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Fractal Properties of the Universe

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The large-scale structure of the Universe is revealed to be characterized by a range of power-laws. The power-laws are evidences of fractality because they may be interpreted through a conception of the Universe as an assembly of self-similar space–time domains. We accept the hypothesis that the matter of the Universe is described by the scalar charged meson field possessing the rotary symmetry. On basis of the hypothesis, the fractal cosmological model with scale invariant Lagrange's field equation and Einstein's equation permitting physical explanation of these properties is constructed. The field energy densities (which are constant) and the space–time metrics of different domains differ in constant factors only. Therefore, the space–time domains are geometrically similar and evolve similarly. Fractal properties of initial cosmological density perturbations remain and lead to presence of the fractal properties of the Universe's large-scale structure which formed from them. The nonsingular, compacted, pulsating and doubly-connected cosmological model as a partial solution for the homogeneous, isotropic and flat case is constructed. A background radiation power spectrum has been computed. The spectrum is shown to be close to the observable angular power spectrum of the SDSS-quasar distribution on the celestial sphere.

Key words and phrases: quasars, large-scale structure, fractal dimension, complex field, rotary symmetry, fractal properties of the large-scale structure, fractal cosmological model, background radiation.

1. Introduction

The large-scale structure of the Universe is the structure of the galaxy and quasar distribution in whole observable Universe's volume. The structure is sponge-like and composed of regions of a higher galaxy number density (filaments and planes of galaxies and galaxy clusters) bordering huge voids. Scales of these structures are equal to tens and hundreds of megaparsecs (Mpc).

Investigation of geometrical properties of the large-scale structure of the Universe through galaxies, galaxy clusters, quasars and the CMB temperature anisotropy is an important area at present for understanding of galaxy and Universe evolution. Estimation of distances to galaxies is complicated by low quality of spectra, by uncertainty of galaxy peculiar motion and by dependence on a cosmological model determining distance as a function of redshift. Consequently, investigation of three-dimensional galaxy distribution gives not quite reliable results. Therefore, study of statistical and topological properties of the large-scale structure through two-dimensional galaxy distribution is relevant despite obvious mistakes related to overlapping of clustering areas.

The purposes of the present work are:

- the data processing on the quasar distribution on the celestial sphere according to the SDSS DR7 catalogue and on the CMB temperature anisotropy according to WMAP-7;
- revealing of the fractal properties of the Universe's large-scale structure;
- construction of the fractal cosmological model permitting physical explanation of these properties;
- computation of a background radiation power spectrum within the fractal model framework.

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The SDSS DR7 catalogue [1] containing 105,783 quasars with redshifts $0.0645 \leqslant z \leqslant 5.4608$ was used for investigation of the quasar distribution. The catalogue was compiled by a wide-angle, narrow-angle and pencil beam surveys. For the purpose of the present work the area of the celestial sphere with the equatorial coordinates $9^h < \alpha < 16^h$, $0^\circ < \delta < 55^\circ$ covered by the wide-angle survey was chosen. The main fractal properties of the large-scale structure are represented in Section 2. These properties may be interpreted through a conception of the Universe as an assembly of self-similar space domains.

In Section 3 it is shown that the hypothesis of Gaussian (thermal) spectrum of initial cosmological density perturbations implies they possessed fractal properties, their correlation function was a power-law. The perturbations lead to the Universe's large-scale structure formation due to gravitational instability. However, the equations of the General Theory of Relativity are not scale invariant. Due to this, the fractal properties may not remain during gravitational fluctuations evolution, generally.

The hypothesis of the rotary symmetry of the charged scalar meson matter field (complex field) ψ which then possesses the form $\psi = \Psi e^{i\varphi}$ is considered in Section 4. The Universe is composed of space–time domains related by the discrete scaling: $\Psi \leftrightarrow \tilde{\Psi} / \gamma$, $\varphi \leftrightarrow \gamma\tilde{\varphi}$. On basis of this hypothesis the cosmological model with scale invariant (i.e. invariant under the scaling) Lagrange's field equation and Einstein's equation is constructed. The general solution of the equations is derived. The field energy densities E and $\tilde{E}$ (which are constant) and the space–time metrics $g_{mn}(\psi)$ and $\tilde{g}_{mn}\left(\tilde{\psi}\right)$ differ in a constant factor only. Therefore, these space–time domains are geometrically similar and evolve similarly. The fractal properties of the initial density perturbations remain and lead to presence of the fractal properties of the Universe's large-scale structure.

The partial solution of the equations in homogeneous and isotropic case is adduced in Section 5. An equation for the scale factor is derived and solved. The cosmological model is nonsingular, the Universe is found to be compacted, pulsating and doubly-connected.

In Section 6 the anisotropy of a background radiation within framework of this model is considered. When photons pass through the space domains their energies change due to gravitational frequency shift. An observer receives them and detects spots with different brightness on the celestial sphere. The power spectrum of the brightness anisotropy is calculated and revealed to be close to the observed angular power spectrum of the SDSS-quasar distribution on the celestial sphere. Only qualitatively it conforms to the angular power spectrum of CMB.

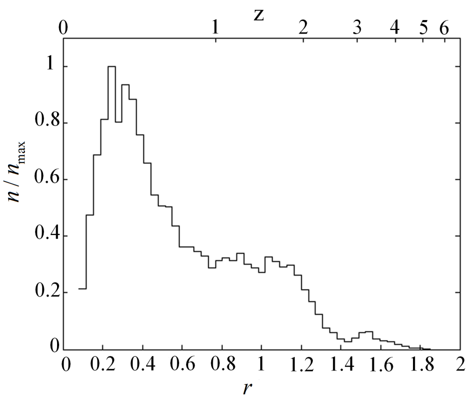
2. Fractal Properties of the Large-Scle Structure

We performed the SDSS-quasars distribution analysis in framework of the standard cosmological model with parameters: $H_0 = 70\ km \cdot s^{-1} \cdot Mpc^{-1}$ is the Hubble constant, $\Omega_M = 0.3$, $\Omega_\Lambda = 0.7$ are the dimensionless density parameters of dust and Λ-term respectively [2,3]. Cosmological distance to a quasar with redshift z (comoving distance) is determined by the formula [4]:

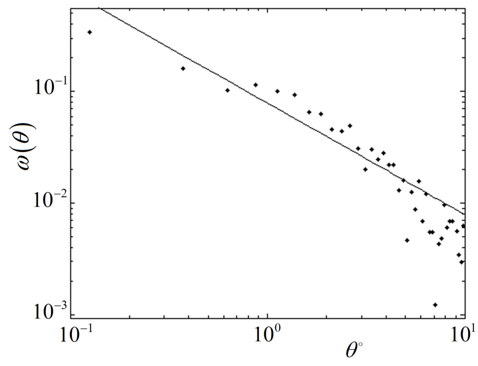
$$r = \frac{c}{H_0} \int\limits_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_M (1+z')^3 + \Omega_\Lambda}}.$$

Distances measured in the Hubble distance c/H_0 are used below.

A variation of SDSS-quasars number density in an element of comoving volume of a spatial cone of the SDSS catalogue is shown on fig.1a (the number density is normalized to a maximal value). As one can see, there was an epoch of high galaxy activity in the Universe evolution at redshift range $0.35 < z < 2.30$ which has been chosen for the further investigation. For decrease of significance of evolution effects this range has been divided into six layers, possessing qualitatively resembling shape of the luminosity function. This implies that the same common properties of the large-scale structure are displayed in every layer.



1a)



1b)

Figure 1. a) Number density of SDSS-quasars in an element of comoving volume; b) The correlation function of SDSS quasars for the layer $1.04 < z < 1.36$

1. The correlation dimension computation methods are given in papers [5–7], for example. This value characterizes quasar clumping degree and difference of the quasar distribution from a homogenous and isotropic one. The dependence of a quasar number $N(r)$ in a sphere on its radius r for the chosen redshift range is described by a power-law:

$$N\,(< r) \sim r^{d_c}, \tag{1}$$

where the exponent (correlation dimension) is equal to $d_c = 2.17$ [2,3].

The power-law (1) is usually considered as an indication of fractality of a spatial sources distribution. Similar $N\,(r)$ dependence is typical for galaxies ($d_c \approx 1.15 \div 2.25$) and it is a large-scale structure common law [7,8].

2. An analogous relation arises for two-dimensional quasar distribution on the celestial sphere between a number of quasars with angular distances less than ϑ and $\sin(\vartheta/2)$:

$$N\,(< \vartheta) \sim \left(\sin\frac{\vartheta}{2}\right)^{\alpha}, \tag{2}$$

where $\alpha = 1.49 \div 1.56$ for different redshift layers [5,6].

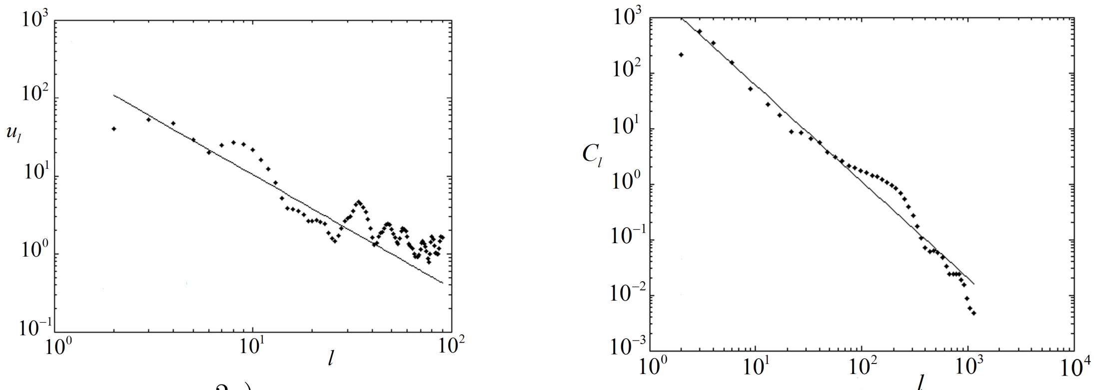
3. For each layer, the luminosity function parameters and the angular correlation function $\omega\,(\vartheta)$ of the observable quasar distribution as well as the angular power spectrum u_l for its expansion in spherical functions series have been computed using methods described in [2,3,9,10]. The power spectrum is usually used for comparison of large-scale structure formation models with observations because the power spectrum depends on an adopted cosmological model. This fact enables to estimate cosmological parameters.

Both the angular correlation function and the angular power spectrum are revealed to possess the similar shape in every layer. They may be described by the power laws

$$\omega\,(\vartheta) \sim \vartheta^{1-\gamma}, \tag{3}$$

$$u_l \sim l^{\gamma_u - 3}, \tag{4}$$

respectively with the layer-averaged parameters $\langle\gamma\rangle = 2.08$ and $\langle\gamma_u\rangle = 1.92$. Here, l is a multipole moment number. Examples of the correlation function and of the power spectrum plot for the layer $1.04 < z < 1.36$ are shown on fig.1b and fig.2a respectively [2,3].



2a) 2b)

Figure 2. Angular power spectrum of: a) SDSS quasars in the layer $1.04 < z < 1.36$; b) CMB

4. CMB photons show us the Universe as it was at the recombination epoch. The WMAP experiment proves convincingly that CMB temperature angular fluctuations exist. The angular power spectrum for the expansion of the CMB temperature anisotropy in a spherical functions series is a power-law as well [2, 3]:

$$C_l \sim l^{-1.74}. \tag{5}$$

The power spectrum plot is shown on fig.2b. It qualitatively corresponds to the power spectrum of the quasar distribution (4): $u_l \sim l^{-1.08}$.

5. For estimation of quasar clump sizes the spherical wavelet transform [11] was used. The wavelet analysis permits to single out quasar clump areas: the wavelet coefficients are positive in these areas. Large quasar groups are discovered in papers [2, 3]. For study of fractal properties of the large quasar groups we should determine a number of groups N_c with a certain angular size ϑ_c. Their relation has been revealed to be a power-law:

$$N_c(\vartheta_c) \sim \vartheta_c^{-2.08}. \tag{6}$$

These power-laws are evidences of fractality because they may be interpreted through a conception of the Universe as an assembly of self-similar space domains. These large quasar groups mark the domains. Let each i-th domain with a size r_i contains m_i pairs of objects (galaxies or quasars). Self-similarity implies the sizes of domains and numbers of pairs form geometric progressions:

$$r_i = qr_{i-1} = r_0 q^{i-1}, \quad m_i = pm_{i-1} = m_0 p^{i-1},$$

where $i = 1, 2, 3, ...$ and $p > 1$, $q > 1$ are constant numbers. The total number of galaxies in the sample and the radius of the sample volume are equal

$$N = 2\sum_{i=1}^{N/2} m_i = 2m_0 \sum_{i=1}^{N/2} p^{i-1} = 2m_0 \frac{p^{N/2} - 1}{p - 1} \approx 2m_0 p^{\frac{N}{2}-1}, \quad r_{N/2} = r_0 q^{\frac{N}{2}-1}.$$

Then we obtain the relation (1):

$$N = 2m_0 \left(\frac{r_{N/2}}{r_0}\right)^{\frac{\ln p}{\ln q}} \sim \left(r_{N/2}\right)^{\frac{\ln p}{\ln q}}.$$

3. Fractal Spectrum of Initial Cosmological Density Perturbations

We accept a hypothesis that these fractal properties are consequences of fractal properties of initial matter density perturbations which further led to star, galaxy and cluster formation due to gravitational instability [12, 13]. Now we observe traces of these fractal properties through quasars.

This interpretation follows from the hypothesis of Gaussian (thermal) spectrum of the initial density perturbations. Let's consider a spherical volume V containing mass M in a continuous medium. Probability of appearance of thermal density fluctuation near this mass is defined by the formula [14, 15]

$$W \sim \exp\left[-\frac{c_v}{2}\left(\frac{\delta T}{T}\right)^2 - \frac{M}{2kT}\left(\frac{\partial P}{\partial \rho}\right)_T\left(\frac{\delta V}{V}\right)^2 - \frac{R_{\min}}{kT}\right],$$

where ρ is medium density, P is pressure, c_v is medium heat capacity at constant volume, δT and δV are independent fluctuations of temperature and volume respectively, $R_{\min}$ is minimal work necessary for reversible removal of a mass δM for a distance δr in the gravity field of the mass M. In case of radial displacement the work is equal to

$$R_{\min} \approx G\frac{M \cdot \delta M}{r^2}\delta r = \frac{4\pi}{3}G\rho r \cdot \delta M \cdot \delta r$$

in Newtonian approximation, where $\delta M = 4\pi\rho r^2 \delta r$ is a mass of a spherical layer. In this case, probability of a thermal fluctuation is equal

$$W = \frac{1}{2\pi\Delta_T\Delta_r}\exp\left[-\frac{1}{2{\Delta_T}^2}\left(\frac{\delta T}{T}\right)^2 - \frac{1}{2{\Delta_r}^2}\left(\frac{\delta r}{r}\right)^2\right],$$

where the variances are equal to

$${\Delta_T}^2 = {c_v}^{-1}, \quad {\Delta_r}^2 = \left\{\frac{12\pi\rho}{kT}\left[\left(\frac{\partial P}{\partial \rho}\right)_T r^3 + \frac{8\pi G\rho}{9}r^5\right]\right\}^{-1}.$$

The root-mean-square relative density fluctuation (fluctuation spectrum) are equal to

$$\Delta_\rho = \sqrt{\left\langle\left(\frac{\delta\rho}{\rho}\right)^2\right\rangle} = 3\sqrt{\left\langle\left(\frac{\delta r}{r}\right)^2\right\rangle} = 3\left\{\frac{12\pi\rho}{kT}\left[\left(\frac{\partial P}{\partial \rho}\right)_T r^3 + \frac{8\pi G\rho}{9}r^5\right]\right\}^{-1/2}. \quad (7)$$

At spatial scales for which pressure gradients are important, i.e. when the first term in the brackets dominates, we have the "white noise" spectrum (Zel'dovich–Harrison spectrum): $\Delta_\rho \sim r^{-1{,}5}$. At large scales for which gravity effects are important, i.e. when the second term dominates, one has the following spectrum $\Delta_\rho \sim r^{-2{,}5}$. The spectrum (7) is scale invariant because the fraction $\frac{\delta r}{r}$ doesn't change under scale transformations. This thermal fluctuations spectrum is an example of fractal spectrum.

The root-mean-square relative density fluctuation is an estimate of the correlation function according to the correlation function definition [10]: $\xi \approx \Delta_\rho$. For a random fluctuation average number of neighbour fluctuations within distance less then r may be estimated as

$$\langle N\rangle \approx 4\pi \langle n\rangle \int\limits_0^r (1+\xi)\,\tilde{r}^2 d\tilde{r},$$

where n is a mean fluctuations number density. In case of fluctuations clumping and $\xi \geqslant 1$ there are fractal laws like (1): $\langle N\rangle \sim r^{1,5}$ for the white noise and $\langle N\rangle \sim r^{0,5}$ at large scales.

The observable correlation dimension value $d_c \approx 1{,}15 \div 2{,}25$ may follow from the spectrum

$$\sqrt{\left\langle \left(\frac{\delta\rho}{\rho}\right)^2 \right\rangle} \sim r^{-1,85} \div r^{-0,75},$$

therefore, it permits not only the white noise spectrum.

Thereby, the fractal laws (1)–(6) are expected to exist in Newtonian approximation. However, it is not quite so in the general theory of relativity because Einstein's tensor is not invariant under scale transformation of the Riemannian space–time [16, 17]. Generally, if the large-scale structure evolution is described by Einstein's gravity theory the fractal properties may not conserve, even if the initial fluctuations had the thermal spectrum.

4. General Solution of Lagrange's and Einstein's Equations for the Complex Field with the Rotary Symmetry

We consider a hypothesis that the matter of the Universe is described by the charged scalar meson field (complex field) which possesses the rotary symmetry [12,13]:

$$\psi\psi^* = \Psi^2 = \text{const}, \tag{8}$$

(where the asterisk denotes complex conjugation and Ψ is the field amplitude related to the field charge $Q \sim \Psi^2$). The dynamic system of gravity and complex ψ fields is described by Einstein-Hilbert action within general relativity framework:

$$S = -\frac{c^3}{16\pi G}\int \left(R - \frac{8\pi G}{c^4}L\right)\sqrt{-g}\,\mathrm{d}^4x,$$

where G is Newton's gravity constant, R is scalar curvature, $g < 0$ is determinant of the metric tensor. We use the following form of complex field Lagrangian:

$$L = \frac{1}{hc}\left(g^{ik}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi^*}{\partial x^k} - U_0\psi\psi^*\right), \tag{9}$$

where U_0 is a constant field potential parameter, h is Planck's constant, c is light velocity. The field Lagrange's equation is

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^k}\left(\sqrt{-g}g^{ik}\frac{\partial\psi}{\partial x^i}\right) = -\frac{\partial U}{\partial\psi^*}. \tag{10}$$

In Einstein's equation

$$R_i^k - \frac{1}{2}R\delta_i^k = \kappa T_i^k \tag{11}$$

energy-momentum tensor of the complex field is equal

$$T_i^k = \frac{\partial\psi}{\partial x^i}\frac{\partial L}{\partial\left(\frac{\partial\psi}{\partial x^k}\right)} + \frac{\partial\psi^*}{\partial x^i}\frac{\partial L}{\partial\left(\frac{\partial\psi^*}{\partial x^k}\right)} - \delta_i^k L = \frac{1}{hc}g^{kl}\left(\frac{\partial\psi}{\partial x^l}\frac{\partial\psi^*}{\partial x^i} + \frac{\partial\psi}{\partial x^i}\frac{\partial\psi^*}{\partial x^l}\right) - \delta_i^k L,$$

where R_i^k is Ricci tensor, $\kappa = 8\pi G\,/\,c^4$ is Einstein's gravity constant, δ_i^k is the Kronecker delta.

It's easily seen that a field of the form of

$$\psi = \Psi \mathrm{e}^{i\varphi}, \quad \psi^* = \Psi \mathrm{e}^{-i\varphi}, \tag{12}$$

satisfies the condition (8), where field phase $\varphi\left(x^i\right)$ is a differentiable function. Then the equation (10) is satisfied by the general solution

$$g_{ik} = \frac{1}{U_0}\left(4\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^k} + \frac{\partial\varphi}{\partial x^i}a_k + \frac{\partial\varphi}{\partial x^k}a_i\right), \tag{13}$$

$$\Gamma_{kl}^i = \frac{1}{U_0}\frac{\partial^2\varphi}{\partial x^k \partial x^l}\left(g^{im}\frac{\partial\varphi}{\partial x^m} + a^i\right),$$

where derivative $\frac{\partial\varphi}{\partial x^i}$ and covariant vector a_i must satisfy equations

$$g^{ik}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^k} = U_0, \quad g^{ik}\left(\frac{\partial\varphi}{\partial x^i}\right)_{;k} = 0, \tag{14}$$

$$a_{i;k} = 0, \quad a_i a^i = -3U_0, \quad \frac{\partial\varphi}{\partial x^i}a^i = 0.$$

Covariant a_i and contravariant a^k vectors satisfy equations:

$$\frac{\partial a_i}{\partial x^l} = -3\frac{\partial^2\varphi}{\partial x^i \partial x^l}, \quad \frac{\partial a^k}{\partial x^l}a_k = 3a^k\frac{\partial^2\varphi}{\partial x^k \partial x^l}. \tag{15}$$

This general solution contains isotropic and anisotropic solutions because the derivative may depend on direction.

The Lagrangian (9) is equal to 0. The Ricci tensor and the energy-momentum tensor for this solution are equal to:

$$R_{ik} = \frac{\partial\Gamma_{ik}^l}{x^l} - \frac{\partial\Gamma_{il}^l}{x^k} + \Gamma_{ik}^l\Gamma_{lm}^m - \Gamma_{il}^m\Gamma_{km}^l = \frac{1}{U_0}\left(\frac{\partial^2\varphi}{\partial x^i \partial x^k}\frac{\partial a^l}{\partial x^l} - \frac{\partial^2\varphi}{\partial x^i \partial x^l}\frac{\partial a^l}{\partial x^k}\right) +$$

$$+\frac{1}{{U_0}^2}\left(\frac{\partial^2\varphi}{\partial x^i \partial x^k}\frac{\partial^2\varphi}{\partial x^m \partial x^l} - \frac{\partial^2\varphi}{\partial x^i \partial x^l}\frac{\partial^2\varphi}{\partial x^m \partial x^k}\right)\left(g^{ln}\frac{\partial\varphi}{\partial x^n} + a^l\right)a^m,$$

$$T_{ik} = \frac{2}{hc}\Psi^2\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^k}.$$

Functions $\frac{\partial\varphi}{\partial x^i}$, a_i, a^i are derived from equations (11) and (15).

The energy density is constant and positive:

$$E = \frac{1}{hc}\left(g^{ik}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi^*}{\partial x^k} + U_0\psi\psi^*\right) = \frac{2}{hc}U_0\Psi^2 > 0.$$

Therefore, the solution (12)–(13) corresponds to a stationary field condition.

Let the Universe is composed of space–time domains related by the discrete scaling:

$$\Psi \leftrightarrow \tilde{\Psi} \,/\, \gamma, \quad \varphi \leftrightarrow \gamma\tilde{\varphi}. \tag{16}$$

The Christoffel symbols and the Ricci tensor R_{ik} don't change under this transformation. Vector a_i, metric tensor, enrgy density, mixed components of energy-momentum and Ricci tensors are multiplied by constant factors:

$$a_i \leftrightarrow \gamma\tilde{a}_i, \quad g_{ik}\left(\psi\right) \leftrightarrow \gamma^2 \frac{\tilde{U}_0}{U_0}\tilde{g}_{ik}\left(\tilde{\psi}\right), \tag{17}$$

$$E \leftrightarrow \frac{1}{\gamma^2}\frac{U_0}{\tilde{U}_0}\tilde{E}, \quad R_i^k \leftrightarrow \frac{1}{\gamma^2}\frac{U_0}{\tilde{U}_0}\tilde{R}_i^k, \quad T_i^k \leftrightarrow \frac{1}{\gamma^2}\frac{U_0}{\tilde{U}_0}\tilde{T}_i^k.$$

Therefore, Einstein's and Lagrange's equations don't change. These domains are geometrically similar and evolve similarly. The fractal properties of the initial density perturbations remain and lead to presence of the fractal properties of the Universe's large-scale structure.

The phase path of the fields ψ and ψ^* is a circle (8):

$$\psi\psi^* = {\psi_1}^2 + {\psi_2}^2 = \Psi^2, \quad \psi = \psi_1 + i\psi_2, \quad \psi^* = \psi_1 - i\psi_2.$$

The function φ is a degree of rotation round the circle. Length of a circle arc i.e. an interval of a set $\{\psi_1, \psi_2\}$ is equal

$$\mathrm{d}F^2 = \left(\mathrm{d}\psi_1\right)^2 + \left(\mathrm{d}\psi_2\right)^2 = \mathrm{d}\psi\mathrm{d}\psi^* = \Psi^2 \frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^k}\mathrm{d}x^i\mathrm{d}x^k.$$

The relation between the phase space interval $\mathrm{d}F$ and the space–time interval $\mathrm{d}s$ is

$$\mathrm{d}F^2 = \frac{1}{4}\Psi^2\left[U_0\mathrm{d}s^2 - \left(a_i\frac{\partial\varphi}{\partial x^k} + a_k\frac{\partial\varphi}{\partial x^i}\right)\mathrm{d}x^i\mathrm{d}x^k\right]. \tag{18}$$

The first equation (15) has the following solution:

$$a_i = -3\frac{\partial\varphi}{\partial x^i} + d_i,$$

where d_i is a constant vector $\left(\dfrac{\partial d_i}{\partial x^k} = 0\right)$. The expression (18) shows that the vector d_i may be chosen so that the phase space interval is proportional to the time interval: $\mathrm{d}F \sim \mathrm{d}t$. Therefore, the solution (12)–(13) may describe a time-pulsating cosmological model. As energy density of the system is finite this model must be nonsingular [12,13].

5. Partial Solution for Homogeneous and Isotropic Case

Let's consider a partial solution corresponding to a homogeneous, isotropic and flat case $\left(\dfrac{\partial\varphi}{\partial x^1} = \dfrac{\partial\varphi}{\partial x^2} = \dfrac{\partial\varphi}{\partial x^3} = \dfrac{\partial\varphi}{\partial x}\right)$: $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2\left[\left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2\right]$.

Mixed Ricci tensor components and scalar curvature are equal:

$$R_0^0 = -\frac{1}{c^2}\left[3\left(\frac{a_t}{a}\right)_t + 3\left(\frac{a_t}{a}\right)^2\right], \quad R_1^1 = R_2^2 = R_3^3 = -\frac{1}{c^2}\left[\left(\frac{a_t}{a}\right)_t + 3\left(\frac{a_t}{a}\right)^2\right],$$

$$R = -\frac{1}{c^2}\left[6\left(\frac{a_t}{a}\right)_t + 12\left(\frac{a_t}{a}\right)^2\right],$$

where a is a scale factor of the model, index t denotes partial derivative with cosmological time. We may add a total derivative with time of any function to the Lagrangian:

$$L = \frac{1}{hc}\left(g^{ik}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi^*}{\partial x^k} - U_0\psi\psi^*\right) + \frac{\mathrm{d}F}{\mathrm{d}t},$$

and Einstein's equation (11) with the energy-momentum tensor

$$T_{ik} = \frac{2}{hc}\Psi^2\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^k} - g_{ik}\frac{\mathrm{d}F}{\mathrm{d}t}$$

comes to the following two equations:

$$3\left(\frac{a_t}{a}\right)^2 = \frac{2\kappa}{hc}\Psi^2\left(\frac{\partial\varphi}{\partial t}\right)^2 - \kappa c^2\frac{\mathrm{d}F}{\mathrm{d}t},$$

$$\frac{1}{c^2}\left[2\left(\frac{a_t}{a}\right)_t + 3\left(\frac{a_t}{a}\right)^2\right] = -\frac{2\kappa}{hc}\Psi^2\frac{1}{a^2}\left(\frac{\partial\varphi}{\partial x}\right)^2 - \kappa\frac{\mathrm{d}F}{\mathrm{d}t}. \tag{19}$$

Lagrange's equation (10) turns into two equations:

$$\left(\frac{1}{c}\frac{\partial\varphi}{\partial t}\right)^2 - \frac{3}{a^2}\left(\frac{\partial\varphi}{\partial x}\right)^2 = U_0,$$

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{3}{a^2}\frac{\partial^2\varphi}{\partial x^2} + \frac{3}{c^2}\frac{a_t}{a}\frac{\partial\varphi}{\partial t} = 0. \tag{20}$$

Four equations (19)–(20) determine four functions: a, $\frac{\partial\varphi}{\partial t}$, $\frac{\partial\varphi}{\partial x}$, $\frac{\partial^2\varphi}{\partial x\partial t}$. Equations (19) and the first equation (20) lead to the equation determining scale factor a:

$$\left(\frac{a_t}{a}\right)_t + 2\left(\frac{a_t}{a}\right)^2 - \frac{\kappa c}{3h}U_0\Psi^2 = -\frac{2\kappa c^2}{3}\frac{\mathrm{d}F}{\mathrm{d}t}. \tag{21}$$

The hyperbolic solution of this equation corresponds to the case of zero field Lagrangian and $\frac{\mathrm{d}F}{\mathrm{d}t} = 0$: $a = a_0\sqrt{\cosh\left(\frac{t}{\tau}\right)}$, where $\tau = \left(\frac{2\kappa c}{3h}U_0\Psi^2\right)^{-\frac{1}{2}}$. If we choose $\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{1}{hc}U_0\Psi^2$ the equation has the periodic solution

$$a = a_0\sqrt{\cos\left(\frac{t}{\tau} + \phi\right)} \tag{22}$$

with period $2\pi\tau$. The scale factor (22) turns into zero at the moment t_* when $\frac{t_*}{\tau} + \phi = \frac{\pi}{2} \pm \pi n$. The solution (22) is not singular within the interval $0 \leqslant \frac{t}{\tau} \leqslant 2\pi$ if $\frac{t_*}{\tau} \geqslant 2\pi$. This condition permits to choose the phase: $\frac{\pi}{2} \pm \pi n - \phi > 2\pi$. Therefore the model is not singular if

$$\phi < -\frac{3\pi}{2} \pm \pi n. \tag{23}$$

The equation (21) has the integral for the periodic solution (22) that is a map of the phase path (8) in the space–time:

$$\left(\frac{a_0}{a}\right)^4 - 4\tau^2\left(\frac{a_t}{a}\right)^2 = 1.$$

Derivatives of the phase coordinate φ are equal:

$$\frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 = \frac{1}{4}U_0\left[\left(\frac{a_0}{a}\right)^4 + 1\right], \quad \frac{1}{a^2}\left(\frac{\partial\varphi}{\partial x}\right)^2 = \frac{1}{12}U_0\left[\left(\frac{a_0}{a}\right)^4 - 3\right].$$

One can define the metric tensor in the form analogous to the general definition (13) through these expressions. Further, the expressions imply that the parameter a_0 is a maximal scale factor value for the solution (22): $a \geqslant \frac{1}{\sqrt[3]{3}}a_0$. The comoving radial coordinate of the horizon is equal:

$$r(t) = \int\limits_0^t \frac{c\mathrm{d}t}{a} = \frac{2c\tau}{a_0}F\left(\frac{\frac{t}{\tau}+\phi}{2}, 2\right),$$

where $F\left(\frac{\frac{t}{\tau}+\phi}{2}, 2\right)$ is an elliptic integral of the first kind possessing recurring values with period $2\pi\tau$.

Since the horizon comoving radial coordinate values are repeated, a model with pulsating space–time corresponds to the solution (22). This model is compacted, i.e. the total space volume is finite and the evolution in time is a periodic process of space expansion and contraction. In the presence of the phase restriction (23) the space contracts to minimal nonzero volume. The two-dimensional analogy of such space–time is a torus with variable thickness where parallels are lines of time (lines of constant space coordinates) and meridians are space coordinate lines. Analogous compacted model has been constructed in the paper [18] and possible astrophysical consequences of space volume finiteness are discussed there. It has been showed there that dynamical entropy of the complex field is increasing during space pulsating [12, 13].

6. Anisotropy of a Background Radiation

Light signals transfer along isotropic geodesics of space–time (13). An isotropic vector satisfies equations:

$$p_i p^i = 0, \quad p_{i;k} = 0.$$

Under solution (13) these equations are satisfied by the vector

$$p_i = \sqrt{3}\frac{\partial\varphi}{\partial x^i} + a_i. \tag{24}$$

Under the discrete scaling (16) this vector transforms by the rule:

$$p_i \leftrightarrow \gamma\tilde{p}_i. \tag{25}$$

Therefore, an isotropic vector remains isotropic under the scaling.

Self-similarity of space–time domains described by the solution (13) permits consideration of an assembly of such domains because transition from any domain to another resolves itself into dilatation or compression of an interval. Directions of isotropic geodesics don't change. Let's consider transfer of photons of a background radiation within such fractal structure. We take into account gravitational influence of every domain only and leave out of account direct interaction with substance (absorption and scattering). In this case, energy of transferring photon changes due to gravitational frequency shift. Let the background radiation is homogeneous and isotropic, and an observer receives photons which passed through the fractal structure. Brightness differs from one domain to another because energy of photons changes being multiplied by a scaling factor for each domain according to the expression (25). Therefore, the observer notices spots of different brightness in the distribution of the background radiation brightness on the celestial sphere.

An angular distance between a center of the j-th spot and any point of the spot is equal

$$\vartheta = \arccos\left(\cos\delta_j \cos\delta \cos\left(\alpha - \alpha_j\right) + \sin\delta_j \sin\delta\right),$$

where (δ, α) and (δ_j, α_j) are declinations and right ascensions of the point and of the spot center respectively. The whole spot is described by Legendre polynomial of degree l_j and its representation through spherical harmonics (the addition theorem):

$$P_{l_j}\left(\cos\vartheta_j\right) = \frac{4\pi}{2l_j+1} \sum_{m=-l_j}^{m=+l_j} Y^{l_j}_{-m}\left(j\right) Y^{l_j}_{m}\left(\alpha, \delta\right). \tag{26}$$

The multipole number l_j and the spot's size ϑ_j are related by the expression $\theta_j \approx \dfrac{\pi}{l_j} = \dfrac{180^\circ}{l_j}$. The polynomial (26) is of the bell shape in the range $0 \leqslant \vartheta_j \leqslant \theta_j$ with maximum equal to 1 when $\vartheta_j = 0$.

Summarized brightness distribution of the background radiation may be expressed now as

$$F\left(\alpha, \delta\right) = \sum_{j=1}^{N} \gamma_j P_l\left(j, \alpha, \delta\right), \tag{27}$$

where the scaling factor γ_j takes into account change of photons' energy when leaving the j-th domain. For determination of the power spectrum of the background radiation brightness distribution anisotropy the function (27) should be expanded in a spherical harmonics series:

$$F\left(\alpha, \delta\right) = \sum_{m,l} a_{ml} Y^{l}_{m}\left(j, \alpha, \delta\right).$$

The power spectrum is a function $C_l = \dfrac{1}{2l+1} \sum_{m=-l}^{m=l} \left|a_{ml}\right|^2$. Using definitions (26) and (27) we can determine the serial expansion coefficients a_{ml}:

$$a_{ml} = \frac{4\pi}{2l+1} \sum_{j=1}^{N} \gamma_j \sum_{m=-l}^{m=l} Y^{l}_{-m}\left(j\right).$$

The normalization condition for the spherical harmonics on a whole sphere is used here:

$$\int Y^{l_j}_{-m} Y^{l}_{m} d\Omega = \delta_{ll_j}.$$

In the simplest case of symmetric spots, the weight factors γ_j are proportional to the spot's angular size, $\gamma_j \sim \theta_j \approx \dfrac{\pi}{l_j}$, and determine the dependence of the expansion coefficients on the multipole numbers: $a_{ml} \sim \sum_{j=1}^{N} \gamma_j \sim \sum_{j=1}^{N} \dfrac{1}{l_j}$. In this case, the power spectrum may be close to the power-law:

$$C_l \sim \frac{1}{2l+1}\left(\sum_{j=1}^{N}\frac{1}{l_j}\right)^2 \sim l^{-1}. \tag{28}$$

The model power spectrum (28) closely corresponds to the power spectrum of SDSS-quasars (4).

7. Conclusion

The main results of the present work are following.

– Revealing of fractal properties of the large-scale structure of the Universe which are described by the power-laws (1)–(6). The fractal dimension value for the large quasar group distribution in sizes is compared to that of polygonal path of Brownian particle (length distribution of segments). This analogy indicates that initial density perturbations from which large quasar groups arise, apparently, had a thermal spectrum.
– Interpretation of these properties through a conception of the Universe as an assembly of self-similar space–time domains related by the scaling (16).
– Construction on basis of this hypothesis of the fractal cosmological model with scale invariant (i.e. invariant under the scaling) Lagrange's and Hilbert-Einstein equations permitting physical explanation of these properties.
– Construction of the nonsingular, compacted, pulsating and doubly-connected cosmological model as a partial solution for the homogeneous, isotropic and flat case.
– Computation of a background radiation power spectrum within the fractal cosmological model. The spectrum is shown to be close to the observable angular power spectrum of the SDSS-quasar distribution on the celestial sphere. It differs from the average power spectrum of the observable CMB anisotropy (WMAP-7). This fact will be a subject of investigation in further works.

Computation of the cosmological density perturbation evolution due to gravitational instability within fractal cosmological model framework will be the next step in this investigation.

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Фрактальные свойства Вселенной

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Обнаружено, что крупномасштабная структура Вселенной характеризуется рядом степенных зависимостей. Эти степенные законы являются признаками фрактальности, потому что их можно объяснить, если представить Вселенную как совокупность самоподобных пространственно-временных областей. Выдвигается гипотеза, что материя Вселенной описывается скалярным заряженным мезонным полем с вращательной симметрией. На основе этой гипотезы построена фрактальная космологическая модель с масштабно инвариантными уравнениями Лагранжа и Эйнштейна, которая позволяет дать физическую трактовку фрактальных свойств крупномасштабной структуры. Плотности энергии (являющиеся постоянными) и метрические тензоры различных пространственно-временных областей отличаются лишь постоянным множителем. Следовательно, эти области геометрически подобны и эволюционируют одинаково. Фрактальные свойства начальных космологических флуктуаций плотности сохраняются и приводят к наличию фрактальных свойств у крупномасштабной структуры, которая из них образовалась. Построена несингулярная, компактная, пульсирующая и двусвязная космологическая модель как частное решение для однородного, изотропного и плоского случая. Выведен спектр мощности фонового излучения в данной модели. Этот спектр близок к наблюдаемому угловому спектру мощности распределения SDSS-квазаров на небесной сфере.

Ключевые слова: квазары, крупномасштабная структура, фрактальная размерность, комплексное поле, вращательная симметрия, фрактальные свойства крупномасштабной структуры, фрактальная космологическая модель, фоновое излучение.