

(a) periodic $bc = per : y^{(j)}(0) = y^{(j)}(\omega)$, $j = 0, 1, \ldots, 2k-1$;

(b) semiperiodic $bc = ap : y^{(j)}(0) = -y^{(j)}(\omega)$, $j = 0, 1, \ldots, 2k-1$;

(c) Dirichlet $bc = dir : y(0) = \cdots = y^{(2k-2)}(0) = 0$, $y(\omega) = \cdots = y^{(2k-2)}(\omega) = 0$.

Thus, $D(L_{bc}) = \{y \in W_2^{2k}[0,\omega] : y \text{ satisfies conditions } bc\}$.

In this paper, using ideas from [1], we construct a new method to give a detailed spectral analysis for operators L_{bc}. We obtain the following results. In the next theorem, $\theta = 0$ for $bc = per$ and $\theta = 1$ for $bc = ap$.

Theorem 1. *The operators L_{bc}, $bc \in \{per, ap\}$, have compact resolvents and there exists $m \in \mathbb{Z}_+$ such that the spectrum $\sigma(L_{bc})$ has the form $\sigma(L_{bc}) = \sigma_{(m)} \cup (\cup_{n \geqslant m+1} \sigma_n)$, where $\sigma_{(m)}$ is a finite set with number of points not exceeding $2m+1$ and $\sigma_n = \{\widetilde{\lambda}_n^-\} \cup \{\widetilde{\lambda}_n^+\}$, $n \geqslant m+1$. The eigenvalues $\widetilde{\lambda}_n^\mp$, $n \geqslant m+1$, have the* *following asymptotic representation:*

$$\widetilde{\lambda}_n^{\mp} = \left(\frac{\pi(2n+\theta)}{\omega}\right)^{2k} - q_0 - \frac{2\omega^{2k}}{\pi^{2k}} \sum_{\substack{j=1\\ j\neq n}}^{\infty} \frac{q_{n-j}q_{j-n}}{(2j+\theta)^{2k}-(2n+\theta)^{2k}} \mp$$

$$\mp \left(q_{-2n-\theta} + \frac{\omega^{2k}}{\pi^{2k}} \sum_{\substack{j\in\mathbb{Z}\\ j\neq n,\ j\neq -n-\theta}} \frac{q_{-n-j-\theta}q_{j-n}}{(2j+\theta)^{2k}-(2n+\theta)^{2k}}\right)^{\frac{1}{2}} \left(q_{2n+\theta} + \right.$$

$$\left. + \frac{\omega^{2k}}{\pi^{2k}} \sum_{\substack{j\in\mathbb{Z}\\ j\neq n,\ j\neq -n-\theta}} \frac{q_{n+j+\theta}q_{n-j}}{(2j+\theta)^{2k}-(2n+\theta)^{2k}}\right)^{\frac{1}{2}} + \xi_{bc}(n),$$

where the sequence $\xi_{bc}\colon m+\mathbb{N}\to(0,\infty)$ *satisfies the estimate:* $|\xi_{bc}(n)|\leqslant C_\theta \alpha_n/n^{4k-3}$. *Here,* (α_n) *is a square summable sequence,* q_n, $n\in\mathbb{Z}$, *are the Fourier coefficients of the potential* q, *and* $C_\theta > 0$ *is some constant.*

There is an analogous result on asymptotic behaviour of eigenvalues for the operator L_{dir}. We also prove

Theorem 2. *The operator* $-L_{bc}$, $bc \in \{per, ap, dir\}$, *is the generator of an analytic semigroup.*

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Boundedness and Finite-Time Stability for Multivalued Doubly Nonlinear Evolutionary Systems Generated by Microwave Heating Problems

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We investigate a class of evolutionary variational equations in general Hilbert spaces. The variational equations are considered as general control or feedback systems consisting of a linear part and a nonlinear part. A powerful method for the qualitative investigation of such systems are frequency theorems for equations of evolutionary type. Using some properties of the transfer operator for the linear part of a given control system, the frequency theorem gives sufficient conditions for the existence of Lyapunov functionals for dissipativity, global stability, and instability of nonlinear systems.

In this talk, we extend the frequency domain approach to the investigation of global attractors and finite-time stability for systems generated by variational equations[1, 2].

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Poisson's Theorem for Building First Integrals of Multidimensional Differential Systems

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Consider the system of equations in total differentials (or the Pfaff system)

$$dx_i = \sum_{j=1}^{m} X_{ij}(t,x)\,dt_j\,, \quad i=1,\dots,n, \quad X_{ij}\in C^1(D), \quad D\subset \mathbb{R}^{m+n}, \tag{1}$$

and the corresponding Hamiltonian system of equations in total differentials

$$dx_i = \sum_{j=1}^{m} \partial_{y_i} H_j(t,x,y)\,dt_j\,, \quad dy_i = -\sum_{j=1}^{m} \partial_{x_i} H_j(t,x,y)\,dt_j\,, \quad i=1,\dots,n, \tag{2}$$

with the Hamiltonians

$$H_j\colon (t,x,y) \to \sum_{i=1}^{n} X_{ij}(t,x)\,y_i$$

for all $(t,x,y)\in D\times\mathbb{R}^n, \quad j=1,\dots,m.$

In this work, the analytical relations (the existence of first integrals and partial solutions, fulfillment of conditions of complete solvability) between these differential systems are established. We use these relations to prove the Poisson theorem for building first integrals for system (1) (Theorem 1). Moreover, statements of the existence of additional first integrals for this system are obtained.

Theorem 1. *Suppose twice continuously differentiable functions*

$$F_1\colon (t,x,y)\to F_1(t,x)$$

and

$$F_2\colon (t,x,y)\to F_2(t,x,y) \text{ for all } (t,x,y)\in D'\times Y, Y\subset\mathbb{R}^n,$$

are first integrals of Hamiltonian system (2)*. Then the Poisson bracket*

$$F\colon (t,x)\to \big[F_1(t,x),\, F_2(t,x,y)\big]\big|_{y=\partial_x F_1(t,x)}$$

for all $(t,x)\in D'\subset D$ *is a first integral of system* (1) *of equations in total differentials.*

The proof of this theorem is based on Poisson's theorem for a Hamiltonian system in total differentials [1]. Note also that Theorem 1 improves some results from [2].

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Degree of Locally Condensing Perturbations of Fredholm Maps with Positive Index and Its Application

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In the talk, a construction of the topological degree for a new class of perturbations of C^1 Fredholm maps with positive index, namely, for the class of locally condensing perturbations, will be given.

We suppose that we have a Banach space E and a measure of noncompactness ψ on E, a metric space $\hat{X}$ and an open subset $X \subset \hat{X}$, and an oriented Fredholm structure X_Φ on X with model space $E \times R^q$.

Suppose that $h\colon \overline{X} \to E$ is a continuous map such that $h(x) \neq 0$ for $x \in \partial X$ and the set $S = h^{-1}(0)$ is compact. Suppose also that h can be written in the form $h = f - g$, where f and g are continuous maps satisfying the following conditions:

1. $f\colon \overline{X} \to E$ is a continuous proper map such that the restriction $f\colon X \to E$ is a Fredholm map of class C^1 with $\mathrm{ind} f = q > 0$ admissible with respect to X_Φ;

2. there exists an open neighborhood $\overline{\mathcal{O}}$ of the set S such that the restriction $g\colon \overline{\mathcal{O}} \to E$ is f-condensing with respect to ψ.

Applying the finite-dimensional reduction method, we define the degree $d(h, X_\phi, 0)$ as an element of the GL_c-framed cobordism group of the Fredholm structure X_Φ. The constructed degree does not depend on the representation of the map g in the form $f - g$ with f and g satisfying the above conditions.

The constructed degree possesses usual properties including the homotopy invariance for Fredholm homotopies of class C^1 which are admissible with respect to X_Φ.

In the case of the Kuratowski measure of noncompactness, we give the property of g to be locally f-condensing in terms of the derivative $Df(x)$. This allows us to simplify the scheme of application of the degree.

We consider a system of ordinary differential equation with Hopf's boundary conditions and apply the constructed degree to prove the existence theorem for it.

Properties of Solutions of Integro-Differential Equations Arising in Heat and Mass Transfer Theory

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The main goal of our work is to study the asymptotic behavior of solutions to the Gurtin–Pipkin type integro-differential equations on the base of spectral analysis of their symbols. The Gurtin–Pipkin equation arises in such domains of mechanics and physics as heat transfer theory, theory of viscoelastic media, and kinetic theory of gases. Since the Gurtin–Pipkin type integro-differential equations arise in numerous applications, it is reasonable and natural to study integro-differential equations with unbounded operator coefficients being the operator models for partial integro-differential equations of this type. They have the form

$$\frac{du(t)}{dt} + \int_0^t \mathcal{K}(t-s)A^2u(s)ds = f(t), \quad t \in \mathbb{R}_+,$$

where A is a positive self-adjoint operator acting in a separable Hilbert space H with a compact inverse. Assume that the kernel $\mathcal{K}(t)$ is a scalar function that is determined empirically and admits the representation

$$\mathcal{K}(t) = \int_0^\infty \frac{e^{-t\tau}}{\tau} d\mu(\tau),$$

where $d\mu$ is a positive measure corresponding to an increasing right-continuous distribution function μ. The integral is understood in the Stieltjes sense.

The well-posedness of initial-value problems for integro-differential equations with unbounded operator coefficients in Hilbert spaces is established. Spectral analysis of the operator functions being the symbols of these integro-differential equations, is provided. Strong solutions of these equations are represented as sums of terms corresponding to the real and nonreal parts of the spectrum of these operator functions (see [1]). The resulting representations are new for the considered class of integro-differential equations.

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Normal Forms and Localization of Banach Spaces

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We will discuss various results on conjugation of maps with their normal forms in a neighborhood of a fixed point. In particular, we will show that a differentiable at a point or an α-Hölder linearizations can be obtained in a neighborhood of a hyperbolic fixed point without a non-resonance assumption. While it is known that non-resonant maps can smoothly conjugate with their normal forms of any order on Euclidean spaces, absence of resonances does not guarantee smoothness of conjugation on non-smooth Banach spaces. We will investigate the questions of localization of non-smooth spaces and the degree of smoothness of conjugation on Banach spaces. The presentation will be based on the joint works with G. Belitskii, B. Hasselblatt and M. Guysinsky (see [1, 2, 3, 4, 5]).

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Hilbert–Schmidt Matrix Approach to Fourier Filtering in Functional-Differential Equations: Theory and Applications

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The Hilbert–Schmidt class $\mathcal{C}_2$ in a Hilbert space H is the set of operators $A \in \mathcal{L}(H)$ such that the norm $\|A\|_2 = (\sum_{j,k=1}^{\infty} |\langle Af_k, \phi_j\rangle_H|^2)^{1/2}$ is bounded for any orthonormal bases $\{f_k\}, \{\phi_k\}$ in H and it does not depend on the chosen pair of bases [1]. In particular, $\|A\|_2 = (\sum_{j,k=1}^{\infty} |a_{jk}|^2)^{1/2}$, where $a_{jk} = \langle A\phi_k, \phi_j\rangle_H$. The Banach space $\mathcal{C}_2$ is a useful tool in representing acceptable sets of matrix filters in Fourier filtering problems.

Fourier filtering is a widely used technique in nonlinear optical systems with nonlocal feedback having a direct effect on the Fourier spectrum of a light wave [2]. In the thin annular aperture approximation, the dynamics of phase modulation $u = u(x, t)$ is governed by the 2π-periodic boundary-value problem on the circle ($x \in [0, 2\pi]$, $t \geqslant 0$) for the functional-differential equation (FDE)

$$\partial_t u + u - D\partial_{xx}^2 u = K\,|\Phi_Q(A_{in}\exp\{iu\})|^2$$

with the initial condition $u(x,0) = u_0(x) \in H = L_2(0,2\pi)$, $A_{in} \in C^1[0,2\pi]$, $K > 0$, $D > 0$. Here we introduce the operator

$$\Phi_Q(g) = \sum_{k,l\in\mathbb{Z}} Q_{kl}\langle g, e_l\rangle_H e_k$$

of matrix Fourier filtering with the matrix filter $Q = E + P$, where E is the identity matrix, $P \in \mathcal{C}_2$, $e_k(x) = (2\pi)^{-1/2}\exp\{ikx\}$. Note that in previous papers the authors dealt only with the diagonal matrix filters, the so-called filters-multipliers [3].

In this report, we present the new statement of the optimal matrix Fourier filtering problem and give its mathematical basis: existence and Lipschitz continuous dependence on the filter for the energy class solution of the FDE, solvability of optimal filtering problem for various classes of matrix Fourier filters, and convergence of the gradient projection method for optimization of the target functional. We apply the Andronov-Hopf bifurcation approach to demonstrate wide range of possibilities for the use of matrix Fourier filtering as a novel tool for controlled pattern formation. We obtain general conditions on matrix filters giving rise to rotating or oscillating waves, present descriptive examples of matrix filters that provide the excitation of nontrivial bifurcation solutions, and demonstrate close correspondence between the results of direct numerical simulation and theoretically predicted shapes of the solutions. Some of these results are presented in [4].

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Construction of a Singular Set for the Solution of the Hamilton–Jacobi–Bellman Equation with a Two-dimensional Phase Variable

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We consider the Cauchy boundary value problem for the Hamilton–Jacobi–Bellman equation

$$\frac{\partial\varphi(t,x)}{\partial t} + H(D_x\varphi(t,x)) = 0, \quad \varphi(T,x) = \sigma(x), \tag{1}$$

where $t \in [0,T]$, $x \in \mathbb{R}^2$, and $D_x\varphi(t,x) = \Big(\dfrac{\partial\varphi(t,x)}{\partial x_1}, \dfrac{\partial\varphi(t,x)}{\partial x_2}\Big) = s$.

Put $\Pi_T = \big\{(t,x) : t \in [0,T], x \in \mathbb{R}^2\big\}$.

We investigate problem (1) under the following assumptions:

(A1) the function $H(s)$ is continuously differentiable and concave with respect to the variable s;

(A2) the function $D_sH(s)$ is defined on $\mathbb{R}^2$ and is Lipschitz continuous with respect to the variable s, i.e. there exists a constant $L > 0$ such that

$$||D_sH(s^{'}) - D_sH(s^{''})|| \leqslant L||s^{'} - s^{''}||$$

for any $s^{'}$, $s^{''} \in \mathbb{R}^2$;

(A3) the function $\sigma(x)$ is continuously differentiable.

We consider generalized (minimax or viscosity) solutions $\varphi(\cdot)$ to problem (1) in the class of piecewise smooth functions (see, for instance, [1] p. 42).

Definition 1. The function $\varphi(\cdot)$ is a generalized solution to problem (1) if and only if

$$a + H(s) \geqslant 0 \quad \text{for any} \quad (t,x) \in \Pi_T, \quad (a,s) \in D^+\varphi(t,x),$$

where $D^+\varphi(t,x)$ is the supperdifferential of the function φ (see [1], p. 38).

Definition 2. The singular set Q of a generalized solution $\varphi(\cdot)$ to problem (1) is the set of points $(t,x) \in \Pi_T$ where the function φ is not differentiable.

Necessary conditions for the state $(t,x) \in \Pi_T$ to be a point of the singular set Q are obtained in terms of the input data $H(s), \sigma(x)$ of problem (1).

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Hopf Bifurcation of Rotating Waves in Delayed Parabolic Functional-Differential Equations of Nonlinear Optics in the Presence of Symmetry Groups

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We consider a model of a nonlinear optical system with delayed feedback and narrow annulus aperture. The dynamics of the system is governed by the periodic initial-boundary value problem for a delayed semilinear reaction-diffusion equation on the circle [1],

$$\begin{aligned} &u_t + u = Du_{\theta\theta} + I_{\text{feedback}}, \quad \theta \in [0, 2\pi), \quad t > 0, \\ &u|_{\theta=0} = u|_{\theta=2\pi}, \quad u_\theta|_{\theta=0} = u_\theta|_{\theta=2\pi}, \end{aligned}$$

where $I_{\text{feedback}} = I_{\text{feedback}}(u(t-T))$.

It is known that a careful choice of local and nonlocal coupling mechanisms can lead to excitation of rotating-wave solutions. For instance, diffusion, interference, temporal delay, and rotation of the light field constitute a flexible set of control parameters. In this case, $I_{\text{feedback}} = I_{\text{feedback}}(u(\theta + \Delta, t - T))$.

We construct a normal form of the Hopf bifurcation on a center manifold [2] and, based on the analysis of normal form coefficients, study stability properties of rotating waves. We compare and contrast two qualitatively different cases: $\Delta \neq 0$ and $\Delta = 0$.

In the presence of rotation, the system exhibits $SO(2)$ symmetry and the sign of Δ prescribes the "preferred" direction of the system. Hence, stable clockwise (or counterclockwise, depending on $\mathrm{sgn}\Delta$) rotating waves arise through the supercritical Hopf bifurcation [3].

Meanwhile, if $\Delta = 0$, then the system is $O(2)$-symmetric and both directions are equal. This leads to the following two main possibilities [4]:

1. Bistability of clockwise and counterclockwise rotating waves. This is an example of competitive, winner-takes-all dynamics.

2. Stable standing wave. This is an example of cooperative dynamics.

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Elliptic Functional Differential Equations with Incommensurable Contractions

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The "commensurability" of transformations has been a crucial assumption in the study of solvability and regularity of solutions for elliptic functional differential equations in domains, while equations with incommensurable transformations are much less studied. In the talk, we discuss solvability of the Dirichlet problem for the equation

$$-\sum_{i,j=1}^{n} \left(a_{ij}u_{x_i}(x) + b_{ij}u_{x_i}(x/p) + c_{ij}u_{x_i}(x/q)\right)_{x_j} = f(x) \quad (x \in \Omega). \tag{1}$$

Here Ω is a bounded domain in $\mathbb{R}^n$, containing the origin, $a_{ij}, b_{ij}, c_{ij} \in \mathbb{C}$, $p, q > 1$, and $f \in L_2(\Omega)$. The principal condition that $\ln p/\ln q \notin \mathbb{Q}$ is imposed. Note that equations of form (1) were studied in [1] for the case of multiplicatively commensurable p and q.

Based on the study of the corresponding B^*-algebra of functional operators, we establish an algebraic criterion for the Gårding-type inequality guaranteeing the “nice” properties of the Dirichlet problem. Its analysis shows that, in contrast to the case of multiplicatively commensurable contractions, the property of satisfying the Gårding-type inequality (usually called *the strong ellipticity*) is stable with respect to small perturbations of the contraction parameters p and q, cf. [2]. However, an equation of form (1) that is not strongly elliptic can drastically change its properties when passing, for example, from, $p = 1$ to $p > 1$ arbitrarily close to 1.

As a complementary conclusion, we observe that the spectral properties of functional operators with contractions are unstable with respect to small perturbations of scaling parameters.

A famous prototype for (1), *the pantograph equation* $\dot{y} = ay(\lambda t) + by(t)$, emerges in such diverse areas as astrophysics, engineering, and biology.

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Nonlinear Stability of a Stratified Gas Cloud in the Field of Coriolis Force

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We consider the system of non-isentropic polytropic gas dynamics equations in a uniformly rotating reference frame for unknown functions $\rho \geqslant 0$, $p \geqslant 0$, $U = (U_1, U_2, U_3)$, and S (density, pressure, velocity, and entropy), namely,

$$\rho(\partial_t U + (U, \nabla)U + le_3 \times U + ge_3) = -\nabla p,$$

$$\partial_t \rho + \operatorname{div}(\rho U) = 0, \qquad \partial_t S + (U, \nabla S) = 0.$$

The functions depend on time t and on a point $x \in \mathbb{R}^3$, $e_3 = (0, 0, 1)$ is the “upward” unit vector, $l = \mathrm{const} > 0$ is the Coriolis parameter, and g is the acceleration due to gravity (points in $-e_3$ direction). The state equation is $p = \rho^\gamma e^S$, $\gamma \in (1, 2]$, and the hydrostatic balance $\partial_{x_3} p = -g\rho$ is assumed. The gas cloud corresponds to solutions with finite mass, momentum, and energy.

It is well known that the gas dynamics equations and their modifications possess solutions with spatially-uniform velocity gradients. Under this assumption on the velocity profile, the gas dynamics equations can be reduced to a quadratically nonlinear system of ordinary differential equations in a phase space of large dimension. This system is of independent mathematical interest and its solution is very complicated.

In the incompressible case, the analytical solution of the two-dimensional Euler equations with spatially-uniform velocity gradients are known as the Kirchhoff vortex. This vortex demonstrates a steady rotation without changing its shape. In the compressible case, the solutions with this property describe the expansion of a finite mass of gas in vacuum. Unlike the incompressible case, the compressible gas cloud does not admit a steady rotation without external forces. For the irrotational case, the problem was studied thoroughly by Sedov (1954), Ovsyannikov (1956), Bogoyavlensky (1983), Dyson (1968), Anisimov and Lysikov (1970), Gaffet (2001), etc.

Nevertheless, the nonlinear stability issue is very complicated and it is studied insufficiently despite of many efforts. For example, Anisimov and Inogamov (1974) proved that the 3D gas ellipsoid is unstable and tends to shrink into a plane ellipse.

We completely solve the problem of stability for 2D and quasi-2D gas clouds. Namely, we show that the presence of the Coriolis force implies the existence of a family of gas clouds in the form of a stationary vortex, which is nonlinearly stable in the Lyapunov sense. This family cannot exist in the irrotational case. In particular, our results imply that the rotation of the coordinate frame stabilizes the compressible vortex.

Time-Dependent Processes in Physics, Chemistry, and Life Sciences

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A variety of interdisciplinary results is discussed, which is intended to motivate mathematicians to work jointly on applied topics that are state-of-the-art in natural sciences. Two typical areas of current research are discussed at the International Workshop “Differential Equations and Interdisciplinary Investigations”:

Time-dependent processes occurring on different time scales are reported. These can be as short as on the attosecond time scale, if simple model systems are considered, such as isolated nanoparticles that are exposed to ultra-short laser pulses. This gives access to the dynamics of elastic and inelastic scattering processes of electrons that emitted due to photoionization [1]. Modeling of the experimental findings gives deep insights into the dynamics of the processes involving electron scattering. The emission of electrons also leads to changes in elastic light scattering patters (Mie scattering) occurring in the femtosecond time regime. This requires for nanoscopic matter the use of X-rays, as obtained from free electron laser sources.

Time-dependent processes occurring in biological media are finally discussed. This concerns drug penetration into healthy and diseased human and murine skin. Experimental results are obtained from X-ray microscopy for label-free probing of drugs and drug formulations. The experimental findings are modeled by a diffusion model that is based on the Fokker-Planck equation [2]. It allows us by an inversion approach to derive in combination with the experimental results the crucial parameters influencing drug penetration and to understand in great detail the barriers of skin affecting drug penetration. Key parameters are evidently the diffusion coefficient and the free energy.

Both parameters change significantly as a function of penetration depth so that suitable approaches can be derived for optimizing drug penetration strategies. From the experimental point of view polymeric drug nanocarriers are suitable chemical systems for enhancing and controlling drug penetration and to investigate time-resolved drug penetration profiles that are suitable to enhance the molecular understanding of drug penetration processes.

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On a Generalized Samarskii–Ionkin Type Problem for the Poisson Equation

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The Dirichlet and the Neumann boundary value problems play a key role in the theory of harmonic functions. Another important problem, called periodic boundary value problem, arises when one considers the problem in a segment or in a multidimensional parallelepiped. A new class of the boundary value problems for the Poisson equation in a multidimensional ball $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ was introduced for the first time in [1, 2]. These problems are analogous to the classical periodic boundary value problems.

If we turn to the non-classical problems, then the Samarskii–Ionkin problem is among the most popular ones. This problem arose in 1970s in connection with the study of the processes in plasma.

In this talk, an analog of the Samarskii–Ionkin type boundary value problem for the Poisson equation in the multidimensional ball is considered:

$$-\Delta u = f(x),\ x \in \Omega,$$

$$u(x) + \alpha u(x^*) = \tau(x),\ \frac{\partial u}{\partial r}(x) - \frac{\partial u}{\partial r}(x^*) = \nu(x),\ x \in \partial\Omega_+.$$

Here, as usual, $\partial\Omega_+$ is the part of the sphere $\partial\Omega$ where $x_1 \geqslant 0$. Each point $x = (x_1, x_2, ..., x_n) \in \Omega$ is matched by its "opposite" $x^* = (-x_1, \alpha_2 x_2, ..., \alpha_n x_n) \in \Omega$, where the indices $\alpha_j, j = 2, \ldots, n$, take one of the values ± 1. Clearly, if $x \in \partial\Omega_+$, then $x^* \in \partial\Omega_-$.

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Sobolev Spaces of Functions on a Hilbert Space Endowed with a Shift-Invariant Measure

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Let E be a real separable Hilbert space and $\mathcal{E} = \{e_k\}$. We consider a class $\mathcal{M}$ of measures on the space E being additive nonnegative functions on some ring $\mathcal{R}$ of subsets of the space E, invariant with respect to shifts to arbitrary vectors of the space E (see [1]). According to the theorem by A. Weil, none of these measures is countably additive and σ-finite.

For any measure λ of the above class $\mathcal{M}$, the space $\mathcal{H} = L_2(E, \mathcal{R}, \lambda, \mathbf{C})$ of all square integrable functions is defined as the completion of the space of equivalence classes of step functions. For an arbitrary vector $h \in E$, the group $\mathbf{S}_{th}$, $t \in \mathbb{R}$, of the unitary shift operators $\mathbf{S}_{th}u(x) = u(x + th)$ in the space $\mathcal{H}$ is considered (see [2]).

Necessary and sufficient conditions on a vector $h \in E$, ensuring the strong continuity of the group $\mathbf{S}_{th}$, $t \in \mathbb{R}$, are obtained.

Theorem 1. *For any gaussian measure ν on E with the trace-class correlation operator $\mathbf{D}$, the one-parametric family of operators*

$$\mathbf{U}_{\mathbf{D}}(t) = \int\limits_E \mathbf{S}_{\sqrt{t}h} d\nu(h),\ t \geqslant 0,$$

is a semigroup of self-adjoint contractions. If, in addition, $\mathbf{D}^a$ is a trace-class operator for some $a \in (0, 1/2)$, then the semigroup $\mathbf{U}_{\mathbf{D}}$ is strongly continuous.

Let $\mathcal{E} = \{e_k\}$ be the orthonormal basis of eigenvectors of a nonnegative operator $\mathbf{D}$ of trace class, and $\mathbf{D}e_k = d_k e_k$, $k \in \mathbb{N}$. If $u \in \mathcal{H}$, $j \in \mathbb{N}$, and there is an element $\partial_j u \in \mathcal{H}$ such that $\lim\limits_{s\to 0} \|s^{-1}(\mathbf{S}_{se_j}u - u)\|_{\mathcal{H}} = 0$, then $\partial_j u$ is called the derivative of the element u in the direction e_j.

For l equal 1 or 2, the linear space $W^l_{2,\mathbf{D}}(E)$ is defined by

$$W^l_{2,\mathbf{D}}(E) = \{u \in \mathcal{H} :\ \forall\ j \in \mathbb{N}\ \exists\ \partial^l_j u \in \mathcal{H},\ \sum_{j=1}^{\infty} d_j \|\partial^l_j u\|^2_{\mathcal{H}} < \infty\}.$$

Theorem 2. *Let $\mathbf{D}$ be a positive operator in the space E. If $\mathbf{D}^a$ is a trace-class operator for some $a \in (0, 1/4)$, then the space $W^2_{2,\mathbf{D}}(E)$ is the domain of the generator $\Delta_{\mathbf{D}}$ of the semigroup $\mathbf{U}_{\mathbf{D}}$ and $\Delta_{\mathbf{D}}u = \sum\limits_{j=1}^{\infty} d_j \partial^2_j u \quad \forall\ u \in W^2_{2,\mathbf{D}}(E)$.*

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Asymptotic Formulas for High-Order Ordinary Differential Operators with Distribution Coefficients

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We consider the ordinary differential expressions

$$l(y)=\sum_{k=0}^{m}(-1)^{m-k}(\tau_k(x)y^{(m-k)}(x))^{(m-k)}+$$

$$+i\sum_{k=0}^{m-1}(-1)^{m-k-1}\left[(\sigma_k(x)y^{(m-k-1)})^{(m-k)}+(\sigma_k(x)y^{(m-k)})^{(m-k-1)}\right]$$

of order $n=2m\geqslant 2$ on the finite segment $x\in[0,1]$. The coefficients of this expression are assumed to be such that $\tau_0>0$, $\tau_0\in W^{1,1}[0,1]$, $\tau_k^{(-k)}$, $\sigma_k^{(-k)}\in L^2[0,1]$, $k\geqslant 1$. We consider the equation $l(y)=\lambda\varrho(x)y$ with $\varrho(x)>0$, $\varrho\in W^{1,1}[0,1]$, and a complex spectral parameter $\lambda\to\infty$. As in the classical case, this equation is equivalent to a system of n first order differential equations of the form

$$\mathbf{y}'=\lambda\rho(x)\mathbf{B}\mathbf{y}+\mathbf{A}(x)\mathbf{y}+\mathbf{C}(x,\lambda)\mathbf{y}$$

with a constant matrix $\mathbf{B}$ and summable matrices $\mathbf{A}(x)$ and $\mathbf{C}(x,\lambda)$. We establish asymptotic formulas of exponential type for its fundamental system of solutions. This work was supported by RNF (grant no. 171101215).

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Elliptic Operators Associated with Groups of Quantized Canonical Transformations

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To a representation of a Lie group G by quantized canonical transformations Φ_g, $g\in G$, on a smooth closed manifold M, we associate a class of *G-operators* of the form

$$D=\int_G D_g\Phi_g dg : H^s(M)\longrightarrow H^{s-m}(M), \tag{1}$$

where D_g is a smooth family of (pseudo)differential operators on M. In the work, we give the notion of a symbol in this setting and obtain the Fredholm property for elliptic operators (1), see [1].

Note that operators (1) are interesting from many points of view. For instance, they are of interest in noncommutative geometry since their symbols form essentially noncommutative algebras which are crossed products by G.

We also note that only G-operators associated with shift operators Φ_g (changes of variables) induced by the action of the group on the main manifold were previously considered in the literature. We study a considerably more general situation of quantized canonical transformations. One of the motivations for this generalization stems from recent work [2] by C. Bär and A. Strohmaier on the index of boundary value problems for the Dirac operator in Lorentzian geometry (where the problem is reduced to a Toeplitz analogue of operator (1) on the boundary), and also from work [3] by S. Walters on noncommutative orbifolds.

Finally, our results give finiteness theorems as special cases in many known theories (e.g., in the transversally elliptic theory (Atiyah and Singer), in the theory of operators with shifts (Antonevich and Lebedev), in the case of G-operators associated with actions of compact Lie groups (Savin and Sternin [4])).

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Estimation of the Domain of Attraction for Constrained TS Delay Systems via LMI

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This work aims to estimate the domain of attraction (DA) of the equilibrium point (the origin) for nonlinear systems with time delay. It is assumed that there is a representation of the system by a continuous-time Takagi–Sugeno (TS) system on a subset of the state space containing the origin (see e.g. [1]). The advantage of the method is that stability conditions can be determined by solving some generalized eigenvalue minimization problems (GEVPs) or a system of linear matrix inequalities (LMIs), which can be handled efficiently.

Another important issue in stability analysis is how to estimate the DA. For ODEs such estimates can be obtained based on Lyapunov functions. Specifically, for a Lyapunov function which guarantees the local stability, any sublevel set of this function is an inner estimate of the DA if the set belongs to the region where the function is positive definite and its derivative with respect to the system is negative definite. For delay equations, due to the fact that the space of initial values is infinite dimensional, a proper definition of the DA remains an open problem. Some authors use generalized concepts of the DA to be a subset of a certain functional space. However, it is a difficult task to construct an estimate for such DA, as well as to check in practice whether the initial function is contained in the set. It is more convenient to define the so-called direct DA that is a subset of the finite-dimensional space of instant states (see, e.g., [2]). Then it would be reasonable to use the modification of the direct Lyapunov method with classical auxiliary functions (with the Razumikhin conditions) rather than Krasovskii's functionals, and to employ sublevel sets of Lyapunov functions as inner estimates of the direct DA.

Moreover, the usage of a quadratic function for a TS system leads to stability conditions in terms of LMIs or GEVPs. If the LMIs are feasible, then the resulted quadratic function seems to be a global Lyapunov function for the system. In this case, however, asymptotic stability conditions stand good only within the set where the convex sum property of TS systems holds. Therefore, when dealing with TS systems, what makes the problem more challenging is that additional constraints should be considered because the system can usually be represented in the TS form only on some subset of the state space ("modeling region"). Also, the system may have physical constraints precisely reflected in the states belonging to some modeling region. So the problem arises to obtain the largest possible Lyapunov-based estimate of the DA of a nonlinear system with the asymptotically stable origin and subject to the given constraints.

In this work, GEVPs and LMIs are presented to obtain such estimates subject to some kinds of constraints which are turned into LMIs. Some numerical examples are studied. Notice that both GEVPs and LMIs can be efficiently handled via available software, e.g. MatLab.

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On Homogenization for Locally Periodic Strongly Elliptic Operators

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In homogenization theory, one is interested in studying asymptotic properties of solutions to differential equations with rapidly oscillating coefficients. We will consider such a problem for the matrix strongly elliptic operator $\mathcal{A}^\varepsilon = -\operatorname{div} A(x, x/\varepsilon)\nabla$, where A is Hölder continuous of order $s \in [0, 1]$ in the first variable and is periodic in the second one. We do not require that $A^* = A$, so $\mathcal{A}^\varepsilon$ is not necessarily self-adjoint.

It is well known that the resolvent $(\mathcal{A}^\varepsilon - \mu)^{-1}$ converges, in some sense, as ε tends to 0.

In this talk, we will discuss results regarding convergence in the uniform operator topology on $L_2(\mathbb{R}^d)^n$, i.e., the strongest type of operator convergence. We present two terms in the approximation for $(\mathcal{A}^\varepsilon - \mu)^{-1}$ and the first term in the approximation for $\nabla(\mathcal{A}^\varepsilon - \mu)^{-1}$. Particular attention will be paid to the rates of approximation.

Orthogonal-Projection Method for Solving Stationary Differential Equations of Heat and Mass Transfer in Semi-Infinite Region

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Stationary differential equations of heat and mass transfer describe different physical processes that do not depend on the distribution of particles in time, as a result, for example, of their diffusion or temperature distribution from a stationary source in condensed medium. The Galerkin projection method proved to be effective in solving the differential equation of heat transfer in a semi-infinite region. The original problem is solved in the cylindrical coordinate system, and the solution is sought in the form of partial sums of Fourier series in the dual system of modified Laguerre functions. The corresponding residuals are estimated. In more detail, we look for a solution of the stationary heat equation

$$\operatorname{div}\left[\operatorname{grad}\Delta p\,(M)\right] = -\rho\,(M)$$

subject to the boundary conditions

$$\left.\frac{\partial \Delta p\,(M)}{\partial z}\right|_{z=0} = S\Delta p\,(x, y, 0)\,, \quad \Delta p\,(\infty, \infty, \infty) = 0.$$

Here the function $\Delta p\,(M)$ describes heat distribution, $M\,(x, y, z)$ is an arbitrary point, $x, y \in (-\infty, \infty)$, $z \in [0, \infty)$, and S is a constant. The function $\rho\,(M)$ describes heat sources. We consider the case where the three-dimensional model can be reduced to the two-dimensional one. As soon as the boundary condition at infinity is set, one can use the Galerkin projection method to solve this problem. It has been proved that the residual converges to zero on average.

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Asymptotic Solution of the Linearized Korteweg–de Vries Equation over Variable Background

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Consider the linearized one-dimensional Korteweg–de Vries equation with variable coefficients

$$\psi_t + (C(x,\,t)\psi)_x + h^2\psi_{xxx} = 0 \tag{1}$$

with small parameter h. For this equation we pose the initial value problem with localized data

$$\psi|_{t=0} = V\left(\frac{x-\xi}{\mu}\right), \tag{2}$$

where another small parameter μ describes the localization of the initial function.

The localized function V can be represented with the help of the Maslov canonical operator [1]. This representation allows one to implement the full theory of the Maslov canonical operator [2] for constructing the asymptotic solution of the initial value problem.

In the case of the constant coefficient $C(x,\,t)$, the Green function for (1) is known and it has the form of the Airy function.

In the case of the variable coefficient, one can determine the wave front using the ideas of [3] and construct the asymptotics of the leading wave near this front. This asymptotics can also be represented via the Airy function and is similar to the Green function for the constant coefficient.

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The Reversible KAM Theory: Context 2 Is Catching up with Context 1

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A dynamical system is said to be *reversible* with respect to a smooth involution G of the phase space if this system is invariant under the transformation $(p, t) \mapsto (Gp, -t)$, where p is a point of the phase space and t is the time (i.e., if G casts the system considered into the system with the reverse time direction). Hamiltonian and reversible systems are two main classes of dynamical systems one deals with in the finite dimensional Kolmogorov–Arnol′d–Moser (KAM) theory. The reversible KAM theory (founded in 1965–67 independently by J. Moser and by Yu. N. Bibikov – V. A. Pliss) studies quasi-periodic regimes of motion in non-integrable reversible systems (where the tori filled up with these regimes are invariant under the reversing involution).

Up to now, the KAM theory for reversible systems is nearly as developed as the Hamiltonian KAM theory. Nevertheless, almost all results in the reversible KAM theory pertain to the so-called *reversible context 1* where $\operatorname{codim} \mathcal{T} \leqslant 2 \dim \operatorname{Fix} G$. Here $\operatorname{Fix} G$ is the fixed point manifold of the reversing involution G (all the connected components of $\operatorname{Fix} G$ are assumed to be of the same dimension) and $\operatorname{codim} \mathcal{T}$ is the phase space codimension of the invariant torus $\mathcal{T}$ considered. The opposite *reversible context 2* where $\operatorname{codim} \mathcal{T} > 2 \dim \operatorname{Fix} G$ remained completely unexplored until 2011 [1] although this context is not only interesting by itself but also essential for studies of the destruction of unperturbed invariant tori with resonant frequencies in context 1.

However, very recently some important results in the reversible context 2 have been obtained (for analytic systems), and now this context is swiftly “catching up” with context 1. In particular, one of the most fundamental theorems for the reversible context 1 describes the persistence of lower dimensional invariant tori (in the possible presence of external parameters) under the so-called Broer–Huitema–Takens nondegeneracy condition [2]. The parallel result for the reversible context 2 has been proven in [3] as a rather easy corollary of the persistence of invariant tori with singular normal behavior in context 1 [4]. The main result of [3] straightforwardly implies various significant consequences, in particular, theorems on the so-called partial preservation of frequencies and Floquet exponents of invariant tori in the reversible context 2 and on non-autonomous perturbations depending on time quasi-periodically. The characteristic feature of context 2 is that the generic existence of invariant tori carrying quasi-periodic motion requires the system to depend on many (at least $\operatorname{codim} \mathcal{T} - 2 \dim \operatorname{Fix} G + 1$) external parameters.

In the talk, I plan to explain the drastic differences between the reversible KAM contexts 1 and 2 and to outline the recent achievements in context 2.

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Distances between Classes of Sphere-Valued Maps

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Certain spaces of Sobolev maps taking values in spheres can be decomposed into classes according to the singularities of each map.

Two important examples are $W^{1,1}(\Omega; S1)$ and $W^{1,2}(\Omega; S2)$, where Ω is a smooth bounded domain in $\mathbb{R}^N$.

We shall present several results and open problems regarding two natural notions of distance between different classes.

This is a joint work with Haim Brezis and Petru Mironescu.

On a Solution of the Hamilton–Jacobi Equation with the Noncoercive Hamiltonian

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The following Hamilton–Jacobi equation with state constraints is considered:

$$\partial u/\partial t + H(x, \partial u/\partial x) = 0, \quad t \geqslant 0, \quad x \in [-1; 1], \tag{1}$$

$$H(x,p) = -f(x) + 1 - \frac{1+x}{2}e^{2p} - \frac{1-x}{2}e^{-2p}, \tag{2}$$

$$u(0,x) = u_0(x), \quad x \in [-1; 1]. \tag{3}$$

The problem arises in the Crow–Kimura model of molecular evolution [1].

Hamiltonian (2) is noncoercive, and the known theorems on the existence of a viscosity solution are inapplicable to problem (1)–(3). A concept of continuous generalized solutions to the problem on the bounded closed set $\overline{\Pi}_T = [0; T] \times [-1; 1]$ was suggested in [2] on the base of viscosity and minimax solutions to the auxiliary Dirichlet problems. The instant T is determined from the condition of continuity on the segment $[0, T]$ for characteristics starting from the initial manifold. Such solutions exist but are not unique.

So, the problem of constructing a prescribed structure generalized solution is considered. Sufficient conditions for the existence of such a solution are obtained, and the method of its construction is provided [3]. Conditions are specified under which the solution can be continued to the entire infinite interval $t \leqslant 0$.

Simulation results are given.

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Integrable System with Dissipation on Tangent Bundle of Two-Dimensional Manifold

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We study nonconservative systems for which the usual methods of study, e.g., Hamiltonian systems, are inapplicable. Thus, for such systems, we must "directly" integrate the main equation of dynamics. We generalize previously known cases and obtain new cases of the complete integrability in transcendental functions of the equation of dynamics of a lower- and multi-dimensional rigid bodies in nonconservative force fields.

Of course, the construction of the theory of integration for nonconservative systems (even of low dimension) is a quite difficult task in the general case. In a number of cases, where the systems considered have additional symmetries, we succeed in finding first integrals through finite combinations of elementary functions (see [1]).

We obtain a series of complete integrable nonconservative dynamical systems with nontrivial symmetries. Moreover, in almost all cases, all first integrals are expressed through finite combinations of elementary functions. These first integrals are transcendental functions of their variables, where the transcendence is understood in the sense of complex analysis and means that the analytic continuation of a function to the complex plane has essentially singular points. This fact is caused by the existence of attracting and repelling limit sets in the system (for example, attracting and repelling focuses).

We introduce a class of autonomous dynamical systems with one periodic phase coordinate possessing certain symmetries typical for pendulum-type systems. We show that this class of systems can be naturally embedded in the class of systems with variable dissipation with zero mean. The latter indicates that the dissipation in the system is equal to zero on average for the period with respect to the periodic coordinate. Although either energy pumping or dissipation can occur in various domains of the phase space, they are balanced in a certain sense. We present some examples of pendulum-type systems on lower-dimension manifolds relevant to dynamics of a rigid body in a nonconservative field [1, 2].

Then we study certain general conditions of the integrability in elementary functions for systems on the tangent bundles of two-dimensional manifolds. Therefore, we propose an interesting example of a three-dimensional phase portrait of a pendulum-like system describing the motion of a spherical pendulum in a flowing medium (see [2, 3]).

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On the Asymptotic Limit of the Effectiveness of Reaction-Diffusion Equations in Perforated Media

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The talk focuses on the study of the asymptotic behaviour as $\varepsilon \to 0$ of the solution to the boundary value problem associated with the p-Laplace operator in a domain with an ε-periodically repeated inclusions on the boundary where a nonlinear Robin-type condition is specified. It is assumed that the size of the particles a_ε is of order ε^α, where $1 < \alpha < n/(n-p)$.

Let Ω be a bounded domain in $\mathbb{R}^n$ and G_ε be a set of particles. Define the sets $\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}$, $S_\varepsilon = \partial G_\varepsilon$, and $\partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon$. Consider the problem

$$\begin{cases} -\Delta_p u_\varepsilon = f(x), & x \in \Omega_\varepsilon, \\ \partial_{\nu_p} u_\varepsilon + \beta(\varepsilon)\sigma(u_\varepsilon) = 0, & x \in S_\varepsilon, \\ u_\varepsilon = 1, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\partial_{\nu_p} u \equiv |\nabla u|^{p-2}(\nabla u, \nu)$, ν is the outward unit normal vector to S_ε, and $f \in L^{p'}(\Omega)$. We assume that σ is a continuous nondecreasing function with $\sigma(0) = 0$ satisfying the following growth condition: $|\sigma(u)| \leqslant C(1 + |u|^{p-1})$, $C > 0$. In (1), $\beta(\varepsilon)$ represents the so-called adsorption coefficient. It is possible that $\beta(\varepsilon) \to \infty$ as $\varepsilon \to 0$. So, on the one hand, the inclusions are tiny. On the other hand, there are strong processes on their boundaries, and various relations between the parameters a_ε and $\beta(\varepsilon)$ lead to different asymptotic behaviour of the solution.

This problem appears in chemical engineering in the design of fixed-bed reactors (see, for example, [1]). A quantity of great interest in the applications is the *effectiveness*, which can be expressed as

$$\mathcal{E}_\varepsilon = \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} \sigma(u_\varepsilon) dS \tag{2}$$

in the nonhomogeneous case and as

$$\mathcal{E} = \frac{1}{|\Omega|} \int_\Omega \sigma(u) dx \tag{3}$$

in the homogenized case. It represents the ratio of the actual amount of reactant consumed per unit time in Ω to the amount that would be consumed if the interior concentration were everywhere equal to the ambient concentration. A high effectiveness is desirable in most applications. The objective of this paper is twofold. First, the homogenized problem is constructed and the theorem is proved stating that the solution of the original problem converges to the solution of the homogenized problem as $\varepsilon \to 0$. Second, we prove that $\mathcal{E}_\varepsilon \to \mathcal{E}$ as $\varepsilon \to 0$.

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Existence and Uniqueness of Solutions to a System of Nonlinear Integro-Differential Equations with Two-Point Boundary Conditions

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In this paper, a system of nonlinear integro-differential equations with nonlocal boundary conditions is investigated. We study existence and uniqueness of solutions to nonlinear integro-differential equations of the type

$$\frac{dx}{dt} = f(t, x(t)) + \int\limits_0^t K\left(t, \tau, x\left(\tau\right)\right) d\tau, \quad 0 \leqslant t \leqslant T,$$

with the two-point boundary conditions

$$Ax(0) + Bx(T) = C,$$

where $A, B \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{n\times 1}$ are given matrices with $\det(A + B) \neq 0$, and $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $K : [0,T] \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ are given functions. By $C\left([0,T], \mathbb{R}^n\right)$ we denote the Banach space of all continuous vector functions on $[0,T]$ with the norm $\|x\| = \max\left\{|x\left(t\right)| : t \in [0,T]\right\}$, where $|\cdot|$ is a norm in $\mathbb{R}^n$. The considered boundary-value problem is reduced to an equivalent operator equation with the operator acting from $C\left([0,T], \mathbb{R}^n\right)$ to $C\left([0,T], \mathbb{R}^n\right)$. Using various fixed point theorems, we prove the existence and uniqueness theorems for the boundary value problem.

Similar results for ordinary differential equations were obtained in [1, 2].

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Path-Independent Integrals for Interfacial Cracks in Composites

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In this talk, we present some rigorous results for a novel model of fibre-reinforced composites. We consider a two-dimensional homogeneous anisotropic elastic body with an embedded thin semirigid inclusion in equilibrium under the action of external forces. The semirigid inclusion is an anisotropic thin structure that stretches along one direction and moves like a rigid body possessing both rotational and translatory motion along the perpendicular direction. It is supposed that partial interfacial debonding occurs, and a pre-existing interfacial crack subjected to nonpenetration conditions is considered. The model is reformulated in a weak form such that information on all components of the displacement field is incorporated into one variational inequality suitable for our further analysis. We assume that the crack can propagate along the interface only, and thus the crack path is known a priori. We are interested in an energy criterion on the basis of which one may decide whether or not the crack will propagate for the given external forces. Following the Griffith energy concept, the energy release rate associated with perturbation of the crack along the interface is introduced to describe crack propagation. We prove that this energy release rate is well defined and derive simultaneously an explicit representation for it. Further, we investigate a regularity question for the weak solution, adapting suitably different quotient approximations for weak derivatives. The regularity result allows us to obtain representations of the energy release rates associated with local translation and expansion of the crack in terms of path-independent energy integrals along smooth curves surrounding one or both crack tips. These path-independent integrals are analogues of the Eshelby–Cherepanov–Rice J-integral and the Knowles–Sternberg M-integral from fracture mechanics. In comparison with the classical formulae, new terms appear depending on the displacements of the thin semirigid inclusion. Finally, we deduce some relations between the path-independent integrals obtained and discuss briefly the situation in which appropriate traction-free conditions on the cracks faces are imposed.

The talk is based partially on recent papers [1, 2].

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Singular Solution of the General Euler–Poisson–Darboux Equation

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Let $\mathbb{R}^n_+=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n:\ x_1>0,\ldots,x_n>0\}$ and Ω be an open set in $\mathbb{R}^n$ symmetric with respect to each hyperplane $x_i=0$, $i=1,...,n$. We denote $\Omega_+=\Omega\cap\mathbb{R}^n_+$ and $\overline{\Omega}_+=\Omega\cap\overline{\mathbb{R}}^n_+$, where $\overline{\mathbb{R}}^n_+=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n,\ \ x_1\geqslant 0,\ldots,x_n\geqslant 0\}$. We consider the space $C^m(\Omega_+)$ of all m times continuously differentiable functions in Ω_+, $m\geqslant 1$. Let $C^m(\overline{\Omega}_+)$ be the set of functions from $C^m(\Omega_+)$ such that all their derivatives in x_i for all $i=1,...,n$ are continuous up to the sets $x_i=0$. Functions from $C^m(\overline{\Omega}_+)$ are called even with respect to x_i $(i=1,...,n)$ if $\left.\frac{\partial^{2k+1}f}{\partial x_i^{2k+1}}\right|_{x=0}=0$ for all non-negative integers k (see [1], p. 21). The class $C^m_{ev}(\overline{\Omega}_+)$ consists of all functions from $C^m(\overline{\Omega}_+)$ even with respect to each of the variables x_i, $i=1,...,n$. By $\gamma=(\gamma_1,\ldots,\gamma_n)$ we denote a multi-index with $\gamma_i>0$, $i=1,...,n$, and $|\gamma|=\gamma_1+\ldots+\gamma_n$.

Let $x\in\mathbb{R}^n_+$ and $t\in(0,\infty)$. Following [2], we construct a solution to the problem

$$\left(\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}\right)v(x,t)=\left(\frac{\partial^2}{\partial t^2}+\frac{k}{t}\frac{\partial}{\partial t}\right)v(x,t),\quad 0<k<1, \tag{1}$$

$$v(x,0)=0,\qquad \lim_{t\to 0}t^k\frac{\partial v}{\partial t}=\varphi(x). \tag{2}$$

Let q be the smallest natural number such that $2q-k\geqslant n+|\gamma|-3$ and $\varphi\in C^q_{ev}(\mathbb{R}^n_+)$. If $2q-k>n+|\gamma|-3$, then we have the following formula for the solution of (1)–(2):

$$v(x,t)=\frac{\Gamma\left(\frac{3-k}{2}\right)\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{2-k+2q-n-|\gamma|+1}{2}\right)}{2^{n+q}(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)\Gamma\left(\frac{2-k+2q}{2}\right)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q\times$$

$$\times\left(t^{1-k+2q}\int\limits_{B_1^+(n)}[{}^\gamma T^{ty}\varphi(x)](1-|y|^2)^{\frac{2-k+2q-n-|\gamma|-1}{2}}y^\gamma dy\right).$$

Next, we give the formula

$$v(x,t)=\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q\left(t^{n+|\gamma|-2}M^\gamma_\varphi(x;t)\right)$$

for the solution to singular Cauchy problem (1)–(2) for the general Euler–Poisson–Darboux equation in the case $2q-k=n+|\gamma|-3$. Here $M_\varphi^\gamma(x;t)$ is the weighted spherical mean (see [3]).

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Qualitative Properties of Large and Very Singular Solutions to Quasilinear Parabolic Equations

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We develop the energy method for the study of formation of dynamic singularities of solutions of initial-boundary value problems in $(0,T)\times\Omega$, $\Omega\subset\mathbb{R}^n$, $n\geqslant 1$, $T<\infty$, for quasilinear parabolic equations of diffusion–strong nonlinear absorption type with degenerating absorption potential. The already existing method of study of initial singularities (see [1, 2] and references therein) gives a sharp condition on the character of degeneration of the potential, guaranteeing the existence or nonexistence of very singular and large solutions with singularities localized at the initial hyperplane $t=0$.

A new version of the energy method allows us to obtain sharp upper local estimates of solutions in a neighborhood of the final hyperplane $t=T$, where the absorption potential degenerates along some manifold $\Gamma\subset\overline{\Omega}$, $\Gamma\cap\partial\Omega\neq\varnothing$. For example, for large solutions taking infinite Dirichlet values on $(0,T]\times\partial\Omega$, we obtain a sharp condition on the degeneration rate of the potential at $t=T$, guaranteeing nonpropagation of singularity toward interior of Ω along the manifold of the potential degeneration and obtain sharp upper estimates of the final profile of the solution as $t\to T$ (see [3] and [4] for some previous results).

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Differential Operators with Distribution Coefficients. Regularization Method and Asymptotics of Fundamental Solutions

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We know at the present time how to treat the Sturm–Liouville or 1d-Schrodinger operator with a distribution potential from the Sobolev space $W^{\theta,2}_{loc}(a,b), (a,b) \subset \mathbb{R}$ with $\theta \geqslant -1$. The objective is to define high order differential operators with singular coefficients.

The classical theory of symmetric ordinary differential operators allows one to work with differential expressions of the form

$$\ell_{2n}[y] := \sum_{k=0}^{n}(-1)^{n-k}(p_k\, y^{(n-k)})^{(n-k)}+ \\ + i\sum_{k=0}^{n-1}(-1)^{n-k-1}\{(q_k\, y^{(n-k-1)})^{(n-k)} + (q_k\, y^{(n-k)})^{(n-k-1)}\}, \quad (1)$$

provided that the coefficients p_k, q_k and $(p_0)^{-1}$ are locally integrable functions. This theory was developed in the works of D. Shin, N. Glazman, M. G. Krein, A. Zettle, N. Everitt et al. Surprisingly, the theory is not developed for non-symmetric differential expressions with summable coefficients. We shall extend the frames of the classical theory to define operators associated with non-symmetric differential expressions

$$\tau(y) = \sum_{k,s=0}^{n} \left(r_{ks}y^{(n-k)}\right)^{(n-s)} \quad (2)$$

whose coefficients r_{ks} are distributions of finite order singularity (depending on the indices k, s).

We shall discuss several approaches to this problem. Some of them work for partial differential operators as well. The most important approach is based on the so-called regularization procedure. This procedure provides several advantages. In particular, one can write out important asymptotic formulae for the solutions of the equation $\tau(y) - \rho^n y = 0$ and use them for the construction and the estimate of the resolvents of the corresponding operators.

The talk is based on the joint papers with K. A. Mirzoev and A. M. Savchuk.

Integrated Mathematical Model of the Heart, Heart Valves and Large Vessels

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1D models of the blood flow have often been used during past 15 years. They allow reasonable computational cost for studying regional, systemic and closed circulation in the various vascular pathologies. Multiscale approaches including mixes of 0D, 1D, and 3D spatial models were widely applied for increasing correctness of pulse wave propagation and patient-specific anatomical detalisation [1, 2]. The heart ejection profile is generally set as a time-defined function or introduced as windkessel or controlled inflow profile models. The most detailed models of the heart functioning deal with action-potential propagation and 3D flow simulations coupled with mechanical contraction. These models are very complex and computationally expensive. They are rarely used for modelling closed circulation. In this work, an integrated model of the 0D heart chambers with valves and 1D vessels is presented. It allows computationally reasonable conservative coupling with 1D hemodynamic models.

The heart chambers are modelled as four elastic spheres joined by short rigid tunnels and connected to the aorta, vena cava, pulmonary artery and vein at the atrial inlets and ventricles outlets. The equation set includes chamber volume dynamics, chamber — chamber and vessel — chamber flow conservation, and Poiseuille's pressure drop condition. Boundary condition at the heart inlets and outlets includes mass conservation and numerical discretisation of compatibility conditions along outgoing characteristics of 1D haemodynamics in the connecting vessels. Simultaneous solution of all these equations is required for balancing flow between the heart chambers and connecting vessels. An iterative procedure is used at every time step for this coupling taking the cross section and velocity value from the connecting vessels as an initial approximation. The following iteration produces new pressure drop values that can be used to update cross section, flow and other model variables.

The model is applied to study heart ejection profile during heart-valves pathologies (not fully closed, not fully open).

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To the Bohl–Perron Theorem for Hybrid Linear Functional Differential Systems with Aftereffect

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The stability theory for differential equations with aftereffect has been systematically developed since the middle of the twentieth century. Among the methods emerged in this context, one should highlight the Lyapunov–Krasovskii functionals, the Razumihkin functions, integral and differential inequalities, monotone operators generating functions with a parameter, and the Azbelev W-method. In the last ten years, the study of hybrid functional differential systems with aftereffect (HFDSA) has begun. The source was the papers by E. Fridman, J.-J. Loiseau, M. Cardelli, X. Dusser, K. Gu, V.I. Kharitonov, J. Chen, C. Bonnet, J. Partington, R. Rabah, G.M. Sklyar, A.V. Rezounenko, S.I. Niculescu, P. Fu, V.M. Marchenko and J.-J. Loiseau.

The study of stability of solutions to HFDSA is far from complete. In [1], V.M. Marchenko and J.-J. Loiseau investigated the problem of stability of solutions to linear stationary hybrid differential-difference systems. Necessary and sufficient conditions for the exponential stability of these systems were obtained.

The general theory of functional differential equations presented in [2] made it possible to give a clear and concise description of the basic properties of solutions, including their stability properties. At the same time, wide classes of HFDSA systems that are relevant to applications, namely, hybrid linear functional differential equations with aftereffect, are not formally covered by the constructed theory and in many respects remain beyond the sight of those who use functional differential and difference systems with aftereffect for simulation of real processes.

In articles [3], [4], and [5], hybrid functional differential analogues of the main statements of the theory of functional differential equations for stability problems are proposed. Here we propose the Bohl–Perron theorem for linear HFDSA.

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Applications Of Buschman–Erdélyi Transmutations to Connection Formulas For Differential Equations with Singular Coefficients

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Transmutation theory methods now form an important part of modern mathematics, cf. [1]–[8]. They have many applications to theoretical and applied problems. Let us just itemize some problems in the modern transmutation theory.

1. Theory of Buschman–Erdelyi transmutations. This class of operators hastialher problems including connection formulas for differential equations with singular coefficients.

2. Theory of operator convolutions and commuting operators. Transmutations are closely connected with commutants. And while commutants in different spaces of analytic functions are completely described by the operator convolution theory of I. Dimovski, commutants in standard spaces like C^k are much more difficult to characterize, this was done only recently.

3. Sonine–Dimovski and Poisson–Dimovski transmutations for hyper–Bessel functions and equations.

4. Sonine and Poisson type transmutations for difference–differential operators of Dunkle type.

5. Application of transmutations to the generalized analytic function theory.

6. Methods of fractional integrodifferentiation and integral transforms with special function kernels. In this context, let us mention a composition method for constructing many classes of transmutations.

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Kato Square Root Problem for Elliptic Functional Differential Operators

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In 1961, T. Kato formulated the following problem: "*Is it true that the domain of the square root of a regular accretive operator is always equal to the domain of the square root of the adjoint operator*?" J.-L. Lions obtained sufficient conditions for the fulfillment of the Kato conjecture for abstract regular accretive operators. He also proved that strongly elliptic differential operators with smooth coefficients and the Dirichlet conditions in a bounded domain with smooth boundary satisfy the Kato conjecture. The proof was based on the theorem on smoothness of generalized solutions to elliptic problems, which allows one to represent the domain of a strongly elliptic differential operator in an explicit form, and on the interpolation theory. In 1972, A. McIntosh constructed a counterexample of an abstract regular accretive operator that does not satisfy the Kato conjecture. Later, mathematicians tried to find new classes of operators satisfying the Kato conjecture. For strongly elliptic differential operators with measurable bounded coefficients, a corresponding result was obtained by P. Auscher, S. Hofman, A. McIntosh, and P. Tchamitchian in 2001. The main difficulties were related to the lack of smoothness of generalized solutions. Therefore, it was impossible to represent the domain of an operator explicitly.

In 2002, W. Arendt drew my attention to the fact that strongly elliptic differential-difference operators with the Dirichlet boundary conditions also satisfy the Kato conjecture. The foundations of the theory of these operators were developed in [1]. Due to nonlocal nature of these operators, smoothness of generalized solutions to corresponding equations can be violated inside a domain. In this lecture, we shall give a review of results related to the Kato conjecture for strongly elliptic functional-differential operators [2] and formulate new results concerning the Kato conjecture for elliptic differential-difference equations with degeneration. Elliptic differential-difference equations with degeneration have some astonishing properties. For example, generalized solutions to corresponding equations do not belong even to the Sobolev space of the first order.

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Boundary Value Problem for First-Order Hyperbolic System in the Plane

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Consider the hyperbolic system

$$\frac{\partial u}{\partial y} - A\frac{\partial u}{\partial x} = 0,$$

where the matrix $A \in \mathbb{R}^{n\times n}$ has real eigenvalues $\nu_1 < \nu_2 < \ldots < \nu_n$, in a domain D bounded by a contour Γ. In the talk, we discuss a new setting of the boundary value problem for this system. In the case $n = 3$, this problem was studied in [1].

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On Solvability of Functional Differential Equations with p-Laplacian and Nonsymmetric Difference Operators

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Let $Q \subset \mathbb{R}^n$ be either a bounded domain with boundary $\partial Q \in C^\infty$ or $Q = (0,d)\times G$, where $G \subset \mathbb{R}^{n-1}$ is a bounded domain (with boundary $\partial G \in C^\infty$ if $n \geqslant 3$). If $n = 1$, we put $Q = (0,d)$. We consider the problem

$$-\sum_{1\leqslant i\leqslant n} \partial_i \left(|\partial_i Ru(x)|^{p-2}\partial_i Ru(x)\right) = f(x) \qquad (x \in Q), \tag{1}$$

with the boundary condition

$$u(x) = 0 \qquad (x \notin Q), \tag{2}$$

where $f \in W_q^{-1}(Q)$, $1/q + 1/p = 1$, and $2 < p < \infty$. Here R is a linear difference operator given by the formula

$$Ru(x) = \sum_{h\in\mathcal{M}} a_h u(x+h), \tag{3}$$

where $a_h \in \mathbb{R}$ and $\mathcal{M} \subset \mathbb{Z}^n$ is a finite set of vectors with integer coordinates (the case of commensurable shifts can be considered similarly).

Put $R_Q = P_Q R I_Q : L_p(Q) \to L_p(Q)$, where $I_Q : L_p(Q) \to L_p(\mathbb{R}^n)$ is the operator of extension of functions from $L_p(Q)$ by zero to $\mathbb{R}^n \setminus Q$ and $P_Q : L_p(\mathbb{R}^n) \to L_p(Q)$ is the operator of restriction of functions from $L_p(\mathbb{R}^n)$ to Q. We note that the operator R_Q is nonlocal.

Definition. A function $u \in \mathring{W}_p^1(Q)$ is called a generalized solution to problem (1)–(2) if

$$\sum_{1\leqslant i\leqslant n} \int_Q |\partial_i R_Q u(x)|^{p-2} \partial_i R_Q u(x) \partial_i \xi \, dx = \int_Q f\xi \, dx \qquad \forall \xi \in \mathring{W}_p^1(Q).$$

We assume that R_Q is nondegenerate and has a bounded inverse $\hat{R}_Q = R_Q^{-1}$. Denote by $J : L_p(Q) \to L_q(Q)$ the duality mapping of $L_p(Q)$ $(J(u) = |u|^{p-2}u)$.

Definition. The linear operator $\hat{R}_Q$ is called strongly accretive if there exists $c > 0$ such that

$$\langle J(u), \hat{R}_Q u \rangle \geqslant c\|u\|^p_{L_p(Q)}.$$

Theorem 1. *If R_Q is nondegenerate and R_Q^{-1} is strongly accretive, then problem* (1)–(2) *has at least one generalized solution.*

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Characterization of Associate Function Spaces

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This research was carried out at the RUDN University, Moscow, Russia, and financially supported by the Russian Science Foundation (Project 16-41-02004).

We analyze the problem of characterization of function spaces associated with given function spaces. The situation is rather different for ideal and non ideal function spaces. We provide several examples of such a characterization including the first order weighted Sobolev space on the real line. As an important corollary, we obtain the duality principle, which allows us to reduce, for instance, the weighted inequalities to more convenient ones.

Long-Term Stability of Dynamical Systems with Respect to Stochastic Perturbations

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The effect of small stochastic perturbations on a system of non-autonomous differential equations with locally stable fixed point is considered. The perturbed system is described by the Ito stochastic differential equations such that the noise does not vanish at the equilibrium. It is known that in this case almost all trajectories escape from any bounded domain and the stability with respect to such perturbations on an infinite time interval holds for a relatively narrow class of globally stable dynamical systems. We describe the classes of perturbations such that the stochastic stability of the locally stable equilibrium holds for polynomially and exponentially long time intervals with respect to a small perturbation parameter [1].

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Homogenization of Higher-Order Elliptic Equations with Periodic Coefficients

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In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider the selfadjoint operator $A_\varepsilon = b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$, $\varepsilon > 0$. Here $g(\mathbf{x})$ is a periodic bounded and positive definite matrix-valued function, $b(\mathbf{D})$ is a matrix differential operator of order p. It is assumed that the symbol $b(\boldsymbol{\xi})$ has maximal rank. Then A_ε is strongly elliptic. We study the behavior of the resolvent $(A_\varepsilon - \zeta I)^{-1}$, where $\zeta = |\zeta| e^{i\varphi} \in \mathbb{C} \setminus \mathbb{R}_+$, for small ε. We prove that

$$\|(A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C_1(\varphi)\varepsilon|\zeta|^{-1+1/2p}, \tag{1}$$

$$\begin{aligned} &\|(A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1} - \varepsilon^p K(\varepsilon;\zeta)\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \\ &\leqslant C_2(\varphi)\big(\varepsilon|\zeta|^{-1/2+1/2p} + \varepsilon^p\big)\big(1 + |\zeta|^{-1/2}\big) \end{aligned} \tag{2}$$

for $0 < \varepsilon \leqslant 1$. Here $A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ is the effective operator and $K(\varepsilon;\zeta)$ is a corrector (note that $\|K(\varepsilon;\zeta)\|_{L_2\to H^p} = O(\varepsilon^{-p})$). Estimates (1) and (2) are order-sharp for small ε.

Now, let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class C^{2p}. By $A_{D,\varepsilon}$ (respectively, $A_{N,\varepsilon}$) we denote the operator in $L_2(\mathcal{O};\mathbb{C}^n)$ given by $b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$ with the Dirichlet (respectively, Neumann) boundary conditions. We prove the following error estimates for $0 < \varepsilon \leqslant \varepsilon_0$ (ε_0 is sufficiently small) and $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geqslant 1$:

$$\|(A_{\flat,\varepsilon} - \zeta I)^{-1} - (A^0_\flat - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leqslant C_3(\varphi)\varepsilon|\zeta|^{-1+1/2p}, \tag{3}$$

$$\begin{aligned} &\|(A_{\flat,\varepsilon} - \zeta I)^{-1} - (A^0_\flat - \zeta I)^{-1} - \varepsilon^p K_\flat(\varepsilon;\zeta)\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})} \\ &\leqslant C_4(\varphi)\big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big). \end{aligned} \tag{4}$$

Here $\flat = D, N$, and $A^0_\flat$ is the effective operator; $K_\flat(\varepsilon;\zeta)$ is the corresponding corrector. Estimate (3) is order-sharp for small ε. The order of estimate (4) is worse than the order of (2) because of the boundary layer effect.

The constants $C_j(\varphi)$, $j = 1, \dots, 4$, are uniformly bounded in any sector $\varphi_0 \leqslant \varphi \leqslant 2\pi - \varphi_0$ with an arbitrarily small $\varphi_0 > 0$. The analogs of estimates (3), (4) for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| < 1$, are also obtained (with different dependence on ζ).

Estimates (1) and (2) were obtained in [1] by the operator-theoretic method based on the scaling transformation, the Floquet–Bloch theory, and the analytic perturbation theory. Estimates (3) and (4) were proved in [2], [3] on the base of the results obtained for the problem in $\mathbb{R}^d$, introduction of the boundary layer correction term, and a careful analysis of this term.

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Calculus of Variations in the Large for Magnetic Geodesics

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The periodic problem for magnetic geodesics is quite different form its classical analog for Riemannian geodesics. In particular, the Palais–Smale condition fails for the action functional in general. If the magnetic field is not exact, then the action functional is also multi-valued. In many interesting cases however, it is possible to establish the existence of closed magnetic geodesics. We give a survey of the problem and some recent results.

On Solvability of Functional-Differential Equations with Contractions in Weighted Spaces

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We study elliptic functional-differential equations with contractions and expansions of the arguments in the principal part. The weighted spaces introduced by V. A. Kondratiev for elliptic problems in domains with angular or conical points, see [1], are also useful in the study of boundary value problems for functional-differential equations whose solutions may have power singularities at some points on the boundary boundary or inside the domain.

We consider the equation

$$A_R u \equiv -\sum_{i,j=1}^{2} (R_{ij} u_{x_i})_{x_j} = f(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2, \tag{1}$$

where $R_{ij}v(x) = a_{ij0}v(x_1,x_2) + a_{ij1}v(q^{-1}x_1,px_2) + a_{ij,-1}v(qx_1,p^{-1}x_2),\ p,q>1$, $a_{ij0},a_{ij,\pm1}\in\mathbb{C}\ (i,j=1,2)$, and $f\in H^s_\beta\left(\mathbb{R}^2\right)$.

We study solvability of (1) in the scale $H^s_\beta\left(\mathbb{R}^2\right)$ of weighted spaces, where $H^s_\beta\left(\mathbb{R}^2\right)$ is introduced for a nonnegative integer s as the completion of the set $C_0^\infty(\mathbb{R}^2\setminus\{0\})$ with respect to the norm

$$\|u\|_{H^s_\beta(\mathbb{R}^2)} = \left(\sum_{|\alpha|\leqslant s}\int_{\mathbb{R}^2}|x|^{2(\beta-s+|\alpha|)}|D^\alpha u(x)|^2dx\right)^{1/2}. \tag{2}$$

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Asymptotic Solution of a Linear Wave in a Regular Lattice Created by a Localized Perturbation and New Types of Lagrangian Singularities

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We consider the wave propagation problem in a 2D lattice with variable velocity under assumption that the wave is created by a spatially localized initial perturbation. We assume that the lattice is regular and the lattice points are $x_{n_1,n_2}=(n_1h,n_2h)$, where the lattice increment $h>0$ is a small parameter. For the displacement $u_n(t)$ of the particle sitting at the point x_n, we have the following equation:

$$\ddot{u}_{n_1,n_2} = C^2_{n_1,n_2}\left(u_{n_1+1,n_2}+u_{n_1-1,n_2}+u_{n_1,n_2+1}+u_{n_1,n_2-1}-4u_{n_1,n_2}\right). \tag{1}$$

An approach to the construction of the solving operator (or parametrix) for such type of equations was developed by V. P. Maslov, V. G. Danilov, and P. N. Zhevandrov (see [1, 2]). Their construction is based, in particular, on the representation of Eq. (1) in the form of a pseudodifferential equation and on the use of the so-called nonstandard characteristics.

Here we consider the Cauchy problem with localized initial data

$$u_{n_1,n_2}|_{t=0} = V(n_1h/\mu, n_2h/\mu),\ \dot{u}_{n_1,n_2}|_{t=0},$$

where V is a function decaying fast at infinity, and $1>>\mu>h$.

In recent paper [3], S.Yu.Dobrokhotov and V. E. Nazaikinskii proposed an approach based on modified Maslov's canonical operator defined on the so-called punctured Lagrangian manifolds. It was shown, in particular, that the asymptotic solution in a neighborhood of the leading edge front is connected with the special type of Lagrangian singularities (special caustics) in the phase space. Also the effective formulas for the asymptotic solutions near the regular points from the leading edge front were presented in [3]. In this work, we discuss the structure of Lagrangian singularities and the behavior of the asymptotic solutions in the neighborhood of singular points of the leading edge front. We also discuss the possibility of computer implementation of the asymptotic solutions and transition to solutions of the wave equation, taking the relationship between the parameters h and μ into account.

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Integrability and Chaos: Coexistence on Open Sets

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I plan to discuss several examples of finite-dimensional Hamiltonian systems where chaotic and regular behavior coexist, both occupying open sets in the phase space.

Krylov Test for the Lagrange–Sturm–Liouville Operators

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For a potential of bounded variation q, the zeros of eigenfunctions U_n, $n \in \mathbb{N}$, of the Sturm–Liouville problem

$$\begin{cases} U'' + [\lambda - q]U = 0, \\ U'(0) - hU(0) = 0, \\ U'(\pi) + HU(\pi) = 0, \end{cases} \tag{1}$$

lying in $[0,\pi]$ and numbered in ascending order, will be denoted by

$$0 \leqslant x_{0,n} < x_{1,n} < \ldots < x_{n,n} \leqslant \pi \quad (x_{-1,n} < 0, x_{n+1,n} > \pi). \tag{2}$$

To every continuous function f, assign the Lagrange–Sturm–Liouville operators [1]-[4]

$$L_n^{SL}(f,x) = \sum_{k=0}^{n} f(x_{k,n}) \frac{U_n(x)}{U_n'(x_{k,n})(x-x_{k,n})} = \sum_{k=0}^{n} f(x_{k,n}) l_{k,n}^{SL}(x) \tag{3}$$

interpolating f at the nodes $\{x_{k,n}\}_{k=0}^{n}$.

Theorem 1. *Let $0 \leqslant a < b \leqslant \pi$, $0 < \varepsilon < \frac{b-a}{2}$, and $V_f[a,b]$ be the total variation of a continuous function f on $[0,\pi]$. If $V_f[a,b] < \infty$, then we have*

$$\lim_{n\to\infty} \|f - L_n^{SL}(f,\cdot)\|_{C[a+\varepsilon,b-\varepsilon]} = 0. \tag{4}$$

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Solvability of Vlasov–Poisson Equations with Angle Errors in Magnetic Field in the Half-Space

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Let $T > 0$ and $\mathbb{R}^3_+$ be the half-space, i.e., $\mathbb{R}^3_+ = (0,\infty) \times \mathbb{R}^2$. Then we consider the following Vlasov–Poisson system (P):

$$\begin{cases} -\Delta\varphi(x,t) = 4\pi e \displaystyle\int_{\mathbb{R}^3} \sum_{\beta=\pm1} \beta f^\beta(x,p,t)\,dp, \quad (x,t) \in \mathbb{R}^3_+ \times (0,T), \\ \dfrac{\partial f^\beta}{\partial t} + \dfrac{1}{m_\beta}\left(p, \nabla_x f^\beta\right) + \beta e \left(-\nabla_x\varphi + \dfrac{1}{m_\beta c}\left[p, B(x)\right], \nabla_p f^\beta\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,p,t) \in \mathbb{R}^3_+ \times \mathbb{R}^3 \times (0,T), \quad \beta = \pm1, \\ f^\beta(x,p,0) = f_0^\beta(x,p), \qquad\qquad (x,p) \in \mathbb{R}^3_+ \times \mathbb{R}^3, \quad \beta = \pm1, \\ \varphi(x,t)|_{x_1=0} = 0, \qquad\qquad x' = (x_2,x_3) \in \mathbb{R}^2, \quad t \in (0,T). \end{cases} \tag{P}$$

This system represents a model of charged ions and electrons in plasma with external magnetic field. Here f^β and φ stand for the density of charged ions for $\beta = +1$, electrons for $\beta = -1$, and the potential of electric field, respectively, and are unknown functions; $B = B(x)$ stands for the external magnetic force.

In this work, we consider existence of solutions to (P). We are also interested in how large the existence time T is until which the plasma does not reach the wall $\partial\mathbb{R}^3_+ = \{x_1 = 0\}$. In 2013, [1] establishes solvability of (P) where the magnetic force is parallel to the wall, namely, $B(x) = (0, 0, h)$ for a constant $h > 0$. Moreover, in 2017, [2] expands the time T by effectively using the property $B = (0, 0, h)$.

This talk provides an existence result for (P) where the magnetic force has small angle error and is not parallel to the wall. That is to say, we assume

$$B(x) = (\varepsilon(x_1), 0, h(x_1)) \quad \text{with a constant} \quad B_0 \equiv \left(\varepsilon(x_1)^2 + h(x_1)^2\right)^{1/2},$$

where $\varepsilon, h > 0$ are functions such that ε decreases and converges to 0 as $x_1 \to 0$.

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The Cauchy Problem for the Equation of Longitudinal Oscillations of an Infinite Rod

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For the differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^4 u}{\partial x^2 \partial t^2} - \frac{\partial^2 u}{\partial x^2} + \alpha^2 \frac{\partial^4 u}{\partial x^4} = \beta \frac{\partial^2}{\partial x^2} u^2$$

describing the longitudinal oscillations of a nonlinearly elastic rod, where α, β are given parameters, we study solvability to the Cauchy problem in the space of continuous functions on the axis by reduction to the abstract Cauchy problem in a Banach space.

An explicit form of the solution of the corresponding linear equation is found. The time interval for the existence of the classical solution of the Cauchy problem for a nonlinear equation is established and an estimate for the norm of this local solution is obtained. Conditions for the existence of a global solution are considered.

Battle–Lemarié Type Wavelet Systems with Localization Property

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Fix $n \in \mathbb{N}$ and put $B_0 = \chi_{[0,1)}$. The nth order B-spline is defined recursively by $B_n(x) := (B_{n-1} * B_0)(x)$. It is known [1] that B_n generates spline spaces constituting multiresolution analysis of $L^2(\mathbb{R})$. Wavelet subspaces related to B_n are also generated by some basic functions (wavelets) in the same manner as the spline spaces are generated by B_n. The Battle–Lemarié scaling function ϕ_n and the related wavelet(s) ψ_n are polynomial splines obtained from the B-splines by the process of orthogonalization.

There is a number of papers devoted to Battle–Lemarié wavelet systems $\{\phi_n, \psi_n\}$. Most of them deal with implicit or approximate expressions for these systems. In our work, we give explicit formulae for a class of Battle–Lemarié scaling functions and wavelets of all positive integer orders. The second result is devoted to “localization properties” Φ_n and Ψ_n of the class. Namely, we establish compactly supported sums of translations of Battle–Lemarié wavelets ψ_n. Analogous results are given for the scaling function ϕ_n, and the both are applied to the equivalent norm characteristics in Besov type spaces.

In the case of general $n \in \mathbb{N}$, explicit formulae for our class of $\{\phi_n, \psi_n\}$ have quite complicated forms. Here we consider the case where $n = 1$ and $\hat{B}_1(\omega) = \mathrm{e}^{-i\omega/2}\,[\cos(\omega/4)]^2 \hat{B}_1(\omega/2)$:

$$\phi_1(x) = \phi_1^I(x) \simeq \sum_{l\geqslant 0}(-r_1)^l B_1(x+l),\ \phi_1(x) = \phi_1^{II}(x) \simeq \sum_{l\geqslant 0}(-r_1)^l B_1(x-l-1),$$

$$\psi_1(x) = \psi_1^I(x) \simeq \sum_{k\geqslant 0}(-r_1)^k \sum_{m\geqslant 0}(-r_1)^m \big[\frac{1}{r_1}B_1(2x-2k+m+1)$$
$$-(1+\frac{2}{r_1})B_1(2x-2k+m) + (2+\frac{1}{r_1})B_1(2x-2k+m-1) - B_1(2x-2k+m-2)\big],$$

$$\psi_1(x) = \psi_1^{II}(x) \simeq \sum_{k\geqslant 0}(-r_1)^k \sum_{m\geqslant 0}(-r_1)^m \big[r_1 B_1(2x+2k-m+2)$$
$$-(1+2r_1)B_1(2x+2k-m+1) + (2+r_1)B_1(2x+2k-m) - B_1(2x+2k-m-1)\big],$$

where $r_1 = 2-\sqrt{3}$. In particular, for ϕ_1^I we have $\Phi_1(\cdot) := \phi_1^I(\cdot) + r_1\phi_1^I(\cdot+1) \simeq B_1(\cdot)$. The localization property Ψ_1 for ψ_1 operates with its four translations and shifts [2]:

$$\Psi_1(\cdot) := \psi_1^I(\cdot - \frac{1}{2}) + r_1[\psi_1^I(\cdot) + \psi_1^I(\cdot - \frac{3}{2})] + r_1^2\psi_1^I(\cdot - 1) - \psi_1^{II}(\cdot - 1)$$
$$- r_1[\psi_1^{II}(\cdot - \frac{3}{2}) + \psi_1^{II}(\cdot)] - r_1^2\psi_1^I(\cdot - \frac{1}{2}) \simeq B_1(2\cdot) - 2B_1(2\cdot -1) + B_1(2\cdot -2).$$

Our construction uses some hints from [3]. The research was supported by the Russian Science Foundation (project RSF-DST: 16-41-02004).

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Singular Perturbed System of Differential Equations of Infinite Order and Non-Homogeneous Markov Chain

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Let us consider a large-scale network consisting of infinite number of servers with the Poisson input flow of requests. Each request arriving in the system randomly selects two servers and is instantly sent to the one with the shorter queue. The service time is distributed exponentially with mean $\bar{t}_s = 1$. Let $u_k(t)$ be the share of servers with the queue lengths not less than k. The considered system of the servers is described by an ergodic non-homogeneous Markov chain. There is a stationary probability distribution for the states of the system, the evolution of the values $u_k(t)$ becomes deterministic, and the Markov chain asymptotically converges to a dynamic system whose evolution is described by infinite number of differential-difference equations. We can investigate an infinite system of differential-difference equations with small parameter of the form

$$\begin{cases} \dot{u}_k(t) = u_{k+1}(t) - u_k(t) + \lambda\left((u_{k-1}(t))^2 - (u_k(t))^2\right), \\ k = 0, 1, \ldots, n,\ t \geqslant 0, \\ \mu\dot{u}_k(t) = u_{k+1}(t) - u_k(t) + \lambda\left((u_{k-1}(t))^2 - (u_k(t))^2\right), \\ k = n+1, \ldots,\ t \geqslant 0, \\ u_k(0) = g_k \geqslant 0,\ k = 0, 1, 2, \ldots, \end{cases} \tag{1}$$

where λ is the parameter of the input request intensity for each server, $g = \{g_k\}_{k=1}^{\infty}$ is a numerical sequence ($1 = g_0 \geqslant g_1 \geqslant g_2, \ldots$), μ is a small parameter that introduces a singular perturbation to system (1), the latter allows us to describe the processes of rapid change of the system.

Theorem 1. *If the initial conditions in problem* (1) *satisfy the inequalities* $1 = g_0 \geqslant g_1 \geqslant g_2, \ldots$*, then the solutions to problem* (1) *satisfy the inequalities* $1 = u_0(t) \geqslant u_1(t) \geqslant u_2(t), \ldots$ *for all* $t > 0$.

The proof is based on the continuous dependence of the solutions of system (1) on the initial conditions [1], so the inequalities $1 = g_0 \geqslant g_1 \geqslant g_2, \ldots$ imply the inequalities $1 = u_0(t) \geqslant u_1(t) \geqslant u_2(t), \ldots$ for all $t > 0$ for the solutions.

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Manifolds with Non-Smooth Boundaries, Elliptic Operators, and Symbols

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We consider a certain integro-differential operator A on an m-dimensional compact manifold M with boundary. This operator is defined by a function $A(x,\xi),(x,\xi)\in\mathbb{R}^{2m}$. There are some smooth compact submanifolds M_k of dimension $0\leqslant k\leqslant m-1$ on the boundary ∂M of the manifold M, being boundary singularities. These singularities are described by a local representative of the operator A at a point $x_0\in M$ on the map $U\ni x_0$ in the following way:

$$(A_{x_0}u)(x)=\int\limits_{D_{x_0}}\int\limits_{\mathbb{R}^m}e^{i\xi\cdot(x-y)}A(\varphi(x_0),\xi)u(y)d\xi dy,\quad x\in D_{x_0},$$

where $\varphi:U\to D_{x_0}$ is a diffeomorphism, and the canonical domain D_{x_0} has a distinct form depending on the position of the point x_0 on the manifold M. We consider the following canonical domains D_{x_0}: $\mathbb{R}^m$, $\mathbb{R}^m_+=\{x\in\mathbb{R}^m: x=(x',x_m), x_m>0\}$, and $W^k=\mathbb{R}^k\times C^{m-k}$, where C^{m-k} is a sharp convex cone in $\mathbb{R}^{m-k}$.

Such an operator A will be considered in the Sobolev–Slobodetskii spaces $H^s(M)$, and local variants of such spaces will be the spaces $H^s(D_{x_0})$.

Definition 1. The symbol of an operator A is the operator-function $A(x):M\to\{A_x\}_{x\in M}$ defined by local representatives of the operator A.

Under additional assumptions of smoothness on the function $A(x,\xi)$, one has the following result.

Theorem 1. *The operator A possesses the Fredholm property iff its symbol is composed by invertible operators.*

The simplest variant of this theorem was proved in [1].

Definition 2. The operator A is called elliptic if its symbol is composed by invertible operators.

Remark 1. If the ellipticity property does not hold on submanifolds M_k, then one needs to modify local representatives of the operator A, adding special boundary or co-boundary operators.

Using a partition of unity on the manifold M, the elliptic symbol $A(x)$, and envelopings of I. B. Simonenko, one can construct n operators A_j according to the number of singular submanifolds including the whole boundary ∂M and the manifold M.

Theorem 2. *The index of the Fredholm operator A is given by the formula*

$$Ind\ A = \sum_{j=1}^{n} Ind\ A_j.$$

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Differential Equations and Personalized Computation of Fractional Flow Reserve

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Atherosclerosis of coronary arteries is the most common disease. Its main medical treatment is the invasive endovascular intervention (stenting or shunting). The contemporary gold standard of indication for the intervention at a particular location of the coronary vasculature is the fractional flow reserve (FFR) [1]. FFR is defined as the ratio of the mean pressure distal to a stenosis and the mean pressure in the aorta measured under vasodilating administration. The present methods of FFR measurement are invasive (a non-reusable pressure gauge is delivered to coronary arteries) and expensive.

We have developed a non-invasive method of personalized evaluation of the FFR on the basis of a computationally efficient numerical model of blood flow in the network of coronary arteries. The network is reconstructed from CT and angiographic data [2]. The model describes 1D flow of incompressible fluid in the network of elastic tubes [3]. It includes a system of hyperbolic differential equations (mass and momentum conservation) coupled with a given pressure–cross section functional dependence in each tube (vessel). The equations on tubes satisfy certain algebraic restrictions at tubes junctions.

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The Hamilton–Jacobi Method in the Non-Hamiltonian Situation and Boltzmann Extremals

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The hydrodynamic substitution, which is well known in the theory of the Vlasov equation [1, 2, 3], has recently been applied to the Liouville equation and Hamiltonian mechanics [4, 5, 6, 7, 8]. In [4, 5, 6], Kozlov outlined the simplest derivation of the Hamilton–Jacobi (HJ) equation, and the hydrodynamic substitution simply related this derivation to the Liouville equation [7, 8]. The hydrodynamic substitution also solves the interesting geometric problem of how a surface of any dimension subject to an arbitrary system of nonlinearordinary differential equations moves in Euler coordinates (in Lagrangian coordinates, the answer is obvious). This has created prerequisites for generalizing the HJ method to the non-Hamiltonian situation. The H-theorem is proved for generalized equations of chemical kinetics, and important physical examples of such generalizations are considered: a discrete model of the quantum kinetic equations (the Uehling–Uhlenbeck equations) and a quantum Markov process (a quantum random walk). The time means are shown to coincide with the Boltzmann extremes for these equations and for the Liouville equation [9]. This give possibility to prove existence of analogues of action-angles variables in non-Hamiltonian situation.

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Quasilinear Degenerate Elliptic Equations with Measures Data

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We study the solvability of the following quasilinear problem

$$\begin{aligned} -\mathrm{div}\, A(x,\nabla u) + g(x,u,\nabla u) &= \mu \qquad \text{in } \Omega \\ u &= 0 \qquad \text{in } \partial\Omega, \end{aligned} \tag{1}$$

where A and g are Carathéodory functions satisfying some natural growth assumptions and μ a bounded Radon measure. The solutions are considered in the renormalized sense. If this problem is solvable we say that μ is *g-admissible*.

Our aim is

- to find conditions on g for all bounded measure be g-admissible.
- when $g(x,r,\xi) \sim g_q(r) := \pm|r|^{q-1}r$ with $p-1<q$, to find conditions on μ to be g_q-admissible.
- To obtain condition of removability of compact sets for the operator on the left-hand side of (1).

Our main tools are Wolff and Riesz potentials, Bessel–Lorentz capacities and maximal operators.

The Existence of a Unique Weak Solution to the Problem for the Aggregation Equation with the $p(\cdot)$-Laplacian

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Let Ω be a bounded domain in $\mathbb{R}^n$, containing the origin, and $D^T = \Omega \times (0, T)$ be the corresponding cylindrical domain. Consider the equation

$$\beta(x,u)_t = \operatorname{div}(a(x,u,\nabla u) - \beta(x,u)G(u)) + f(x,u) \tag{1}$$

with the initial and the boundary conditions $u(x,0) = u_0(x) \geqslant 0$, $(a(x,u,\nabla u) - uG(u)) \cdot \nu = 0$ on $\partial\Omega \times (0,T)$, where ν is the outer normal vector. Here $\beta(x,r), f(x,r)$ are Caratheodory functions, β increases with r, $\beta(x,0) = 0$, $G(u) = (G_1(u), ..., G_n(u))$ is an integral operator defined by the formula $G_i(u) = \int\limits_{\Omega} g_i(x,y)b(u(y))dy$, where $g_i(x,y) \in C^1(\overline{\Omega} \times \overline{\Omega})$. The function $b(s) \geqslant 0$, $b(0) = 0$, satisfies the Lipschitz condition: $|b(s_1) - b(s_2)| \leqslant L_k|s_1 - s_2|$, $s_1, s_2 \in [0,k]$, $\forall k > 0$. It is assumed that $\sum_{i=1}^n \nu_i g_i(x,y) \leqslant 0$, $x \in \partial\Omega$, $y \in \Omega$.

Let for some M_0, M_T the following conditions hold:

$$s\beta(x,r) \leqslant r\beta(x,s), \text{ as } 0 < M_0 \leqslant r < s \leqslant M_T, \ x \in \Omega;$$

$$|\beta(x,r)| \leqslant C\beta; \ f(x,r) = \beta(x,r)q(x,r), \ |q(x,r)| \leqslant q_0, \text{ as } |r| \leqslant M_T;$$

$$|G(v)| \leqslant C_G, \quad |\operatorname{div} G(v)| \leqslant N_G, \quad \forall v: \ |v(x)| \leqslant M_T.$$

. Suppose also that

$$|a_j(x,r,y)|^{\overline{p}_j(x)} \leqslant C(F(x) + \sum_{i=1}^n |y_i|^{p_i(x)}), \ \frac{1}{\overline{p}_j} + \frac{1}{p_j} = 1, F(x) \in L_1(\Omega)$$

for all $r \in [0, M_T]$, $y \in \mathbb{R}^n$, $x \in \Omega$, and

$$(a(x,r,y) - a(x,r,z)) \cdot (y - z) \geqslant 0, \ y \neq z; \quad a(x,r,y) \cdot y \geqslant \delta_0 \sum |y_i|^{p_i(x)}, \ \ \forall\, y \in \mathbb{R}^n.$$

Theorem 1. *Let $T = \mu^{-1} \ln|\frac{M_T}{M_0} - 1|$, $\mu = 1 + q_0 + N_G$, and the above assumptions hold. Then there exists a unique weak solution of problem* (1) *in D^T.*

A review of the results concerning the aggregation equation is given in [1].

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Weighted Hardy-Type Spaces of Harmonic and Analytic Functions

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We introduce Hardy-type spaces $e_p(\mathcal{B};\rho)$ of harmonic functions $u(z)$ and analogous spaces $E_p(\mathcal{B};\rho)$ of analytic functions $f(z)$ with weight ρ in planar domains $\mathcal{B}$ with rectifiable boundary. The norm in this spaces is defined as the limit of weighted L_p-norms over a family of contours approximating the boundary $\mathcal{C}$ of the domain $\mathcal{B}$. For this spaces we obtain the analogs of a number of basic results of the theory of classical Hardy spaces. In particular, we show that for these spaces the Riesz equalities for limit values $u(z')$ and $f(z')$ hold, where $z' \in \mathcal{C}$. We prove also that complex polynomials are dense in $E_p(\mathcal{B};\rho)$ and harmonic polynomials are dense in $e_p(\mathcal{B};\rho)$. These results generalize the well-known assertions by V. I. Smirnov [1] and by M. V. Keldysh and M. A. Lavrent'ev [2]. We establish an isometric isomorphism between the spaces $e_p(\mathcal{B};\rho)$ and $L_p(\mathcal{C};\rho)$ for $p > 1$, and find bounds for solutions of the Dirichlet problem and all of their derivatives. With the help of the theorem on density of the harmonic polynomials in $e_p(\mathcal{B};\rho)$, we justify the convergence of the projection method for the Dirichlet problem in $\mathcal{B}$. Similar spaces of analytic functions were studied in [2]–[4], see also survey papers [5] and [6]. Note that constructions involving the values of a function on surfaces "parallel" to the boundary have been used in the study of elliptic boundary value problems in smooth domains, see, for example, [7]–[10].

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Spectral Analysis of Integro-Differential Equations in Hilbert Space and Its Applications

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We study integro-differential equations with unbounded operator coefficients in a Hilbert space. The principal part of these equations is an abstract hyperbolic operator perturbed by summands of Volterra integral operators. Operator models of such type have many applications in the linear viscoelasticity theory, homogenezation theory, heat conduction theory in media with memory, etc. In particular, these integro-differential equations can be realized as the system of integro-partial differential equations:

$$\rho\ddot{u}(x,t) - Lu(x,t) + \int_0^t \Gamma_1(t-s)L_1u(x,s)ds + \int_0^t \Gamma_2(t-s)L_2u(x,s)ds = f(x,t),$$

where $u = \vec{u}(x,t) \in \mathbb{R}^3$ is the displacement vector of viscoelastic anisotropic media, $t > 0$, $x \in \Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, u satisfies the Dirichlet conditions in Ω, $L_1 = \mu \cdot (\Delta u + 1/3 \cdot \mathrm{grad}\ \mathrm{div} u)$, $L_2 = \lambda \cdot \mathrm{grad}\ \mathrm{div} u$, $Lu = (L_1 + L_2)u$ is the Lame operator of the elasticity theory, Γ_1, Γ_2 are memory relaxation functions that are series of decreasing exponents with positive coefficients.

Spectral analysis of operator-valued functions being the symbols of the considered integro-differential equations is performed. The structure and localization of spectra for these operator-valued functions are analyzed (see [1], [2]).

These results are natural generalization of our results obtained in [3].

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Reaction-Diffusion Waves in Physiological Applications

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Reaction-diffusion waves describe numerous processes in physiology including tumor growth, blood coagulation or infection spreading. In this presentation we will discuss some approaches to study the existence of waves using the Leray–Schauder method, topological degree for elliptic operators in unbounded domains and a priori estimates of solutions in properly chosen weighted spaces [1].

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Small Movements and Normal Oscillations in System of Two Pendulums with Cavities Partially Filled with Heavy Viscous Fluids

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We study a new linear problem of mathematical physics generated by a hydromechanical problem of small motion of two rigid bodies with cavities partially filled with heavy viscous fluids. We suppose that the two bodies are connected with each other and with a fixed point of the first body by spherical hinges with friction. In the equilibrium state, the centers of gravities in bodies and points of suspension lie on the same straight line parallel to the vector of gravitational acceleration (the plane surfaces of noncapillary fluids are orthogonal to this vector). After linearization, we come to the initial boundary value problem for mechanical equations of angular momenta and hydrodynamical linear Navier-Stokes equations. Notice that such problem for fluids without free surfaces were studied earlier by E. I. Batyr and N. D. Kopachevsky in [1].

Using approaches of S. G. Krein and N. D. Kopachevsky from [2] and [3], one can transform the original problem to the equation in a Hilbert space $\mathcal{H}$

$$\mathcal{C}\frac{d^2X}{dt^2} + \mathcal{A}\frac{dX}{dt} + \mathcal{B}X = F, \tag{1}$$

for the unknown $X = (\vec{w};\vec{\delta})^t \in \mathcal{H} := H_1 \oplus H_2$. Here $\vec{w} = (\vec{w}_1;\vec{w}_2) \in H_1 := \vec{J}_{0,S_1}(\Omega_1) \oplus \vec{J}_{0,S_2}(\Omega_2)$ is the vector of displacement in each fluid ($\vec{w}_k|_{S_k} = \vec{0}$), $\vec{\delta} = (\vec{\delta}_1;\vec{\delta}_2) \in H_2 := \mathbb{C}^3 \oplus \mathbb{C}^3$ is the vector of angular displacement in the bodies. The bounded operator-matrix $\mathcal{C}$ of kinetic energy is positive and boundedly invertible in $\mathcal{H}$, the unbounded operator-matrix $\mathcal{A}$ of energy dissipation is positive definite in $\mathcal{H}$, and the unbounded operator-matrix $\mathcal{B}$ of potential energy is a self-adjoint operator in $\mathcal{H}$ bounded from below.

We prove that the operator $\mathcal{B}\mathcal{A}^{1/2}$ is bounded. Under this property and a natural requirement to the function F, we prove a theorem on the existence of a unique strong solution. The corresponding spectral problem for $\lambda \neq 0$ can be reduced to the eigenvalue problem for S. G. Krein's operator pencil

$$\mathcal{L}(\lambda)Y := (I - \lambda\mathcal{A}_0 - \lambda^{-1}\mathcal{B}_0)Y = 0, \quad Y = \mathcal{A}^{1/2}X, \tag{2}$$

where $\mathcal{A}_0 := \mathcal{A}^{-1/2}\mathcal{C}\mathcal{A}^{-1/2}$ and $\mathcal{B}_0 := \mathcal{A}^{-1/2}\mathcal{B}\mathcal{A}^{-1/2}$. It consists of two branches of positive eigenvalues with limit points 0 and ∞, and no more than a finite number of mutually adjoint nonreal, zero, and negative eigenvalues. If the operator $\mathcal{B}$ is nonnegative, then the problem has no negative eigenvalues and the equilibrium state of the system is stable in the linear approximation. If the inequality $4\|\mathcal{A}_0\| \cdot \|\mathcal{B}_0\| < 1$ holds, then the problem has no nonreal eigenvalues, and the numbers of negative eigenvalues of the operator $\mathcal{B}$ and the pencil (2) are equal to each other.

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Differential Equations Related to Birth-Death Processes

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Birth-death processes (BDPs) are studied in probability theory and applications (information theory, genetics, chemistry, economics and finance). Consider a continuous-time Markov BDP $\{\xi(t),\ t \geqslant 0\}$ with state space $\mathbb{Z}^+ = \{0, 1, \ldots\}$. Given that $\xi(t) = x \in \mathbb{Z}^+$, the state x remains unchanged during an exponential random time τ_x of rate $h(x) := \lambda(x) + \mu(x)\mathbf{1}(x \geqslant 1)$; afterwards the process jumps instantly to $x \pm 1$ with probabilities $\frac{\lambda(x)}{h(x)}$ and $\frac{\mu(x)}{h(x)}$, respectively. We consider functions $\lambda(x) > 0$ and $\mu(x) > 0$ of the polynomial form: $\lambda(x) = \sum\limits_{i=0}^{l} P_i x^i$, $\mu(x) = \sum\limits_{j=0}^{m} Q_j x^j$ assuming for simplicity that $P_i, Q_j \geqslant 0$ with $P_l, Q_m > 0$ and $\max[l, m] \geqslant 1$. We are interested in rare events with low probability of the form e^{-R}, where R represents a large deviation functional related to a naturally emerging Hamilton (or Lagrange) action.

More precisely, introduce a "large" parameter $T > 0$ and impose the conditions $\xi(0) = a$, $\xi(T) = AT$, where $a, A > 0$ are constants. In our case, the functional R is a solution of the following variational problem:

$$R(T) = \int_0^T \sup_\theta \left[\theta \dot{x}(t) - (e^\theta - 1)\lambda(x(t)) - (e^{-\theta} - 1)\mu(x(t))\right] dt,$$

$$x(0) = a,\ x(T) = AT.$$

In order to find the extremum of R, we look at the boundary-value problem giving an optimal trajectory:

$$\dot{x} = \lambda e^{\theta} - \mu e^{-\theta},\ \dot{\theta} = -\frac{d\lambda}{dx}(e^{\theta} - 1) - \frac{d\mu}{dx}(e^{-\theta} - 1),\quad x(0) = a,\ x(T) = AT.$$

It turns out that, as $T \to \infty$, the form of the solution $x(t)$, $\theta(t)$ and the asymptotic behavior of $R(T)$ depend upon the relation between $\lambda(x)$ and $\mu(x)$. This is a new phenomenon not observed so far in the existing probabilistic literature and of interest for the variational problem.

Let us state some examples (with $l, m \leqslant 1$) where the solution can be found explicitly.

Example 1. $\lambda(x) = P_1 x$, $\mu = Q_1 x$ (with $l = m = 1$, $P_1 = Q_1$). Then $x(t) = C_1(C_2 + P_1 t)(C_2 + 1 + P_1 t)$, $\lim_{T\to\infty} R(T) = C_3$, constants C_1, C_2, C_3 depend on a and A. Here the optimizing function $x(t)$ is quadratic.

Example 2. $\lambda(x) = P_0$, $\mu(x) = Q_1 x$ (with $l = 0$, $m = 1$). Here, for $a \sim 0$, we have $x(t) \sim ATe^{Q_1(t-T)}$, $R \sim AT \ln T$, with T increasing, and the optimal curve $x(t)$ has no limit: it tends to a δ-shaped path.

Smoothness Issues in Differential Equations with State-Dependent Delay, and Processes for Volterra Integro-Differential Equations

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(1) We answer a question which has been around since the first constructions of local invariant manifolds for differential equations with state-dependent delay: In general stable manifolds are not better than once continuously differentiable, also in cases where center and unstable manifolds are C^k-smooth, $k \geqslant 2$.

(2) Then we discuss state spaces for equations with state-dependent delay. In general these equations define continuously differentiable solution operators only on the *solution manifold* in $C^1(I, \mathbb{R}^n)$, $I = [-r, 0]$ or $I = (-\infty, 0]$. But there also are classes of such equations which admit nice solution operators on open subsets of the familiar Banach space $C([-r, 0], \mathbb{R}^n)$, and on the Fréchet space $C((-\infty, 0], \mathbb{R}^n)$ in cases of unbounded delay.

(3) On the last space we obtain processes for Volterra integro-differential equations, which are nonautonomous with unbounded time-dependent delay.

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Well-posedness of a Problem with the Initial Conditions for Parabolic Fractional-Differential Equations with Time Shifts

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In the present paper, we study the statement and well-posedness of a problem with the initial conditions for a model parabolic fractional-difference equation of the form

$$\partial_t^\alpha u(t,x) = \mathcal{L}u(t,x) + f(t,x), \quad t>0, \quad x\in\mathbb{R}^d \tag{1}$$

where

$$\mathcal{L}u(t,x) = -\mathbf{A}u(t,x)+\sum_{k=1}^{N}\{[a_k(u^{(1/2)}(t+h_k,x))]+[c_k(\mathbf{A}u^{(1/2)}(t+h_k,x))]\}-\gamma_0 u(t,x),$$

$$(t,x)\in(0,+\infty)\times\mathbb{R}^d. \tag{2}$$

Here the coefficients $a_k, c_k, h_k, k=\overline{1,N}$, are real numbers, $h<0$, f is a given numerical function on the domain $(0,+\infty)\times\mathbb{R}^d$, and u is the unknown numerical function whose domain is the set $(h,+\infty)\times\mathbb{R}^d$. In relation (2), $\mathbf{A}$ is a self-adjoint positive operator in the space $H=L_2(\mathbb{R}^d)$ with domain $D(\mathbf{A})=W_2^2(\mathbb{R}^d)\subset H$.

The problem is to find a function $u:(h,+\infty)\times\mathbb{R}_d\to\mathbb{R}$ that satisfies Eq. (1) in the domain $(0,+\infty)\times\mathbb{R}_d$ and the initial condition

$$u\,|_{(h,0]\times\mathbb{R}^d}=\varphi. \tag{3}$$

The Riemann-Liouville integral is defined as

$$(I_{a_+}^\alpha f)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x f(t)(x-t)^{(\alpha-1)}, \quad x>a,$$

and

$$(I_{b_-}^\alpha f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b f(t)(x-t)^{(\alpha-1)}, \quad x<b.$$

Given the homogeneous linear partial differential equation

$$\partial_t^{(1/2)}u(t,x)=c\partial_x^{(1/2)}u(t,x), \tag{4}$$

$$u(x,0) = u_0(x), \tag{5}$$

we have

$$u(x,t) = u_0[(x^{1/2} + \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{1}{2}+1)} ct^{1/2})^2] = u_0[(x^{1/2} + \frac{\Gamma(3/2)}{\Gamma(3/2)} ct^{1/2})^2].$$

We denote $\phi(\alpha,\beta,z)$ the simplest Wright function defined (for $\alpha,\beta,z \in C$) by the series

$$\phi(\alpha,\beta,z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} \frac{x^k}{k!}.$$

Theorem. *Let $G(t-\tau)$ be Green's function, then problem* (1)–(3) *is solvable and its solution $u(t,x)$ is given by*

$$u(t,x) = \sum_{k=1}^{N} \int_{-\infty}^{\infty} G_k^{\frac{1}{2}}(x-\tau,t) F_k(\tau) d\tau$$

with

$$G_k^{\alpha}(t,x) = \frac{1}{2}(t+h_k)\phi(-\frac{1}{2}, \frac{1}{2} - k + 1; -\frac{t+h_k}{2} t^{-\frac{1}{2}}).$$

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An ALE Approach to Mechano-Chemical Processes in Fluid-Structure Interactions

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Mathematical modeling and simulation of fluid-structure interaction problems have been in the focus of research for a long time. One should also take chemical reactions into account, which is rather new but for many applications highly important area. In this talk, we formulate a model for fluid-structure interactions including chemical reactions. The penetration of chemical substances from the fluid phase into the solid one and their reactions lead to changes of volume and mechanical properties of the solid structure. Numerical algorithms are developed and used to simulate the dynamics of such a mechano-chemical fluid-structure interaction problem. The arbitrary Lagrangian Eulerian approach (ALE) is chosen to solve the systems numerically. Temporal discretization of the fully coupled monolithic model is accomplished by backward Euler scheme, and spatial discretization by stabilized finite elements. As an example, a plaque formation model is derived as a specific model system for this scenario. Numerical studies confirm the convergence of the fully coupled scheme with respect to the temporal and spatial discretizations, and effective methods are described to maintain mesh qualities under large deformations. The investigation has shown that the chosen ALE approach delivers very reliable numerical results, which in case of the plaque formation model are in good qualitative agreement with clinical observations.

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Finding Moment Functions of Solutions of Differential Equations with Random Coefficients

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Let X be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle, T = [t_0, t_1] \subset \mathbb{R}, A : X \to X$ bounded linear operator.

We consider the problem

$$\frac{dx}{dt} = \varepsilon(t,\omega)Ax + f(t,\omega), x(t_0) = x_0. \tag{1}$$

Here $x \in X, \varepsilon$ is a scalar stochastic process, f is a vector stochastic process, x_0 is a stochastic vector independent of ε and f.

The task is to find the mathematical expectation $Mx(t)$ for a solution of problem (1). Let the characteristic functional

$$\psi(u(\cdot), v(\cdot)) = M(\exp(i\int_T [\varepsilon(s,\omega)u(s) + \langle f(s,\omega), v(s)\rangle]ds))$$

be known for the processes ε and f. Here $i = \sqrt{-1}$, $u(\cdot)$ is integrable on T, and $v : T \to X$ is a vector function integrable on T. Put

$$y(t, u(\cdot), v(\cdot)) = M(x(t)\exp(i\int_T [\varepsilon(s,\omega)u(s) + \langle f(s,\omega), v(s)\rangle]ds)).$$

Then $y(t,0,0)=M(x(t))$.

We obtain the Cauchy problem for $y(t,u(\cdot),v(\cdot))$,

$$\frac{\partial y(t,u(\cdot),v(\cdot))}{\partial t}=-iA\frac{\delta_p y(t,u(\cdot),v(\cdot))}{\delta u(t)}-i\frac{\delta_p\psi(u(\cdot),v(\cdot))}{\delta v(t)},$$

$$y(t_0,u(\cdot),v(\cdot))=M(x_0)\psi(u(\cdot),v(\cdot)).$$

Here $\frac{\delta_p y(t,u(\cdot),v(\cdot))}{\delta u(t)}$ is the partial variational derivative [1]. Let the function

$$\chi(s,t,\tau)=\begin{cases}\operatorname{sign}(\tau-s), & \tau\in(\min(s,t),\max(s,t)),\\ 0, & \tau\notin(\min(s,t),\max(s,t)).\end{cases}$$

The solution of this problem has the form

$$y(t,u(\cdot),v(\cdot))=\psi(u(\cdot)E-i\chi(t_0,t,\cdot)A,v(\cdot))M(x_0)\\ -i\int_{t_0}^{t}\frac{\delta\psi(u(\cdot)E-i\chi(s,t,\cdot)A,v(\cdot))}{\delta v(s)}ds.$$

Here E is the operator the identity transformation in X. As $u=0$ and $v=0$, the mathematical expectation $M(x(t))$ can easily be found,

$$M(x(t))=\psi(-i\chi(t_0,t,\cdot)A,0)M(x_0)-i\int_{t_0}^{t}\frac{\delta\psi(-i\chi(s,t,\cdot)A,0)}{\delta v(s)}ds.$$

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Transformation of Systems of Partial Differential Equations to Systems of Quasilinear and Linear Differential Equations and Their Reduction

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The report deals with first-order PDE systems. Our purpose is to investigate some general properties of first-order PDE systems including the possibility of their simplification (reduction), using the earlier obtained results [1]. In author's previous works, the possibility of reduction of the dimension of overdetermined systems of differential equations was shown. The task was to obtain, as much as possible, and as better as possible, the overrides of broad classes of PDEs. Earlier, overdeterminations of the equations of hydrodynamics and ODEs were obtained, and an assumption was made about the possibility of overriding of any PDE system. In the first half of the report, we give a new way to override any PDE of the first order and, in doing so, try to take into account that the general solutions of this extended system of equations only contain solutions to a pre-defined Cauchy problem. This is advantageous in the sense that then the method of diminishing the dimensionality theoretically can reduce the dimension of these equations up to a complete solution of the Cauchy problem, which can be represented explicitly. In addition, we also establishes a link between Euler's hydrodynamic equations and an arbitrary first-order PDE system.

In the second half of this work, we consider the reduction of PDE systems to just one quasi-linear evolution equation of the second order for one unknown. This increases the dimension as the number of variables, and new problems arise for the study. It is shown that the Cauchy problem for these systems of equations can be reduced to the Cauchy problem for a second-order quasilinear equation but of larger dimension. The question of existence and uniqueness of the solution of this Cauchy problem is not yet solved. We propose the possibility of reducing the Cauchy problem for a PDE system to the Cauchy problem for one higher-dimensional linear differential equation solving which one can find the solution to the original PDE system. It has long been well known that there is a general way of transforming PDE systems to systems of first-order quasilinear differential equations. This fact is used in the proof of the Cauchy–Kovalevskaya theorem. In this work, further progress is made in the study of this issue.

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Stabilization of Nonlinear Systems with a Time-Varying Feedback Control in Critical Cases

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In this talk, we consider a class of nonlinear control systems governed by the following ordinary differential equations:

$$\frac{dx}{dt} = \sum_{j=1}^{m} u_j f_j(x) \equiv f(x,u), \quad x \in X \subset \mathbb{R}^n,\ u \in \mathbb{R}^m,\ m < n, \tag{1}$$

where $x = (x_1, ..., x_n)$ is the state and $u = (u_1, ..., u_m)$ is the control. We assume that all $f_j : X \to \mathbb{R}$ are smooth vector fields, and that $x = 0$ is an internal point of X. It is a well-known fact due to R. W. Brockett that the trivial equilibrium of system (1) cannot be made asymptotically stable by a differentiable state feedback control $u = \phi(x)$, $\phi(0) = 0$, provided that $f_1(0), f_2(0), ..., f_m(0)$ are linearly independent vectors and $m < n$. Note that, for each feedback law of this class, the closed-loop system $\dot{x} = f(x, \phi(x))$ exhibits a critical case of stability (the Jacobian matrix of $f(x, \phi(x))$ at $x = 0$ has zero eigenvalues).

To stabilize the equilibrium $x = 0$, we apply an extended class of controls – *time-varying* feedback laws of the form $u = h(x,t)$,

$$h(x,t) = \sum_{k=-N}^{N} v_k(x) e^{2\pi k i t/\varepsilon}, \tag{2}$$

where the number $N \geqslant 0$ and the coefficients $v_k(x)$ are such that $v_k(0) = 0$ and $v_{-k}(x) = \overline{v_k(x)}$ for all $k = 0, 1, ..., N$, and ε is a positive parameter.

Our basic assumption is that the vector fields $f_1, f_2, ..., f_m$ satisfy Hörmander's condition in a neighborhood of $x = 0$. Thus, system (1) is locally controllable at $x = 0$, and *the existence* of a stabilizing time-varying control follows from J.-M. Coron's theorem. In this work, we propose an approach which allows *to compute the number N and the coefficients $v_k(x)$ so that the solution $x = 0$ of the corresponding closed-loop system* (1), (2) *is exponentially stable.* The proof of this result is based on an extension of Lyapunov's direct method with the use of Volterra series to estimate the decay rate of a Lyapunov function along the trajectories of system (1). As a result, the values of $v_k(x)$ at each point $x \in X$ may be obtained in terms of solution to a certain system of algebraic equations. The solvability of that algebraic system is proved by exploiting the topological degree theory and generalizing the approach of the papers [1, 2]. We present here explicit formulas for the controls under low-order controllability assumptions. The proposed methodology is illustrated by several examples from nonholonomic mechanics. Possible extensions of this approach for control-affine nonlinear systems are discussed as well.

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Экстремали Больцмана и эргодическая проблема по Пуанкаре и Гиббсу

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Н-теорема впервые была рассмотрена Больцманом в [1]. Эту теорему, обосновывающую сходимость решений уравнений типа Больцмана к максвелловскому распределению, Больцман связал с законом возрастания энтропии [2]. Доказательство Н-теоремы не только обосновывает 2-е начало термодинамики, но и делает поведение решения уравнения понятным, так как позволяет узнать, куда сходится решение для данного уравнения при времени, стремящемся к бесконечности.

Мы рассматриваем обобщения уравнений химической кинетики, включающие в себя классическую и квантовую химическую кинетику [3]. Н-теорема для этих обобщений уравнений химической кинетики в случае непрерывного времени исследовалась в [3]. Были изучены обобщенное условие детального равновесия (баланса) и обобщённое условие динамического равновесия (или обобщенное условие Штюккельберга—Батищевой—Пирогова), при выполнении которых справедлива H-теорема. В работах [4, 5] было показано, как выполняется закон роста энтропии для уравнений Лиувилля: энтропия временного среднего больше или равна энтропии начального распределения, хотя вдоль решения она сохраняется. В работах [6, 7] показано, что временные средние для уравнения Лиувилля совпадают с экстремалью Больцмана там, где достигается условный максимум энтропии при фиксированных законах сохранения. Мы доказываем это совпадение для представлений групп, вводя энтропию и изучая ее свойства в теории представлений. Потом мы выясняем, что дает это для эргодической проблемы, получая обобщение и уточнение эргодических теорем Рисса, Биркгофа—Хинчина, фон Неймана и Боголюбова с единой точки зрения. Это обосновывает, проясняет и уточняет метод Гиббса. Это также по-новому проясняет проблемы необратимости, в частности, парадоксы Лошмидта и Пуанкаре.

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О стремлении к нулю величины отклонения аргумента в дифференциально-разностных уравнениях с опережением

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Настоящая работа посвящена исследованию функционально-дифференциального уравнения вида

$$u_t(t) = au(t) + bu(t+h) + f(t), \qquad t > 0, \tag{1}$$

в котором a, b — вещественные постоянные, положительные постоянные h, τ являются отклонениями аргумента (опережением и запаздыванием соответственно), а

f — заданная на полуоси $\mathbb{R}_+ = (0, +\infty)$ непрерывная числовая функция. Требуется определить неизвестную числовую функцию $u : (-\tau, +\infty) \to \mathbb{R}$, удовлетворяющую уравнению (1) и начальному условию

$$u(+0) = u_0. \tag{2}$$

Определение. Решением задачи Коши (1), (2) будем называть функцию $u \in W^1_{2,\gamma}(0, +\infty)$, которая удовлетворяет уравнению (1) на интервале $(0, +\infty)$ и начальному условию (2).

Обозначим через $L_{2,\gamma}((a, b), \mathcal{H}), (-\infty \leqslant a < b \leqslant +\infty)$ пространство вектор-функций со значениями в $\mathcal{H}$, снабженное нормой

$$\| f \|_{L_{2,\gamma}((a,b),\mathcal{H})} = (\int_a^b exp(-2\gamma t) \| f(t) \|^2_{\mathcal{H}} \, dt)^{1/2}, \quad \gamma \geqslant 0$$

Через $W^l_{2,\gamma}(a, b)$ при каждом $l \in \mathbf{N}$ обозначим пространство вектор-функций на интервале (a, b) со значениями в $\mathcal{H}$ таких, что

$$u^{(1-j)l}(t) \in L_{2,\gamma}((a, b), \mathcal{H}), \quad j = 0, 1, \qquad l = 1, 2, ...,$$

Теорема 1. *Пусть$f \in L_{2,\gamma_0}(0, +\infty)$ при некоторых $\gamma_0 \in R$ и пусть $\omega(\gamma) < 1$ на интервале $(\alpha, \beta) \subset R$, где $\omega(\gamma) = \frac{be^{\gamma h}}{\gamma - a}$, $\gamma \in R$. Тогда задача Коши (1), (2) имеет единственное решение u в пространстве $W^1_{2,\gamma}(0, +\infty)$ при всех $\gamma \in [\gamma_0, \beta)$.*

Теорема 2. *Пусть выполнено условие $|c| < \gamma - a$. Тогда существует такое $\tau_0 > 0$, что при всех $\tau \in (0, \tau_0)$ задача с начальными условиями (1)-(2) имеет единственное решение $u_\tau \in W^1_{2,\gamma}(R_+)$, при этом*

$$\lim_{\tau \to +0} \|u_\tau - u_0\|_{W^1_{2,\gamma}(R_+)} = 0,$$

где u_0 — решение ОДУ $u'(t) = (a + c)u(t) + f(t)$, $t > 0$, с начальным условием (2).

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Существование решения уравнения высокого порядка с запаздыванием

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Настоящее сообщение посвящено доказательству существования решения уравнения в частных производных высокого порядка с дискретным отклонением аргумента в младших членах.

Рассмотрим

$$u^k \cdot L_1(u) + L_2(u) + \gamma u(x, t-\tau) = f(x) \cdot g(t), \tag{1}$$

где $u = u(x,t)$ – искомая функция на $\mathbb{R}^2$,

$$L_1 = \sum_{i=1}^{m} \alpha_i \frac{\partial^i}{\partial x^i}, \qquad L_2 = \sum_{j=1}^{n} \beta_j \frac{\partial^j}{\partial t^j},$$

$k, m, n \in N$, $\tau = const > 0$; а $f(x)$, $g(t)$, $\alpha_i = \alpha_i(x)$, $\beta_j = \beta_j(t)$, $\gamma = \gamma(t)$ — заданные достаточно гладкие функции.

Вопрос разрешимости уравнения (1) в случае $g(t) \equiv 0$ редуцирован к вопросу разрешимости обыкновенного дифференциального уравнения и нелинейного дифференциального уравнения с запаздыванием относительно функций $X(x)$ и $T(t)$ соответственно:

$$L_1\left(X\right) = -\lambda X^{1-k}, \qquad L_2\left(T\right) = \lambda T^{k+1} - \gamma T(t-\tau),$$

где λ — числовой параметр.

Для случая $g(t) \neq 0$ справедлива следующая

Теорема 1. *Если* $\dfrac{k+1}{2} \in \mathbb{N}$, *функция* $g(t) \in C\left(-\infty; \infty\right)$ *положительно определена и справедливо равенство*

$$L_2\left[g(t)^{\frac{1}{k+1}}\right] = \lambda g(t) - \gamma g(t-\tau)^{\frac{1}{k+1}},$$

то решение уравнения (1) *существует и может быть представлено в виде* $u = X(x) \cdot T(t)$.

В качестве частного случая в области $\Omega = \{(x,t) : 0 < x < l, -\pi < t < \pi\}$, где $l - const > 0$, для уравнения

$$u \cdot \frac{\partial^2}{\partial x^2}(u_{xx} + \alpha u) - \beta u_t + \gamma u(x, t-\tau) = 0, \tag{2}$$

где $\tau > 0$ при $t > 0$ и $\tau < 0$ при $t < 0$, $\alpha, \beta, \gamma = const > 0$, исследована следующая

Задача 1. *Найти решение* $u(x,t)$ *уравнения* (2) *в* $\Omega \setminus \{t = 0\}$ *из класса* $C^1(\bar{\Omega}) \bigcap C^{4,2}_{x,t}(\Omega)$, *удовлетворяющее следующим условиям:*

$$u(0,t) = u_x(0,t) = u(l,t) = u_x(l,t) = 0,$$

$$u(x,t_0)=\varphi_1(x), \qquad u(x,-t_0)=\varphi_2(x),$$

где $\varphi_j(x)\,(j=1,2)$ *– заданные достаточно гладкие функции, причем* $\varphi_j(0)=\varphi_j(l)=0$*,* $\varphi_1(x)\propto\varphi_2(x)$*,* $t_0=\pi$.

Доказательство разрешимости задачи 1 проведено методом Фурье. Вопрос существования решения задачи редуцирован к вопросу разрешимости двух задач Штурма—Лиувилля для соответствующих обыкновенных дифференциальных уравнений. Исследование полученных задач проведено аналогично [1, 2].

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Асимптотические уравнения в механике сплошных сред и их приложения

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Предложены асимптотические разложения для потенциала скорости, на основе которых выводятся асимптотические уравнения газовой динамики для безвихревых изэнтропических течений газа. Приведем эти уравнения для первого приближения потенциала скорости $\varphi(x,r,\theta,t)$.

а) $$\varphi_{tt}+2V\varphi_{xt}+V^2\varphi_{xx}=a^2\left(\varphi_{xx}+\varphi_{rr}+\frac{1}{r}\varphi_r+\frac{1}{r^2}\varphi_{\theta\theta}\right). \tag{1}$$

Здесь и далее индексы снизу обозначают частные производные по времени t и координатам x, r, θ; V, a — скорость газа и скорость звука в однородном невозмущенном потоке. Уравнение (1) — классическое уравнение линейной теории, которое применяется для описания как дозвуковых, так и сверхзвуковых течений.

б) $$\begin{aligned}&2\varphi_{xt}+(\gamma+1)\varphi_x\varphi_{xx}+2\psi_r\varphi_{xr}+\frac{2}{r^2}\psi_\theta\varphi_{x\theta}+\frac{\gamma-1}{2}\Big(2\psi_t+\psi_r^2+\\&+\frac{1}{r^2}\psi_\theta^2\Big)\varphi_{xx}-\varphi_{rr}-\frac{1}{r}\varphi_r-\frac{1}{r^2}\varphi_{\theta\theta}=-\psi_{tt}-2\psi_r\psi_{rt}-\frac{2}{r^2}\psi_\theta\psi_{\theta t}-\psi_r^2\psi_{rr}-\\&-\frac{1}{r^4}\psi_\theta^2\psi_{\theta\theta}-\frac{2}{r^2}\psi_r\psi_\theta\psi_{r\theta}+\frac{1}{r^3}\psi_\theta^2\psi_r.\end{aligned} \tag{2}$$

Нелинейное уравнение (2) для $\varphi(x,r,\theta,t)$ (в т. ч. нелинейный член $\varphi_x\varphi_{xx}$) описывает трансзвуковые течения газа (течения, содержащие как дозвуковые, так и сверхзвуковые зоны, а также звуковую поверхность — поверхность перехода скорости газа через скорость звука; в установившемся случае эта поверхность является поверхностью параболичности, разделяющей гиперболическую (сверхзвуковую) и эллиптическую (дозвуковую) области). Функция $\psi(r,\theta,t)$ удовлетворяет уравнению Лапласа $\psi_{rr}+\frac{1}{r}\psi_r+\frac{1}{r^2}\psi_{\theta\theta}=0$. В случае $\psi\equiv 0$ из (2) получим уравнение Линя—Рейсснера—Тзяна, переходящее для установившихся течений в уравнение смешанного типа Кармана—Фальковича.

в) $$2V\varphi_{\xi t}+2\beta a^2\varphi_{\xi r}+[(\gamma+1)VM^2\varphi_\xi+(\gamma-1)M^2\psi_t-2V\beta\psi_r]\varphi_{\xi\xi}+ \\ +\frac{1}{r}\beta a^2\varphi_\xi=a^2(\psi_{rr}+\frac{1}{r}\psi_r+\frac{1}{r^2}\psi_{\theta\theta})-\psi_{tt} \qquad (3)$$

Здесь $\xi=x-\beta r$, $\beta=\sqrt{M^2-1}$, $M=V/a$ — число Маха, γ — показатель Пуассона, функция $\psi(r,\theta,t)$ — произвольная. Нелинейное уравнение (3) для $\varphi(\xi,r,\theta,t)$ описывает сверхзвуковое течение в окрестности ударной волны, мало отличающейся от линии Маха $\xi=const$.

Функция $\psi(r,\theta,t)$ в (2), (3) задает поперечное аэродинамическое воздействие.

На основе уравнений (1)–(3) представлены решения некоторых задач газовой динамики.

Принцип максимума Понтрягина как каноническая двойственность в задачах оптимизации

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Принцип максимума был сформулирован Понтрягиным изначально для чисто практических целей — для расчета оптимальной динамики в системах автоматического управления, к которым классические вариационные методы принципиально неприменимы, т.к. множества значений управляющих параметров в таких задачах не являются, как правило, открытыми, и в большинстве случаев они компактны. Для конечномерных задач такое расширение условий приводит к тому обстоятельству, что классическое правило множителей Лагранжа для отыскания стационарных точек рассматриваемой функции, которое, по существу, является разновидностью соотношения двойственности между дифференциалом этой функции и ее производной по направлению в стационарных точках, перестает действовать и приходится искать специфические признаки экстремальности для таких задач, такие, например, как линейное программирование и т.п.

В докладе будет показано, что принцип максимума является манифестацией двойственности между касательным и кокасательным расслоениями фазового пространства для самых общих оптимальных задач и единообразно применимо практически к любой задаче оптимизации без исключений.

Об асимптотике при большом времени решений параболических уравнений с растущими старшими коэффициентами

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Рассмотрим модельную задачу

$$Lu + (b, \nabla u) + c(x)u - u_t = 0, \quad \text{в} \quad R^N \times (0, \infty), N \geqslant 3, \tag{1}$$

$$u(x, 0) = u_0(x),\ x \in R^N, \tag{2}$$

где

$$Lu = \sum_{i,k=1}^{N} a_{ik}(x) u''_{x_i x_k}, (b, \nabla u) = \sum_{i=1}^{N} b_i(x) u'_{x_i}.$$

Предполагается, что 1) $a_{ik} = a_{ki}, i, k = 1, ..., N,$
2) существуют постоянные $\lambda_0 > 0, \lambda_1 > 0, B > 0$ и $\beta > 0$, такие, что

$$\lambda_0^2 b(|x|)|\xi|^2 \leqslant \sum_{i,k=1}^{N} a_{ik}(x)\xi_i\xi_k \leqslant \lambda_1^2 b(|x|)|\xi|^2, \tag{3}$$

где

$$b(r) = 1 + r^2, r^2 = x_1^2 + ... + x^N, \tag{4}$$

$$\sum_{i=1}^{N} |b_i(x)| \leqslant B(1 + r^2)^{1/2}, x \in R^N, \tag{5}$$

$$c(x) \leqslant -\beta^2,\ x \in R^N. \tag{6}$$

Пусть

$$v(n) = \frac{\lambda_1(\beta)}{2 + \frac{1}{n}}, n = 1, 2, ..., \lambda_1(\beta) = \frac{2 - s + \sqrt{D}}{2}, \tag{7}$$

где

$$s = \frac{\lambda_1^2(N-1) + \lambda_0^2 + B}{\lambda_0^2}, D = (2-s)^2 + 4\beta_1^2, \beta_1 = \frac{\beta}{\lambda_1}.$$

Теорема 1. *Если функция (2) ограничена в R^N, коэффициенты $a_{ik}\,(i, k = 1, ..., N)$ в (1) удовлетворяют неравенствам (3) и (4), коэффициенты $b_i(x)\,(i = 1, ..., N)$ удовлетворяют (5), а коэффициент $c(x)$ удовлетворяет (6) при*

$$\beta^2 > \lambda_1^2(s-1), \tag{8}$$

то решение задачи (1), (2) стабилизируется к нулю со скоростью $t^{-v(n)}$, т.е. существует предел

$$\lim_{t\to\infty} t^{v(n)} u(x, t) = 0$$

равномерно по x на любом компакте K в R^N.

В работе [1] изучен случай, когда $b_i(x) = 0, i = 1, ..., N$.

Работа выполнена при финансовой поддержке РФФИ (грант № 15-01-00471).

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Об одном граничном тождестве трехмерных векторных полей и соответствующих краевых задачах

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Любая пара гладких трехмерных полей удовлетворяет безусловному граничному тождеству, связывающему граничные значения этих полей со значениями на границе их же векторов нормальных производных, роторов и дивергенций. Эта связь позволяет для любой вектор-функции из пространства Соболева первого порядка определить «след» некоторой линейной комбинации указанных операций первого порядка — девиатора операторов Лапласа, записанных в стандартной форме и в роторно-дивергентной форме. Указанное граничное тождество выражает свойство симметричности девиатора и приводит к континуальному обобщению известного граничного разложения векторных полей в сумму нормальной и тангенциальной составляющих. Обсуждаются соответствующие краевые задачи.

Зависимость собственных значений оператора сдвига от величины сдвига

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Рассматривается уравнение вида:

$$x'(t) = f(x(t), x(t-1)).$$

Предполагается, что известно периодическое решение $\tilde{x}$ этого уравнения. Через T обозначен период этого решения. Соответствующее линеаризованное уравнение имеет вид

$$v'(t) = a(t)v(t) + b(t)v(t-1).$$

Рассматривается семейство операторов сдвига $M_p: \ C[-1,0] \to C[-1,0]$, которые действуют по формуле $M_p\phi(t) = v^{\phi}(t+p)$, где v^{ϕ} — решение линеаризованного уравнения с начальным условием $v(t) = \phi(t)$ при $t \in [-1,0]$. В работе [1] обсуждался вопрос рациональной аппроксимации оператора монодромии M_T. В работе [2] в качестве аналога рациональной аппроксимации рассматривалось семейство операторов M_p с параметром $p \in \mathbb{R}$.

В докладе каждому собственному значению $\lambda(p_0)$ оператора M_{p_0} ставятся в соответствие собственные значения $\lambda(p)$ операторов M_p при малых $|p-p_0|$. Рассматривается вопрос определения показателя Гельдера α, для которого неравенство $|\lambda(p_0)-\lambda(p)| \leqslant k|p-p_0|^\alpha$ выполнено при малых $|p-p_0|$ и некоторой константе k.

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Регуляризация задачи Коши для систем уравнений эллиптического типа первого порядка

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В данной работе с использованием методики работ [1]-[2] построена регуляризация задачи Коши для матричной факторизации уравнения Гельмгольца в трехмерной неограниченной области.

Пусть $\mathbb{R}^3$ — 3-мерное вещественное евклидово пространство, $x=(x_1,\, x_2,\, x_3) \in \mathbb{R}^3$, $y=(y_1,\, y_2,\, y_3) \in \mathbb{R}^3$, $x'=(x_1,\, x_2) \in \mathbb{R}^2$, $y'=(y_1,\, y_2) \in \mathbb{R}^2$.

Рассмотрим в области G систему дифференциальных уравнений

$$D\left(\frac{\partial}{\partial x}\right)U(x)=0, \tag{1}$$

где $D\left(\frac{\partial}{\partial x}\right)$ — матрица дифференциальных операторов первого порядка.

Обозначим

$$H_\rho(G)=\{\,U(y):\ U(y)\in H(G),\ \ |U(y)|\leqslant \exp\left[o\left(\exp\rho\,|y'|\right)\right],\ y\to\infty,\ y\in G\}, \tag{2}$$

Задача Коши. Пусть $U(y)\in H_\rho(G)$ и

$$U(y)|_S=f(y),\ \ y\in S. \tag{3}$$

Здесь $f(y)$ — заданная непрерывная вектор-функция на S. Требуется восстановить вектор-функцию $U(y)$ в области G, исходя из ее значений $f(y)$ на S.

Пусть $U(y)\in H_\rho(G)$ и вместо $U(y)$ на S задано ее приближение $f_\delta(y)$, соответственно, с уклонением $0<\delta<1$, $\max\limits_S|U(y)-f_\delta(y)|\leqslant\delta$. Положим

$$U_{\sigma\delta}(x)=\int_S N_\sigma(y,\, x)f_\delta(y)ds_y,\ \ x\in G. \tag{4}$$

Теорема. *Пусть* $U(y) \in H_\rho(G)$ *удовлетворяет на части плоскости* $y_3 = 0$ *условию* $|U(y)| \leqslant 1,\ y \in T$.

Тогда справедлива оценка

$$|U(x) - U_{\sigma\delta}(x)| \leqslant C_\rho(x)\sigma\delta^{\frac{x_3}{h}}, \quad \sigma > 1, \quad x \in G. \tag{5}$$

Здесь через $C_\rho(x)$ *мы обозначили функции, зависящие от* x *и* ρ*. Причем в различных неравенствах они различные.*

Следствие. *Предельное равенство*

$$\lim_{\delta \to 0} U_{\sigma\delta}(x) = U(x)$$

имеет место равномерно на каждом компакте из области G.

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Многообразие задач и алгоритмы оценивания интегральной воронки динамической системы

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Для динамической системы рассматривается численно-аналитическое построение оценок интегральной воронки (ИВ) (построение временных сечений фазового потока системы, начинающегося в начальном множестве). Известно, что при получении качественных оценок ИВ (в определенном смысле) решаются разнообразные задачи теории динамических систем, теории устойчивости, теории управления. Оценивание рассматриваемых множеств проводится среди множеств, ограниченных поверхностями уровня различных специальных функций Ляпунова, а также среди пересечений (объединений) описанных множеств.

Численно–аналитическими методами строятся оценки в случаях:

1. единственного инвариантного множества системы внутри начального множества,

2. единственного инвариантного множества системы (не пересекающегося с начальным множеством), некоторая окрестность которого достижима рассматриваемым фазовым потоком на изучаемом отрезке времени,

3. отсутствия в рассматриваемой области инвариантных множеств,

4. наличия в рассматриваемой области нескольких инвариантных множеств.

Используя известные конструкции метода сравнения и теоремы, доказанные автором, построены оригинальные системы сравнения (СС) для оценивания ИВ. Для получения аналитических оценок ИВ достаточно проинтегрировать (численно, аналитически) СС только один раз.

В докладе обсуждаются различные модификации построения СС для улучшения оценок и возможность построения высокоточных оценок (в частных случаях).

Приводятся примеры оценивания ИВ в системах, описывающих движение ЛА.

О разрешимости и асимптотиках решений некоторых сингулярных задач Коши

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Рассматриваются задачи Коши вида

$$\alpha(t)x'(t) = f(t, x(t), x(g(t)), x'(t), x'(h(t))), \quad x(0) = 0,$$

где $x : (0,\tau) \to \mathbb{R}$ — неизвестная функция, $f : D \to \mathbb{R}$ — непрерывная функция, $D \subset (0,\tau) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$,

$$\alpha : (0,\tau) \to (0,+\infty), \quad g : (0,\tau) \to (0,+\infty), \quad h : (0,\tau) \to (0,+\infty)$$

— непрерывные функции, $g(t) \in (0,t]$, $h(t) \in (0,t]$, $t \in (0,\tau)$, $\lim\limits_{t\to+0} \alpha(t) = 0$. Рассматриваются случаи:

$$\lim_{t\to+0} \frac{\alpha(t)}{t} = \sigma, \quad \sigma \in [0,+\infty)\,;$$

$$\lim_{t\to+0} \frac{\alpha(t)}{t} = +\infty.$$

Под решением данной задачи понимается непрерывно дифференцируемая функция $x : (0,\rho] \to \mathbb{R}$ ($\rho \in (0,\tau)$), которая тождественно удовлетворяет исследуемому уравнению при всех $t \in (0,\rho]$, причем $\lim\limits_{t\to+0} x(t) = 0$.

Формулируются достаточные условия, при выполнении которых у каждой из рассматриваемых задач существует непустое множество решений $x : (0,\rho] \to \mathbb{R}$ ($\rho \in (0,\tau)$), где ρ — достаточно мало, каждое из которых обладает определенными свойствами при $t \in (0,\rho]$. Обсуждается вопрос о числе таких решений.

В исследованиях используются методы качественной теории дифференциальных уравнений и функционального анализа.

Дифференциальные операторы на графах и их спектральный анализ

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В докладе рассматриваются дифференциальные операторы второго порядка на графах. Вводится максимальный дифференциальный оператор Λ_{max} на ориентированном графе без петель. Приведены формула Лагранжа для максимального оператора на графе-дереве, а также асимптотически независимый набор решений на графе операторного уравнения $\Lambda_{max}\vec{\Psi} = \lambda\vec{\Psi}$. Затем вводится определение регулярных краевых условий по Бирхгофу для дифференциальных операторов на графах. Исследуются спектральные свойства сужения Λ максимального оператора Λ_{max}, область определения которого задается с помощью регулярных краевых условий по Бирхгофу. В частности, доказаны асимптотические формулы для собственных значений, функции Грина сужения Λ и теорема о разложений по корневым функциям оператора Λ.

Бифуркации в неоднородном уравнении Дуффинга при малых возмущениях

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Уравнение без затухания

$$\ddot{z} + \Omega^2 z = \varepsilon\nu z^3 + \varepsilon a \cos\Omega_1 t,$$

в котором $\Omega^2 > 0,\ \nu,\ a,\ \Omega_1$ — постоянные параметры, $\varepsilon > 0$ — малый параметр, при условии $|\Omega - \Omega_1| \sim \varepsilon$ решено с произвольными начальными условиями с использованием переменных ван дер Поля $b,\ \psi\ \ (z = b\cos\psi)$ во втором приближении метода усреднения.

Для медленных переменных b и фазы $\theta = \psi - \Omega_1 t$ в первом приближении имеем усредненную по быстрым фазам систему уравнений

$$\dot{b} = -\varepsilon\frac{a}{2\Omega}\sin\theta, \quad \dot{\theta} = \varepsilon\left(h - \frac{3\nu}{8\Omega}b^2\right) - \varepsilon\frac{a}{2\Omega}\frac{\cos\theta}{b},$$

где $h = (\Omega - \Omega_1)/\varepsilon$. С использованием интеграла

$$\cos\theta = \frac{h\Omega}{a}\,b - \frac{3\nu}{16a}\,b^3 + \frac{C_1}{b},$$

который следует из системы, получаем уравнение осциллятора

$$\ddot{b} = -\frac{p}{2}\cos\theta\ \dot{\theta} = \frac{\varepsilon^2}{4}\left[\left(-h^2 + \frac{3a\nu C_1}{8\Omega^2}\right)b + \frac{3h\nu}{4}\,b^3 - \frac{27\nu^2}{256\Omega^2}\,b^5 + \frac{a^2}{\Omega^2}\,\frac{C_1^2}{b^3}\right] = F(b).$$

Корни уравнения $F(b) = 0$ дают положения равновесия. При $C_1 \neq 0$ на плоскости параметров

$$\tilde{\alpha} = \frac{\Omega(\Omega - \Omega_1)}{\varepsilon \nu C_1^2}, \qquad \tilde{\gamma} = \frac{\varepsilon a}{\Omega(\Omega - \Omega_1) C_1}$$

существуют шесть областей различных режимов колебаний осциллятора. Интересны области $I)$ $\tilde{\alpha} > 0,\ 0 < \tilde{\gamma} < 4\tilde{\alpha}/9$ и $II)$ $\tilde{\alpha} > 0,\ 0 > \tilde{\gamma} > -4\tilde{\alpha}/3$, в которых потенциал для уравнения осциллятора имеет две ямы. Бифуркационные значения параметров $\tilde{\alpha}$, $\tilde{\gamma}$, соответствующие колебаниям осциллятора на вершине потенциального барьера, связаны квадратным относительно $\tilde{\alpha}$ уравнением

$$P\tilde{\alpha}^2 + Q\tilde{\alpha} + R = 0,$$

где

$$P = \tilde{\gamma} - 4, \quad Q = -\frac{81}{64}\,\tilde{\gamma}^3 + \frac{27}{4}\,\tilde{\gamma}^2 - 6\tilde{\gamma}, \quad R = -\frac{9}{4}\,\tilde{\gamma}^2.$$

Из двух корней уравнения к областям I и II относится корень $\tilde{\alpha} = (-Q + \sqrt{Q^2 - 4PR}\,)/2P$.

Отметим существование бифуркаций как при $\nu > 0$, так и при $\nu < 0$.

Для всех шести областей на основе приведенного выше уравнения осциллятора построены приближенные решения уравнения Дуффинга.

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Об одном уравнении с нелокальными условиями

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Исследуется задача

$$\begin{aligned} &u'' + \tfrac{a_1(x)}{x}u' + \tfrac{a_0(x)}{x^2}u = f(x), \quad x \in (0,1], \\ &u(0) = 0, \quad \int\limits_0^1 u(x)\,dx = 0. \end{aligned} \tag{1}$$

Здесь a_1, a_0 — непрерывные функции на отрезке $[0,1]$. Правая часть уравнения принадлежит пространству $C_\alpha[0,1]$, элементами которого являются непрерывные на полуинтервале $(0,1]$ функции, для которых существует конечный предел $\lim\limits_{x \to 0} x^{1+\alpha} f(x)$. Норма в пространстве $C_\alpha[0,1]$ задается формулой

$$\|f\|_0 = \max_{0 \leqslant x \leqslant 1} |x^{1+\alpha} f(x)|.$$

Решением задачи будем называть непрерывную на отрезке $[0,1]$ функцию, дважды непрерывно дифференцируемую на полуинтервале $(0,1]$ и удовлетворяющую (1).

Получены условия на функции a_1, a_0, при которых рассматриваемая задача имеет единственное решение для любой функции $f \in C_\alpha[0,1]$.

Задача (1) порождает оператор A в пространстве $C_\alpha[0,1]$, заданный формулой

$$Au = u'' + \frac{a_1(x)}{x}u' + \frac{a_0(x)}{x^2}u$$

с областью определения

$$D(A) = \{u \in C[0,1] \cap C^1(0,1] \cap C^2(0,1] : Au \in C_\alpha[0,1],\ u(0) = 0, \int\limits_0^1 u(x)\,dx = 0\}.$$

Показано, что для всех $|\lambda| \geqslant q$, $\mathrm{Re}\lambda > 0$, где $q \geqslant 0$ – некоторая константа, существует резольвента оператора A и справедлива оценка

$$\|(A - \lambda I)^{-1}\|_0 \leqslant \frac{C}{|\lambda|^{1/2}}.$$

Из этой оценки вытекает, что оператор A является производящим оператором полугруппы с особенностями.

Численное тестирование в обратных задачах на графах

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Доклад посвящен восстановлению коэффициентов параболического уравнения на графе-дереве. Пусть $\Omega = \{E, V\}$ обозначает конечный связный компактный метрический граф-дерево, $E = \{e_1, e_2, \ldots, e_N\}$ есть множество ребер и $V = \{\nu_1, \nu_2, \ldots, \nu_{N+1}\}$ — набор вершин, $\{\gamma_1, \gamma_2, \ldots, \gamma_m\} = \partial\Omega \subset V$ — граничные вершины.

В результате преобразований параболическая задача на квантовом графе [1] принимает вид:

$$u_t - u_x x + q(x)u = p(t)h(x), \quad t \in (0,T). \tag{1}$$

$$\begin{cases} \sum_{e_j \sim v} \partial u_j(v,t) = 0 \text{ в каждой вершине } v \in V \setminus \partial\Omega, \text{ и } t \in [0,T] \\ u(\cdot,t) \text{ непрерывны в каждой вершине для всех } t \in [0,T], \end{cases} \tag{2}$$

$$\partial u = f \text{ на } \partial\Omega \times [0,T],\ u|_{t=0} = 0 \text{ на } \Omega. \tag{3}$$

Положим $\mathcal{H} = L^2(\Omega)$ и $\mathcal{F}^T = L^2([0,T];\mathbb{R}^m)$. Существует [2] единственное решение начально-краевой задачи (1)–(2). Здесь $p \in H^1(0,T)$, а оператор отклика $R^T \mathcal{F}^T \to \mathcal{F}^T$ есть $(R^T f)(t) = u^f(\cdot,t)$, $t \in [0,T]$.

Решена обратная задача, которая состоит в восстановлении топологии графа, длин ребер и векторов $q(\cdot)$ и $h(\cdot)$, известных по $R^T f$ для всех $f \in \mathcal{F}^T$. Алгоритм, описанный в [2], позволяет свести обратную задачу для параболического уравнения на дереве к обратной задаче для волнового уравнения на каждом ребре $e \in E$. Мы отождествляем e с интервалом $(0, l)$ и предлагаем практические алгоритмы, которые могут быть реализованы численно. Коды Matlab демонстрируют численное тестирование нашей модели. Потенциал q задается в конкретной области, $x \in (0, 2l)$, где $l = 1$ или π. Для численного тестирования мы рассматриваем такие потенциалы, как $q(x) = 4x$ и $q(x) = \sin(4x)$. В результате получаем сетку соответствующих значений и график, описывающие численное восстановление синусоидального потенциала.

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Нестандартная гидродинамика (двумерные ударные волны)

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Исследуются два примера нестандартной гидродинамики: в существующих на сегодняшний момент методах расчета турбулентности предполагается выполнение чисто математического условия Годунова—Лакса о существовании полного базиса собственных векторов на критическом многообразии кратных корней (симметризация системы). При этом предположении для задачи Римана о распаде разрыва из теоремы Майды следует существование для любых двух точек фазового пространства единственной соединяющей их цепочки устойчивых ударных волн, волн разряжения и контактных разрывов.

В то же время, из эксперимента (Ландау, Пригожин, Ричардсон, ...) хорошо известно о возникновении двухскоростного режима в начальной стадии турбулентности (т. е. бифуркации устойчивой ударной волны), что противоречит теореме Майды. С возможным механизмом возникновения двухскоростного режима (названного Пригожиным катастрофой Римана—Гюгонио), приводящим к нарушению условия Годунова—Лакса, связаны исследуемые ниже примеры.

Для модификации системы уравнений мелкой воды (система для двухкомпонентной смеси) и усеченной модели Эйлера (система для двухкомпонентной смеси с одним уравнением неразрывности) доказано существование неклассических (двумерных) ударных волн в задаче Римана.

Задача Дирихле для нагруженного уравнения смешанного типа

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Рассмотрим нагруженное уравнение смешанного типа

$$Lu = \begin{cases} u_{xx} + u_{yy} - b^2 u(x,y) + C_1(y)u(x,0) = 0, & y > 0, \\ u_{xx} - u_{yy} - b^2 u(x,y) + C_2(y)u(x,0) = 0, & y < 0, \end{cases} \tag{1}$$

в прямоугольной области $D = \{(x,y) : 0 < x < l, -\alpha < y < \beta\}$, α, β, l, $b \geqslant 0$ — заданные положительные действительные числа, $C_1(y)$, $C_2(y)$ — заданные непрерывные функции.

Задача Дирихле. Найти в области D функцию $u\,(x,y)$, удовлетворяющую следующим условиям:

$$u\,(x,\,y) \in C^1\left(\overline{D}\right) \cap C^2\left(D_+ \cup D_-\right); \tag{2}$$

$$Lu\,(x,\,y) \equiv 0, \quad (x,\,y) \in D_+ \cup D_-; \tag{3}$$

$$u\,(0,\,y) = u\,(l,\,y) = 0, \quad -\alpha \leqslant y \leqslant \beta; \tag{4}$$

$$u\,(x,\,\beta) = \varphi\,(x)\,, u\,(x,\,-\alpha) = \psi\,(x)\,, \quad 0 \leqslant x \leqslant l, \tag{5}$$

где $\varphi\,(x)$, $\psi\,(x)$ — заданные достаточно гладкие функции, $\varphi\,(0) = \varphi\,(l) = \psi\,(0) = \psi\,(l)$, $D_+ = D \cap \{y > 0\}$, $D_- = D \cap \{y < 0\}$.

В работах [1], [2] впервые для нагруженного параболо-гиперболического уравнения в прямоугольной области изучена начально-граничная задача методом спектральных разложений, где установлен критерий единственности решения и доказана теорема существования решения этой задачи. Решение построено в виде суммы ряда по собственным функциям соответствующей одномерной задачи на собственные значения.

Задача (2)–(5) при $l = 1$ изучена в работе [3], здесь при всех $b \geqslant 0$ установлены необходимые и достаточные условия единственности решения задачи. Само решение построено в виде суммы ряда Фурье. При доказательстве равномерной сходимости ряда из-за проблемы малых знаменателей накладываются условия малости норм коэффициентов $C_1(y)$ и $C_2(y)$, входящих в уравнение (1) в составе нагруженных слагаемых.

В данной работе, на основе [3, 5] установлен критерий единственности решения задачи (2)–(5) для нагруженного уравнения (1) в прямоугольной области D. Решение построено в виде суммы ряда Фурье. При обосновании сходимости ряда возникает проблема малых знаменателей относительно сторон α/l прямоугольника D_-. В связи с чем установлены оценки для отделенности от малого знаменателя с соответствующей асимптотикой для рациональных и иррациональных значений числа α/l, которые позволили обосновать сходимость построенного ряда в классе регулярных решений (2) и (3). При этом условие малости норм $||C_1|| = \max\limits_{0 \leqslant y \leqslant \beta} |C_1(y)|$ и $||C_2|| = \max\limits_{-\alpha \leqslant y \leqslant 0} |C_2(y)|$ снято.

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Сходимость спектральных разложений для системы Дирака

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Рассматривается оператор Дирака $\mathcal{L}_P$ в пространстве $\mathbb{H} = L_2[0,\pi]\oplus L_2[0,\pi] \ni \mathbf{y}$, порожденный дифференциальным выражением

$$\ell_P(\mathbf{y}) = B\mathbf{y}' + P\mathbf{y}, \quad \text{где}$$

$$B = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \qquad P(x) = \begin{pmatrix} p_1(x) & p_2(x) \\ p_3(x) & p_4(x) \end{pmatrix}, \qquad \mathbf{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}.$$

Функции p_j, $j = 1,2,3,4$, предполагаются суммируемыми на отрезке $[0,\pi]$ и комплекснозначными. Общий вид краевых условий для оператора $\mathcal{L}_P$ задается системой двух линейных уравнений $U(\mathbf{y}) = 0$.

Обозначим через $\{\mathbf{y}_n\}$ систему корневых функций оператора $\mathcal{L}_{P,U}$. Зафиксируем некоторый ортонормированный базис $\{\mathbf{e}_n\}_{n\in\mathbb{Z}}$ в пространстве $\mathbb{H}$ и определим оператор T равенствами $T\mathbf{e}_n = \mathbf{y}_n$, $n \in \mathbb{Z}$. Введем обозначения

$$E_0 = Lin\{\mathbf{e}_n\}_{n\in J_N}, \quad E_1 = \overline{Lin\{\mathbf{e}_n\}_{n\notin J_N}},$$

$$\mathcal{H}_0 = Lin\{\mathbf{y}_n\}_{n\in J_N}, \quad \mathcal{H}_1 = \overline{Lin\{\mathbf{y}_n\}_{n\notin J_N}},$$

где через J_N обозначено множество целых индексов мощности $N \in \mathbb{N}$.

Теорема 1. *Пусть $\mathcal{L}_{P,U}$ — сильно регулярный оператор Дирака с потенциалом $P \in \mathcal{X}$, где $\mathcal{X}$ — компакт в $L_1[0,\pi]$. Пусть краевые условия U и величина $\int_0^\pi (p_4(t)-p_1(t))\,dt$ фиксированы. Тогда найдутся номер $N = N(\mathcal{X}, U)$ и константа $C = C(\mathcal{X}, U)$ такие, что, выбрав множество индексов $J = J_{2N+1} = \{n\}_{-N}^{N}$, для оператора*

$$S\mathbf{f} = \begin{cases} T\mathbf{f}, & \mathbf{f} \in E_1, \\ \sum\limits_{|n|\leqslant N} (\mathbf{f}, \mathbf{e}_n)\varphi_n, & \mathbf{f} \in E_0, \end{cases}$$

где $\{\varphi_n\}_{-N}^{N}$ *— произвольный ортонормированный базис в подпространстве* $\mathcal{H}_0$*, имеем оценку*

$$\|S\| \cdot \|S^{-1}\| \leqslant C.$$

Теорема 2. *Пусть* $\mathcal{L}_{P,U}$ *— сильно регулярный оператор Дирака с потенциалом* $P \in L_{\varkappa}[0,\pi]$*,* $\varkappa \in (1,2]$*,* $\|P\|_{L_\varkappa} \leqslant R$*. Пусть краевые условия* U *и величина* $\int_0^\pi (p_4(t) - p_1(t))\,dt$ *фиксированы. Тогда найдутся номер* $N = N(R,U)$*, множество индексов* J_N *(зависящее от потенциала* P *и краевых условий* U*) и константа* $C = C(R,U)$ *такие, что для оператора*

$$S\mathbf{f} = \begin{cases} T\mathbf{f}, & \mathbf{f} \in E_1, \\ \sum\limits_{n\in J_N} (\mathbf{f}, \mathbf{e}_n)\varphi_n, & \mathbf{f} \in E_0, \end{cases}$$

где $\{\varphi_n\}_{n\in J_N}$ *— произвольный ортонормированный базис в подпространстве* $\mathcal{H}_0$*, выполнена оценка*

$$\|S\| \cdot \|S^{-1}\| \leqslant C.$$

Работа выполнена при финансовой поддержке РФФИ (проект № 16-01-00706).

Коэрцитивная разрешимость нелокальной задачи для параболического уравнения в пространствах Слободецкого

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В произвольном банаховом пространстве E рассматривается нелокальная задача

$$v'(t) + Av(t) = f(t) \quad (0 \leqslant t \leqslant 1),\ v(0) = v(\lambda) + \mu \quad (0 < \lambda \leqslant 1) \tag{1}$$

для абстрактного параболического уравнения с неограниченным сильно позитивным оператором A, имеющим всюду плотную в E область определения $D(A)$ и порождающим аналитическую полугруппу $\exp\{-tA\}(t \geqslant 0)$; μ — элемент некоторого подпространства E, который будет определен ниже.

Введем пространства Л. Н. Слободецкого $W_p^\alpha = W_p^\alpha([0,1],E)$ $(1 \leqslant p < \infty,$ $0 < \alpha < 1/p)$, полученные замыканием множества всех гладких функций $f(t)$, определенных на отрезке $[0,1]$ со значениями в пространстве E, для которых конечна норма

$$\|f\|_{W_p^\alpha}^p = \int\limits_0^1 \|f(t)\|_E^p\,dt + \int\limits_0^1\int\limits_0^1 \frac{\|f(t)-f(\theta)\|_E^p}{|t-\theta|^{1+\alpha p}}\,d\theta dt. \tag{2}$$

Обозначим через $E_{\alpha,p} = E(\alpha,p,A)$ $(0 < \alpha < 1/p,\ 1 \leqslant p < \infty)$ банахово пространство с нормой

$$\|u\|_{E_{\alpha,p}}^p = \int\limits_0^1 z^{-\alpha p}\,\|A\exp\{-zA\}u\|_E^p\,dz. \tag{3}$$

Пусть $E_{\alpha,p}$ – совокупность тех элементов $u \in E$, для которых сходится интеграл (3).

Ранее в работе [1] была доказана коэрцитивная разрешимость задачи Коши

$$v'(t) + Av(t) = f(t) \quad (0 \leqslant t \leqslant 1),\ v(0) = v_0. \tag{4}$$

В настоящей работе доказывается следующая теорема.

Теорема 1. *Пусть* $\mu \in E_{\alpha,p}$, $f \in W_p^\alpha$ *при некоторых* $p \geqslant 1$, $0 < \alpha < 1/p$. *Тогда нелокальная задача* (1) *коэрцитивно разрешима в* W_p^α *и справедливо неравенство коэрцитивности*

$$\|v'\|_{W_p^\alpha} + \|Av\|_{W_p^\alpha} + \max_{0\leqslant t\leqslant 1} \|v(t)\|_{E_{\alpha,p}} \leqslant M\left[\|\mu\|_{E_{\alpha,p}} + \|f\|_{W_p^\alpha}\right] \tag{5}$$

с положительной постоянной M, *не зависящей от* μ *и* f.

Полученный результат обобщается и на случай переменного оператора.

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Гладкие решения линейных функционально-дифференциальных уравнений нейтрального типа

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Рассмотрим начальную задачу с начальной функцией для следующего линейного функционально-дифференциального уравнения нейтрального типа:

$$\dot{x}(t) + p(t)\dot{x}(t-1) = a(t)x(t-1) + b(t)x(t/q) + \bar{f}(t),\ t \in [0,\infty); \tag{1}$$

$$x(t) = g(t),\ t \in [-1,0], \tag{2}$$

где $g(t) \in C^\infty[-1,0]$, $p(t) = p_0 + p_1 t$, $a(t) = a_0 + a_1 t$, $b(t) = b_0 + b_1 t$, $\bar{f}(t) = \sum_{n=0}^{F} \bar{f}_n t^n$.

Сформулируем задачу о гладких решениях.

Задача. *Определить условия существования и способы нахождения начальной функции* $g(t)$, $t \in [-1,0]$, *такой, что порождаемое ею решение начальной задачи* (1)-(2) *обладает в точках, кратных запаздыванию, необходимой гладкостью.*

Данная задача может быть решена на основе метода полиномиальных квазирешений [1], в основе которого лежит представление неизвестной функции в виде полинома некоторой степени $x(t) = \sum_{n=0}^{N} x_n t^n$. При подстановки его в исходную задачу появляется некорректность в смысле размерности полиномов, которая компенсируется путем введения в уравнение невязки. Для невязки получена точная аналитическая формула, характеризующая меру возмущения исходной начальной задачи. Показано, что если для исследуемой начальной задачи (1)-(2) выбрать в качестве начальной функции $g(t)$ полиномиальное квазирешение степени N, то порождаемое решение будет иметь в точках стыковки гладкость не ниже степени N.

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Гиперболизация нелинейного уравнения Шредингера

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Нелинейное уравнение Шредингера (НУШ) обладает чрезвычайно высокой универсальностью и применяется для описания волновых процессов во многих областях физики: в теории поверхностных волн, в моделях эволюции распределений плазменных колебаний, нелинейной оптике, биофизике и т. д Данное уравнение является основным уравнением нелинейной волновой оптики и активно используется, например, при математическом моделировании волоконно-оптических линий связи, волоконных лазеров и различных оптических устройств. Для адекватного численного расчета задач в области волоконно-оптических линий связи или волоконных лазеров с детальным пространственно-временным разрешением число точек по «пространственной» переменной может составлять 10^6–10^7, что приводит к огромным затратам машинного времени и быстрому накоплению ошибки. Практически единственным путем расчета таких задач является параллельная реализация численных алгоритмов. Для их эффективного применения требуются новые математические модели, алгоритмы и математическое обеспечение.

В области гидро и газодинамики в последнее время интенсивно развивается метод гиперболизации [1, 2], хорошо зарекомендовавший себя при адаптации на архитектуру параллельных высокопроизводительных вычислительных систем.

Гиперболизацией уравнений называют добавку дополнительного члена с малым параметром в качестве коэффициента перед второй производной по времени. Наличие такого члена позволяет строить трехслойные явные схемы, обладающие лучшим условием устойчивости, чем традиционные явные схемы для параболических и нестационарных уравнений Шредингеровского типа. Для уравнений “параболического” типа, к которым относятся нестационарное уравнение Шредингера и аналогичные уравнения и системы, разностные схемы обладают очень жесткими условиями устойчивости: $\triangle t < h^2$, что по сути дела при измельчении сетки замедляет решение задачи. Помимо этого, в уравнениях типа НУШ высокие пространственные гармоники не затухают с течением времени, а имеют быстро изменяющиеся фазы, что приводит даже при “относительно мягком” условии устойчивости к явлению случайных фаз [3]. Это явление быстро распространяется до низких гармоник и превращает весь процесс вычислений в фактически случайный процесс. Как для исходного НУШ, так и для гиперболизированного варианта на заданном малом отрезке $(t_0, t_0 + \triangle t)$ построено приближенное решение задачи, которое кроме высокого порядка аппроксимации обладает ещё и тем свойством, что в начальной н конечной точках рассматриваемого малого интервала производные по времени приближенного решения совпадают с производными точного решения.

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Contents

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