UDC 517.925.51

The Algorithm of Reducibility of Inhomogeneous Systems with Polynomially Periodic Matrix on the Basis of Spectral Method

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The paper is devoted to investigation of the class of linear and quasi-linear systems of ordinary differential equations, the matrix of which can be characterized as polynomially periodic. The main aim of this article is to generate a new algorithm of their splitting in order to create equivalent sets with almost diagonal matrix that are simpler to analyze. Another objective is formulating and proving of sufficient stability conditions or asymptotic stability of their trivial decision. The question is topical since the analysis of a considered class of non-autonomous systems with the use of known methods (for example, the method of functions of Lyapunov) is complicated. In addition, the usage of spectral and other methods while solving non-uniform sets might cause extra difficulties. The authors of the paper develop an analytical method which appears to be a summary of known classical theorems. At the heart of the offered algorithm of reducibility lies one of options of splitting method, which is conducted by a spectrum of a defining matrix in studied non-autonomous system (taking into account its splitting on diagonal and non-diagonal part) lies. The present article shows possibilities of reducibility of sets of the specified class depending on structure of a matrix spectrum. This simplifies the analysis of questions of stability. Theorems of stability or asymptotic stability of the trivial decision of the transformed equivalent systems and the relevant initial systems that is development and generalization of a spectral method of research of stability for the class of non-autonomous systems considered in work are proved.

Key words and phrases: quasi-linear system, spectral method, polynomially periodic matrix, splitting method, stability.

1. Introduction

A big class of ordinary differential equations (ODE) sets with a polynomially periodic matrix, that arise at the description of various physical systems or technical processes, is discovered in the article. The usage of prominent methods [1,2] in studying stability of sets appears to be either fundamentally impossible, or slightly effective.

Let us consider the non-linear inhomogeneous set of ODE:

$$\begin{gathered} \dot{x} = t^m A(t)x + f(x,\, t); \quad x(t_0) = x_0, \\ x,\, f \in \mathbb{R}^n; \quad m \geqslant 0 \quad t_0 > 1; \quad f(0,\, t) \equiv 0, \end{gathered} \tag{1}$$

for which matrix $A(t)$ is polynomially periodic and can be presented in the form of series

$$A(t) = \sum_{k=0}^{\infty} A_k(t) t^{-k},$$

where $A_k(t)$ — are rather smooth in some area $\Omega = \{|x| \leqslant R,\, t \geqslant 0\}$ T-periodical square matrices.

The problem of search of conditions, that are imposed on the matrix spectrum $A_0(t)0$ is imposed, by which with the help of a certain algorithm it is possible to transform the system od ODE (1) to a system of differential matrix equation of a

Received 8th August, 2013.

simpler type, that enables us to formulate constructive conditions of stability or asymptotic stability of an isolated trivial solution of the system (1).

The paper contains the generalized splitting method [3, 4], for the sake of convenance let us represent any square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

in the form of a sum of a diagonal and without-diagonal matrices:

$$\overline{A} = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} = \operatorname{diag}\{a_{11}, \dots , a_{nn}\}, \quad \overline{\overline{A}} = A - \overline{A}.$$

2. About Reducibility of Quasi-Linear Non-Autonomous Inhomogeneous Systems with a Polynomially Periodic Matrix

Let us firstly consider the case $m \geqslant 1$.

Theorem 1. *Let for a non-autonomous quasi-linear set of ODE*

$$\begin{gathered} \dot{x} = t^m A(t)x + f(x, t); \quad x(t_0) = x_0, \\ x, f \in \mathbb{R}^n; \quad m \geqslant 1; \quad t_0 > 1; \quad f(0, t) \equiv 0, \end{gathered} \tag{2}$$

the following conditions are met:

1) matrix series $A(t) = \sum\limits_{k=0}^{\infty} A_k(t)t^{-k}$ of T-periodical and rather smooth square matrices $A_k(t)$ converges by $t \geqslant t_0 > 1$;
2) the spectrum $\{\lambda_{0j}(t)\}_1^n$ of matrix $A_0(t)$ satisfies inequalities:

$$\sigma_{jk}(t) \equiv \lambda_{0j}(t) - \lambda_{0k}(t) \neq 0 \quad j, k = 1, 2, \dots , n; \quad j \neq k; \quad t \geqslant t_0 > 1, \tag{3}$$

i.e. matrix $A_0(t)$ has a stable aliquant spectrum.

Then the set (2) *with the help of substitution, which is non-degenerate by rather high $t > t_0 > 1$,*

$$x = S_0(t)H_{(N)}(t)z,$$

$$S_0^{-1}(t)A_0(t)S_0(t) = \Lambda_0(t) = \operatorname{diag}\{\lambda_{01}(t), \dots , \lambda_{0n}(t)\}, \quad H_{(N)}(t) = E + \sum_{k=1}^{N} \overline{\overline{H}}_k(t)t^{-k}$$

becomes a non-autonomous set:

$$\begin{gathered} \dot{z} = t^m Q(t)z + g(z, t); \quad z(t_0) = z_0, \\ Q(t) = \Lambda_{(N)}(t) + t^{-N-1}G_{(N+1)}(t), \quad \Lambda_{(N)}(t) = \sum_{k=0}^{N} \Lambda_k(t)t^{-k}; \\ \|G_{(N+1)}(t)\| \leqslant C, \quad N \geqslant m. \end{gathered} \tag{4}$$

By all that diagonal $\Lambda_k(t)$ and "without-diagonal" $\overline{\overline{H}}_k(t)$ T-periodical matrices ($k = 1, 2, \ldots, N$) are unambiguously defined by an iterative method.

Proof. Let us consistently make two replacements. At the beginning after a non-degenerate T-periodical substitute $x = S_0(t)y$ we will get a set

$$\dot{y} = t^m B(t)y + h(y, t); \quad y(t_0) = y_0,$$

$$B(t) = \Lambda_0(t) + \sum_{k=1}^{\infty} B_k(t)t^{-k}, \quad B_k = S_0^{-1}A_kS_0.$$

Then we will get a set (4) after a non-degenerate by rather high $t > t_0 > 1$ polynomially periodical substitution $y = H_{(N)}(t)z$. This is possible, if matrices $B(t)$, $Q(t)$ and $H_{(N)}(t)$ satisfy the following differential matrix equation:

$$\dot{H}_{(N)} = t^m \left(B(t)H_{(N)}(t) - H_{(N)}(t)Q(t)\right).$$

Making coefficients by the same power equal t, we will get matrix equations:

$$t^{m-k}: \quad \Lambda_0(t)\overline{\overline{H}}_k(t) - \overline{\overline{H}}_k(t)\Lambda_0(t) = \Lambda_k(t) - P_k(t); \quad k = 1, \ldots, N;$$

$$P_1(t) = B_1(t); \quad P_k = B_k(t) + \sum_{j=1}^{k-1} \left(B_j(t)\overline{\overline{H}}_{k-j}(t) - \overline{\overline{H}}_{k-j}(t)\Lambda_j(t)\right); \quad k = 2, \ldots, N;$$

$$t^{-k}: \quad \Lambda_0(t)\overline{\overline{H}}_{m+k}(t) - \overline{\overline{H}}_{m+k}(t)\Lambda_0(t) = \Lambda_{m+k}(t) - P_{m+k}(t);$$

$$P_{m+k} = B_{m+k}(t) + \sum_{j=1}^{m+k-1} \left(B_j(t)\overline{\overline{H}}_{m+k-j}(t) - \overline{\overline{H}}_{m+k-j}(t)\Lambda_j(t)\right) - \dot{\overline{\overline{H}}}_k(t) + k\overline{\overline{H}}_k(t);$$

$$k = 1, \ldots, N - m.$$

We can unambiguously define all T-periodical diagonal matrices out of these linear algebraic matrix equations $\Lambda_k(t)$ and "without-diagonal" matrices $\overline{\overline{H}}_k(t)$ following the algorithm:

$$\Lambda_k(t) = \overline{P}_k(t); \quad \overline{\overline{H}}_k(t) = \{h_{ijk}(t)\};$$

$$P_k = \{p_{ijk}(t)\}; \quad h_{ijk}(t) = -p_{ijk}(t)/\sigma_{ij}(t), \quad i \neq j; \quad i, j = 1, \ldots, n; \quad k = 1, \ldots, N.$$

The theorem is proved. □

Let us consider the next case $m = 0$.

Theorem 2. *Let for a non-autonomous quasi-linear system of ODE*

$$\begin{gathered} \dot{x} = A(t)x + f(x, t); \quad x(t_0) = x_0, \\ x, f \in \mathbb{R}^n; \quad t \geqslant t_0 > 1; \quad f(0, t) \equiv 0, \end{gathered} \tag{5}$$

the following conditions are fulfilled:

1) *matrix series $A(t) = A_0 + \sum\limits_{k=1}^{\infty} A_k(t)t^{-k}$ of T-periodical and rather smooth matrices $A_k(t)$ converges by $t \geqslant t_0 > 1$;*
2) *spectrum $\{\lambda_{0j}\}_1^n$ of matrix A_0 satisfies inequations:*

$$\sigma_{jk} \equiv \lambda_{0j} - \lambda_{0k} \neq i\frac{2\pi q}{T}, \quad j, k = 1, 2, \ldots, n; \quad j \neq k; \quad q = 0, \pm 1, \pm 2, \ldots. \tag{6}$$

Then the system (5) *with the help of polynomially periodical non-degenerate by rather high* $t > t_0 > 1$ *substitute*

$$x = S_0 H_{(N)}(t) z,$$

$$S_0^{-1} A_0 S_0 = \Lambda_0 = \operatorname{diag}\{\lambda_{01}, \ldots, \lambda_{0n}\}, \quad H_{(N)}(t) = E + \sum_{k=1}^{N} H_k(t) t^{-k}$$

becomes a non-autonomous system with an almost diagonal polynomial matrix:

$$\dot{z} = Q(t) z + g(z, t); \quad z(t_0) = z_0,$$
$$Q(t) = \Lambda_{(N)}(t) + t^{-N-1} G_{(N+1)}(t), \quad \Lambda_{(N)}(t) = \sum_{k=0}^{N} \Lambda_k t^{-k}; \quad \|G_{(N+1)}(t)\| \leqslant C, \tag{7}$$

where Λ_k *— constant diagonal matrices,* $H_k(t)$ *— T-periodical matrices, which are unambiguously defined by an iterative method.*

Proof. Under theorem conditions after the substitute $x = S_0 y$ we will get a system

$$\dot{y} = B(t) y + h(y, t); \quad y(t_0) = y_0, \quad B(t) = \Lambda_0 + \sum_{k=1}^{\infty} B_k(t)$$

and after a non-degenerate by rather high $t > t_0 > 1$ polynomially periodical substitute $y = H_{(N)} z$ we get a set (7), if matrices $B(t)$, $Q(t)$ and $H_{(N)}(t)$ satisfy the differential matrix equation

$$\dot{H}_{(N)} = B(t) H_{(N)}(t) - H_{(N)}(t) Q(t). \tag{8}$$

Equaling in (8) the coefficients by the same powers t, we become the following differential equations

$$\dot{H}_k = \Lambda_0 \overline{\overline{H}}_k(t) - \overline{\overline{H}}_k(t) \Lambda_0 + P_k(t) - \Lambda_k, \tag{9}$$
$$P_1(t) = B_1(t);$$
$$P_k(t) = B_k(t) + \sum_{j=1}^{k-1} \left(B_j(t) H_{k-j}(t) - H_{k-j}(t) \Lambda_j\right) + (k-1) H_{k-1}(t);$$
$$k = 2, 3, \ldots, N.$$

We should emphasize, that in (9) the matrix $\overline{\overline{R}}(t) = \Lambda_0 \overline{\overline{H}}_k(t) - \overline{\overline{H}}_k(t) \Lambda_0$ is always non-diagonal and has a simple spectrum owing to condition (6). Let us factorize the equation (9) to diagonal and without-diagonal components, i.e. to two differential matrix equations:

$$\dot{\overline{H}}_k = \overline{P}_k(t) - \Lambda_k \tag{10}$$

and

$$\dot{\overline{\overline{H}}}_k = \Lambda_0 \overline{\overline{H}}_k(t) - \overline{\overline{H}}_k(t) \Lambda_0 + \overline{\overline{P}}_k(t). \tag{11}$$

This equation (10) has the only T-periodical solution $\overline{H}_k(t) = \int_0^t \left(\overline{P}_k(s) - \Lambda_k\right) \mathrm{d}s$, if for every diagonal matrix

$$\Lambda_k = \frac{1}{T} \int\limits_0^T \overline{P}_k(t) \mathrm{d}t, \quad k = 1, 2, \ldots, N.$$

Equation (11) breaks into $(n^2 - n)$ scalar differential equations:

$$\dot{h}_{ijk} = \sigma_{ij} h_{ijk}(t) + p_{ijk}(t),$$

each of them with consideration of (6) has the only T — periodical solution [2]

$$h_{ijk}(t) = \exp\left(\sigma_{ij}(t+T)\right)\left(1 - \exp\left(\sigma_{ij}T\right)\right)^{-1} \cdot \int\limits_{t}^{t+T} \exp\left(-\sigma_{ij}s\right) p_{ijk}(s)\mathrm{d}s.$$

Theorem is proved. □

3. About Stability of Solutions with a Polynomially Periodical Matrix

Theorem 3. *Let under conditions of the first theorem for the set*

$$\begin{gathered} \dot{x} = t^m A(t)x + f(x,\, t); \quad x(t_0) = x_0, \\ x,\, f \in \mathbb{R}^n; \quad m \geqslant 1; \quad t_0 > 1; \quad f(0,\, t) \equiv 0, \end{gathered} \tag{12}$$

the following conditions are fulfilled:

1) spectrum $\{\lambda_j(t)\}_1^n$ *of matrix* $t^m \sum\limits_{k=0}^{m-1} \Lambda_k(t) t^{-k}$ *satisfies inequations:*

$$\operatorname{Re}\lambda_j(t) \leqslant -\sigma_0 t^q + \varphi(t); \quad j = 1,\, 2, \ldots,\, n; \quad q = 0,\, 1, \ldots,\, m; \quad \sigma_0 > 0; \quad \int\limits_{t_0}^{t} \varphi(s) \leqslant C;$$

2) for rather smooth function $f(x,\, t)$ *the estimation is true:*

$$|f(x,\, t)| \leqslant C_0 \left|x\right|^{1+\alpha}, \quad \alpha > 0,\ C_0 > 0,\ |x| \leqslant R,\ t \geqslant t_0.$$

Then the trivial solution of a inhomogeneous set (12) *is asymptotically stable.*

For the corresponding homogeneous set $(f(x,\, t) \equiv 0)$ *under conditions*

$$\operatorname{Re}\lambda_j(t) \leqslant \varphi(t); \quad j = 1,\, 2, \ldots,\, n; \quad \int\limits_{t_0}^{t} \varphi(s) \leqslant C; \quad t > t_0,$$

the trivial solution is stable.

Proof. Let us fix the differential inequation for a square norm of the set (4) solution in case $\operatorname{Re}\lambda_j(t) \leqslant -\sigma_0 t^q + \varphi(t)$,

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}\left|z\right|^2}{\mathrm{d}t} &= t^m \operatorname{Re}\left(z^* \Lambda_{(N)}(t) z\right) + t^{-2} \operatorname{Re}\left(z^* G_{(N+1)}(t) z\right) + \operatorname{Re}\left(z^*,\, g(z,\, t)\right) \leqslant \\ &\leqslant \sum_{1}^{n} \lambda_j(t) \left|z_j\right|^2 + t^{-2} C_1 \left|z\right|^2 + C_2 \left|z\right|^{2+\alpha} \leqslant \left(-\sigma_0 t^q + t^{-2} C_1 + C_2 \left|z\right|^{\alpha}\right) \left|z\right|^2 \leqslant \\ &\leqslant \left(-\sigma_1 t^q\right) \left|z\right|^2; \quad 0 < \sigma_1 < \sigma_0. \end{aligned}$$

We come to estimation: $|z(t)| \leqslant |z_0| \exp\left(\dfrac{\sigma_1}{q+1}(t_0^{q+1} - t^{q+1})\right) \to 0,\ (t \to +\infty)$. In second case, when $\operatorname{Re}\lambda_j(t) \leqslant \varphi(t)$, a stability of the trivial solution of homogeneous set $(f \equiv 0)$ follows from:

$$\frac{1}{2}\frac{\mathrm{d}\,|z|^2}{\mathrm{d}t} \leqslant \left(\varphi(t) + t^{-2}C_1\right)|z|^2.$$

We get the following: $|z(t)| \leqslant |z_0| \exp\left(b(t) + C_1\left(t_0^{-1} - t^{-1}\right)\right) \leqslant C_2|z_0|$. Theorem is proved. □

Theorem 4. *Let under conditions of the second theorem for the set*

$$\begin{gathered}\dot{x} = A(t)x + f(x,\,t); \quad x(t_0) = x_0, \\ x,\, f \in \mathbb{R}^n; \quad m = 0; \quad t_0 > 1; \quad f(0,\,t) \equiv 0,\end{gathered} \tag{13}$$

the following conditions are fulfilled:
1) spectrum of an auxiliary matrix $\{\lambda_j(t)\}_1^n$ $\Lambda_{(1)} = \Lambda_0 + \Lambda_1 t^{-1}$ *satisfies inequalities:*

$$\operatorname{Re}\lambda_j(t) \leqslant -\sigma_0 < 0; \quad j = 1,\,2,\ldots,\,n;$$

2) for a rather smooth function $f(x,\,t)$ *the estimation is true:*

$$|f(x,\,t)| \leqslant C_0\,|x|^{1+\alpha}, \quad \alpha > 0,\ C_0 > 0,\ |x| \leqslant R,\ t \geqslant t_0.$$

Then the trivial solution of a inhomogeneous system (13) *is asymptotically stable. A trivial solution of a corresponding homogeneous set will be stable, if the following inequality for the matrix spectrum is fulfilled:* $\operatorname{Re}\lambda_j(t) \leqslant 0,\ j = 1,\,2,\ldots,\,n.$

Proof. Let us estimate the norm of set (13) solution:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}\,|z|^2}{\mathrm{d}t} &= \operatorname{Re}\left(z^*Q(t)z\right) + \operatorname{Re}\left(z^*g(z,\,t)\right) \leqslant \\ &\leqslant \operatorname{Re}\left(z^*\Lambda_{(1)}(t)z\right) + t^{-2}\operatorname{Re}\left(z^*G_{(N+1)}(t)z\right) + \operatorname{Re}\left(z^*g(z,\,t)\right) \leqslant \\ &\qquad \leqslant \left(-\sigma_0 + C_1t^{-2} + C_2\,|z|^\alpha\right)|z|^2 \leqslant -\sigma_1\,|z|^2; \quad \left(0 < \sigma_1 < \sigma_0\right).\end{aligned}$$

We get the following: $|z(t)| \leqslant C_3|z_0| \exp\left(-\sigma_1(t-t_0)\right) \to 0, \quad (t \to +\infty)$, which proves an asymptotical stability of the trivial solution of inhomogeneous set (13), that is equivalent to the set (12).

For the case of homogenous system, when $\lambda_j(t) \leqslant 0$, we can write down another estimation:

$$\frac{1}{2}\frac{\mathrm{d}\,|z|^2}{\mathrm{d}t} \leqslant C_1t^{-2}\,|z|^2.$$

From this inequation follows: $|z(t)| \leqslant |z_0| \exp\left(C_1\left(t_0^{-1} - t^{-1}\right)\right) \leqslant C_2|z_0|$, which proves the stability of the trivial solution of homogeneous set (12). □

4. Conclusion

The results of this paper, that relate to the Reducibility theorem, develops the splitting method and enables us to get simpler and more convenient for further studying classes of sets of ordinary differential equations. Sufficient non-trivial criterion of stability or asymptotical stability (Theorem 3, 4) can be considered as a summery of the classical Lyapunov theorem about asymptotical stability at a first approximation.

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УДК 517.925.51

Алгоритм приводимости неоднородных систем с полиномиально периодической матрицей на основе спектрального метода

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Изучен класс линейных и квазилинейных неоднородных систем обыкновенных дифференциальных уравнений с полиномиально периодической матрицей при наличии определяющей матрицы различной фиксированной жордановой структуры. Ставится задача для этого нового класса систем разработать алгоритм их приводимости к более простым для анализа эквивалентным системам с почти диагональной матрицей, а также сформулировать и обосновать достаточные условия устойчивости или асимптотической устойчивости их тривиального решения. Эта задача актуальна, так как анализ рассматриваемого класса неавтономных систем известными методами (например, метод функций Ляпунова) затруднён. Кроме этого, рассмотрение неоднородных систем с помощью спектральных и других методов вызывает дополнительные трудности. Поставленная задача решена с помощью разработанного авторами аналитического метода, являющегося обобщением известных классических теорем. В основе предложенного алгоритма приводимости лежит один из вариантов метода расщепления по спектру определяющей матрицы в изучаемой неавтономной системе (с учётом её расщепления на диагональную и бездиагональную часть). Показана возможность приводимости систем указанного класса в зависимости от структуры спектра определяющей матрицы к системам с почти диагональной матрицей, что упрощает анализ вопросов устойчивости. Доказаны теоремы об устойчивости или асимптотической устойчивости тривиального решения преобразованных эквивалентных систем и соответствующих исходных систем, что является развитием и обобщением спектрального метода исследования устойчивости для рассмотренного в работе класса неавтономных систем.

Ключевые слова: квазилинейная система, спектральный метод, полиномиально периодическая матрица, метод расщепления, устойчивость.