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Variational Principles for the Differential Difference Operator of the Second Order

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The purpose of the present paper is to investigate the potentiality of the operator of differential difference equations and to construct of the functional, if the given operator is a potential on a given set relatively to the some bilinear form.

Key words and phrases: differential difference equations, functional differentional equations, inverse problem of the calculus of variations, variational principles, equations with deviating arguments.

1. Introduction

The construction of variational principles in the investigation of the differential equations $N(u) = f$ is connected with the inverse problem of the calculus of variations. The investigation of the solutions of this inverse problem as the construction of the functional $F[u]$ for which the set of critical (extremal or stationary) points coincides with that of a solution of the given equation $N(u) - f = 0$.

Differential difference equations or functional difference equations have already appeared in mathematical papers of the XVIIIth century, for example, in the Euler solution of the problem connected with a search of the general form of a line similar to its evolute.

The search of a functional F that admits some given equations as its Euler-Lagrange equations is known as the classical inverse problem of the calculus of variations. Since the end of the XIXth century there has been a great deal of activity in this field (see Helmholtz [1], Volterra [2], Santilli [3], Tonti [4], Filippov, Savchin and Shorokhov [5] and refs. therein).

There is a practical need to develop different approaches to the construction of integral variational principles for equations with deviating arguments.

It is possible to investigate the problem of the construction of the variational multiplier if the operator of the given equation is not potential on the given set with respect to some bilinear form.

The main aim of this paper is to investigate the potentiality of the operator $N(u)$ of the differential difference equation and to construct the functional $F[u]$, if the operator $N(u)$ is potential on the given set relatively to some bilinear form.

2. Some auxiliary notations and definitions

Let U, V be normed linear spaces over the field of real numbers $\mathbb{R}$, and O_U, O_V be their zero elements.

Take any operator $N : D(N) \to R(N)$, where $D(N) \subseteq U$, $R(N) \subseteq V$. A limit

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}[N(u + \epsilon h) - N(u)] = \delta N(u, h), \quad u \in D(N), \quad (u + \epsilon h) \in D(N),$$

if it exists, is called the Gâteaux differential of N at the point u. If it is linear relative to h, then the operator $\delta N(u, \cdot) : U \to V$ is called the Gâteaux derivative of N at u and will be denoted by N'_u. Its domain of definition $D(N'_u)$ consists of elements $h \in U$ such that $(u + \epsilon h) \in D(N)$ for all ϵ sufficiently small.

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Let us consider the equation $N(u) = O_V$, $u \in D(N)$ with the Gâteaux differentiable operator N, and a convex set $D(N)$.

In order to consider the existence of its variational formulation we need a non-degenerate bilinear form

$$\langle u, v\rangle = \int\limits_{\Omega}\int\limits_{t_0}^{t_1} u(x,t), v(x,t)\mathrm{d}t\mathrm{d}x. \tag{1}$$

Definition. The operator $N : D(N) \to V$ is said to be potential on the set $D(N)$ relative to a given bilinear form $\langle\cdot,\cdot\rangle : V \times U \to R$, if there exists a functional F_N : $D(F_N) = D(N) \to R$ such that $\delta F_N[u,h] = \langle N(u), h\rangle \quad \forall u \in D(N), \quad \forall h \in D(N'_u)$.

The functional F_N is called the potential of the operator N, and in turn the operator N is called the gradient of the functional F_N. As it is known (see Volterra [2]) the condition for potentiality of the operator N takes the form

$$\langle N'_u h, g\rangle = \langle N'_u g, h\rangle \quad \forall u \in D(N), \quad \forall h, g \in D(N'_u). \tag{2}$$

Under this condition the potential F_N is given by

$$F_N[u] = \int\limits_0^1 \langle N(u_0 + \lambda(u - u_0)), u - u_0\rangle \,\mathrm{d}\lambda + \mathrm{const},$$

where u_0 is a fixed element of $D(N)$.

Let us consider the differential equation with deviating arguments

$$N(u) \equiv \sum_{\lambda=-1}^{1} a_\lambda(x,t)\frac{\partial^2 u}{\partial t^2}(x,t+\lambda\tau) - b_\lambda^{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t+\lambda\tau)+$$
$$+ f(x,t+\lambda\tau, u(x,t+\lambda\tau), u_{x_j}(x,t+\lambda\tau), u_t(x,t+\lambda\tau)) = 0, \tag{3}$$

where u is an unknown function; $(x,t) \in Q = \Omega \times (t_1,t_2)$; $t_2 - t_1 > 2\tau$; $a_\lambda(x,t) \in C^{0,2}_{x,t}(\overline{Q})$, $b_\lambda^{ij}(x,t) \in C^{2,0}_{x,t}(\overline{Q})$, $\forall i,j = \overline{1,n}$.

The domain of definition $D(N)$ is given by the equality

$$D(N) = \Big\{u \in U = C^{2,2}_{x,t}(\overline{\Omega} \times [t_0 - \tau, t_1 + \tau]) :$$
$$\frac{\partial^k u(x,t)}{\partial t^k} = \varphi_{1k}(x,t),\ (x,t) \in E_1 = \overline{\Omega} \times [t_0 - \tau, t_0],\ k = 0,1,$$
$$\frac{\partial^k u(x,t)}{\partial t^k} = \varphi_{2k}(x,t),\ (x,t) \in E_2 = \overline{\Omega} \times [t_1, t_1 + \tau],\ k = 0,1,$$
$$\left.\frac{\partial^\nu u}{\partial x^\nu}\right|_{\Gamma_\tau} = \psi_\nu,\ \nu = 0,1\Big\}, \tag{4}$$

where $\Omega \subset \mathbb{R}^n$, $\Gamma_\tau = \partial\Omega \times (t_0 - \tau, t_1 + \tau)$, φ_{10}, φ_{20}, ψ_ν — are given sufficiently smooth functions, $\varphi_{jk} = \dfrac{\partial^k \varphi_{j0}}{\partial t^k}$, $(j = 1,2;\ k = 0,1)$.

The formulation of the problem:

1) to investigate the potentiality of the operator N of the equation (3) on the set $D(N)$ (4) relatively to the some bilinear form (1);

2) if the operator N is potential, then to construct the variational principle for the operator N of the equation (3).

The investigation of the potentiality of the operator N of the equation (3).

Theorem. *For the potentiality of the operator N (3) on the given set $D(N)$ (4) with respect to the bilinear form (1), it is necessary and sufficient that the following conditions hold:*

$$a_\lambda(x,t) = a_{-\lambda}(x, t+\lambda\tau), \quad b^{ij}_\lambda(x,t) = b^{ij}_{-\lambda}(x, t+\lambda\tau) \quad \forall\lambda = -1, 0, 1,$$
$$I_\lambda f(x,t,u,u_{x_j},u_t) = f(x,t,u,u_{x_j},u_t),$$
$$I_\lambda f(x,t,u,u_{x_j},u_t) = \frac{\partial a_\lambda(x,t)}{\partial t} u_t(x,t+\lambda\tau) - \frac{\partial b^{ij}_\lambda(x,t)}{\partial x_j} u_{x_j}(x,t+\lambda\tau) + I_\lambda \psi(x,t,u), \tag{5}$$

where ψ are sufficiently smooth functions, $I_\lambda g(x,t) = g(x, t+\lambda\tau)$.

Proof. We denote

$$N'_u h = \sum_{\lambda=-1}^{1} a_\lambda(x,t)\frac{\partial^2 h}{\partial t^2}(x,t+\lambda\tau) - b^{ij}_\lambda(x,t)\frac{\partial^2 h}{\partial x_i \partial x_j}(x,t+\lambda\tau) +$$
$$+ \left\{ \frac{\partial f}{\partial u} h + \frac{\partial f}{\partial u_{x_j}} h_{x_j} + \frac{\partial f}{\partial u_t} h_t \right\}(x,t+\lambda\tau)$$
$$\forall u \in D(N),\ \forall h, g \in D(N'_u),\ \forall i,j = \overline{1,n}. \tag{6}$$

Taking into account formulas (1) and (6), we get

$$\langle N'_u h, g\rangle = \sum_{\lambda=-1}^{1} \int_\Omega \int_{t_0}^{t_1} \Bigg\{ a_\lambda(x,t)\frac{\partial^2 h}{\partial t^2}(x,t+\lambda\tau) - b^{ij}_\lambda(x,t)\frac{\partial^2 h}{\partial x_i \partial x_j}(x,t+\lambda\tau) +$$
$$+ \left\{ \frac{\partial f}{\partial u} h + \frac{\partial f}{\partial u_{x_j}} h_{x_j} + \frac{\partial f}{\partial u_t} h_t \right\}(x,t+\lambda\tau) \Bigg\} g(x,t)\mathrm{d}t\mathrm{d}x$$
$$\forall u \in D(N),\ \forall h, g \in D(N'_u), \quad \forall i,j = \overline{1,n}. \tag{7}$$

We denote in the items of the formula (7) thus

$$I_1 \equiv \sum_{\lambda=-1}^{1} \int_\Omega \int_{t_0}^{t_1} a_\lambda(x,t)\frac{\partial^2 h}{\partial t^2}(x,t+\lambda\tau) g(x,t)\mathrm{d}t\mathrm{d}x,$$

$$I_2 \equiv \sum_{\lambda=-1}^{1} \int_\Omega \int_{t_0}^{t_1} b^{ij}_\lambda(x,t)\frac{\partial^2 h}{\partial x_i \partial x_j}(x,t+\lambda\tau) g(x,t)\mathrm{d}t\mathrm{d}x,$$

$$I_3 \equiv \sum_{\lambda=-1}^{1} \int_\Omega \int_{t_0}^{t_1} \frac{\partial f}{\partial u_{x_j}} h_{x_j}(x,t+\lambda\tau) g(x,t)\mathrm{d}t\mathrm{d}x,$$

$$I_4 \equiv \sum_{\lambda=-1}^{1} \int_\Omega \int_{t_0}^{t_1} \frac{\partial f}{\partial u_t} h_t(x,t+\lambda\tau) g(x,t)\mathrm{d}t\mathrm{d}x,$$

$$I_5 \equiv \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0}^{t_1} \frac{\partial f}{\partial u} h(x, t+\lambda\tau) g(x,t) \mathrm{d}t\mathrm{d}x.$$

We know that

$$\begin{gathered} \int_{t_0-\tau}^{t_0} h(x,t)\mathrm{d}t = \int_{t_1}^{t_1+\tau} h(x,t)\mathrm{d}t = 0, \\ \int_{t_1-\tau}^{t_1} h(x,t-\tau)\mathrm{d}t = \int_{t_0}^{t_0+\tau} h(x,t+\tau)\mathrm{d}t = 0 \quad \forall x \in \Omega, \end{gathered} \tag{8}$$

from the formula $I_1 - I_5$ we get

$$I_1 = \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0}^{t_1} h(x,t+\lambda\tau) D_t^2(a_\lambda(x,t)g(x,t))\mathrm{d}t\mathrm{d}x = \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0}^{t_1} h(x,t+\lambda\tau) \times \\ \times \left(\frac{\partial^2 a_\lambda(x,t)}{\partial t^2} g(x,t) + 2\frac{\partial a_\lambda(x,t)}{\partial t}\frac{\partial g(x,t)}{\partial t} + \frac{\partial^2 g(x,t)}{\partial t^2} a_\lambda(x,t) \right) \mathrm{d}t\mathrm{d}x.$$

By using the change $t' = t + \lambda\tau$, we obtain

$$I_1 = \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0+\lambda\tau}^{t_1+\lambda\tau} h(x,t') \left(\frac{\partial^2 a_\lambda(x,t'-\lambda\tau)}{\partial t^2} g(x,t'-\lambda\tau) + \right. \\ \left. + 2\frac{\partial a_\lambda(x,t'-\lambda\tau)}{\partial t}\frac{\partial g(x,t'-\lambda\tau)}{\partial t} + \frac{\partial^2 g(x,t'-\lambda\tau)}{\partial t^2} a_\lambda(x,t'-\lambda\tau) \right) \mathrm{d}t\mathrm{d}x.$$

Denoting t' by t and taking into account formula (8), we get

$$I_1 = \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0}^{t_1} h(x,t) \left(\frac{\partial^2 a_\lambda(x,t-\lambda\tau)}{\partial t^2} g(x,t-\lambda\tau) + \right. \\ \left. + 2\frac{\partial a_\lambda(x,t-\lambda\tau)}{\partial t}\frac{\partial g(x,t-\lambda\tau)}{\partial t} + \frac{\partial^2 g(x,t-\lambda\tau)}{\partial t^2} a_\lambda(x,t-\lambda\tau) \right) \mathrm{d}t\mathrm{d}x$$

or

$$I_1 = \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0}^{t_1} h(x,t) \left(\frac{\partial^2 a_{-\lambda}(x,t+\lambda\tau)}{\partial t^2} g(x,t+\lambda\tau) + \right. \\ \left. + 2\frac{\partial a_{-\lambda}(x,t+\lambda\tau)}{\partial t}\frac{\partial g(x,t+\lambda\tau)}{\partial t} + \frac{\partial^2 g(x,t+\lambda\tau)}{\partial t^2} a_{-\lambda}(x,t+\lambda\tau) \right) \mathrm{d}t\mathrm{d}x. \tag{9}$$

$$I_2 = \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0}^{t_1} h(x,t+\lambda\tau) D_{x_i x_j}(b_\lambda^{ij}(x,t)g(x,t))\mathrm{d}t\mathrm{d}x = \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0}^{t_1} h(x,t+\lambda\tau) \times$$

$$\times\left(\frac{\partial^2 b_\lambda^{ij}(x,t)}{\partial x_i\partial x_j}g(x,t)+2\frac{\partial b_\lambda^{ij}(x,t)}{\partial x_i}\frac{\partial g(x,t)}{\partial x_j}+\frac{\partial^2 g(x,t)}{\partial x_i\partial x_j}b_\lambda^{ij}(x,t)\right)\mathrm{d}t\mathrm{d}x.$$

Reasoning by analogy as for I_1, we obtain

$$I_2=\sum_{\lambda=-1}^{1}\int_\Omega\int_{t_0}^{t_1}h(x,t)\left(\frac{\partial^2 b_{-\lambda}^{ij}(x,t+\lambda\tau)}{\partial x_i\partial x_j}g(x,t+\lambda\tau)+\right.$$
$$\left.+2\frac{\partial b_{-\lambda}^{ij}(x,t+\lambda\tau)}{\partial x_i}\frac{\partial g(x,t+\lambda\tau)}{\partial x_j}+\frac{\partial^2 g(x,t+\lambda\tau)}{\partial x_i\partial x_j}b_{-\lambda}^{ij}(x,t+\lambda\tau)\right)\mathrm{d}t\mathrm{d}x$$
$$\forall i,j=\overline{1,n}. \quad (10)$$

$$I_3=-\sum_{\lambda=-1}^{1}\int_\Omega\int_{t_0}^{t_1}h(x,t+\lambda\tau)\times$$
$$\times D_{x_j}\left(\frac{\partial f(x,t+\lambda\tau,u(x,t+\lambda\tau),u_{x_j}(x,t+\lambda\tau),u_t(x,t+\lambda\tau))}{\partial u_{x_j}}g(x,t)\right)\mathrm{d}t\mathrm{d}x=$$
$$=-\sum_{\lambda=-1}^{1}\int_\Omega\int_{t_0}^{t_1}h(x,t+\lambda\tau)\left(\frac{\partial f(x,t+\lambda\tau,u(x,t+\lambda\tau),u_{x_j}(x,t+\lambda\tau),u_t(x,t+\lambda\tau))}{\partial u_{x_j}}g_{x_j}(x,t)+\right.$$
$$\left.+\,g(x,t)D_{x_j}\left(\frac{\partial f(x,t+\lambda\tau,u(x,t+\lambda\tau),u_{x_j}(x,t+\lambda\tau),u_t(x,t+\lambda\tau))}{\partial u_{x_j}}\right)\right)\mathrm{d}t\mathrm{d}x.$$

By using the change $t'=t+\lambda\tau$

$$I_3=-\sum_{\lambda=-1}^{1}\int_\Omega\int_{t_0+\lambda\tau}^{t_1+\lambda\tau}h(x,t')\left(\frac{\partial f(x,t',u(x,t'),u_{x_j}(x,t'),u_t(x,t'))}{\partial u_{x_j}}g_{x_j}(x,t'-\lambda\tau)+\right.$$
$$\left.+\,g(x,t'-\lambda\tau)D_{x_j}\left(\frac{\partial f(x,t',u(x,t'),u_{x_j}(x,t'),u_t(x,t'))}{\partial u_{x_j}}\right)\right)\mathrm{d}t\mathrm{d}x.$$

Denoting t' by t and taking into account formula (8), we get

$$I_3=-\sum_{\lambda=-1}^{1}\int_\Omega\int_{t_0}^{t_1}h(x,t)\left(\frac{\partial f(x,t,u(x,t),u_{x_j}(x,t),u_t(x,t))}{\partial u_{x_j}}g_{x_j}(x,t-\lambda\tau)+\right.$$
$$\left.+\,g(x,t-\lambda\tau)D_{x_j}\left(\frac{\partial f(x,t,u(x,t),u_{x_j}(x,t),u_t(x,t))}{\partial u_{x_j}}\right)\right)\mathrm{d}t\mathrm{d}x.$$

or

$$I_3=-\sum_{\lambda=-1}^{1}\int_\Omega\int_{t_0}^{t_1}h(x,t)\left(\frac{\partial f(x,t,u(x,t),u_{x_j}(x,t),u_t(x,t))}{\partial u_{x_j}}g_{x_j}(x,t+\lambda\tau)+\right.$$
$$\left.+\,g(x,t+\lambda\tau)D_{x_j}\left(\frac{\partial f(x,t,u(x,t),u_{x_j}(x,t),u_t(x,t))}{\partial u_{x_j}}\right)\right)\mathrm{d}t\mathrm{d}x. \quad (11)$$

$$I_4 = -\sum_{\lambda=-1}^{1}\int_{\Omega}\int_{t_0}^{t_1} h(x,t+\lambda\tau)\times$$
$$\times D_t\left(\frac{\partial f(x,t+\lambda\tau,u(x,t+\lambda\tau),u_{x_j}(x,t+\lambda\tau),u_t(x,t+\lambda\tau))}{\partial u_t}g(x,t)\right)\mathrm{d}t\mathrm{d}x =$$
$$= -\sum_{\lambda=-1}^{1}\int_{\Omega}\int_{t_0}^{t_1} h(x,t+\lambda\tau)\left(\frac{\partial f(x,t+\lambda\tau,u(x,t+\lambda\tau),u_{x_j}(x,t+\lambda\tau),u_t(x,t+\lambda\tau))}{\partial u_t}g_t(x,t)+\right.$$
$$\left.+\,g(x,t)D_t\left(\frac{\partial f(x,t+\lambda\tau,u(x,t+\lambda\tau),u_{x_j}(x,t+\lambda\tau),u_t(x,t+\lambda\tau))}{\partial u_t}\right)\right)\mathrm{d}t\mathrm{d}x.$$

By using the change $t' = t+\lambda\tau$

$$I_4 = -\sum_{\lambda=-1}^{1}\int_{\Omega}\int_{t_0+\lambda\tau}^{t_1+\lambda\tau} h(x,t')\left(\frac{\partial f(x,t',u(x,t'),u_{x_j}(x,t'),u_t(x,t'))}{\partial u_t}g_t(x,t'-\lambda\tau)+\right.$$
$$\left.+\,g(x,t'-\lambda\tau)D_t\left(\frac{\partial f(x,t',u(x,t'),u_{x_j}(x,t'),u_t(x,t'))}{\partial u_t}\right)\right)\mathrm{d}t\mathrm{d}x.$$

Denoting t' by t and taking into account formula (8), we get

$$I_4 = -\sum_{\lambda=-1}^{1}\int_{\Omega}\int_{t_0}^{t_1} h(x,t)\left(\frac{\partial f(x,t,u(x,t),u_{x_j}(x,t),u_t(x,t))}{\partial u_t}g_t(x,t-\lambda\tau)+\right.$$
$$\left.+\,g(x,t-\lambda\tau)D_t\left(\frac{\partial f(x,t,u(x,t),u_{x_j}(x,t),u_t(x,t))}{\partial u_t}\right)\right)\mathrm{d}t\mathrm{d}x.$$

or

$$I_4 = -\sum_{\lambda=-1}^{1}\int_{\Omega}\int_{t_0}^{t_1} h(x,t)\left(\frac{\partial f(x,t,u(x,t),u_{x_j}(x,t),u_t(x,t))}{\partial u_t}g_t(x,t+\lambda\tau)+\right.$$
$$\left.+\,g(x,t+\lambda\tau)D_t\left(\frac{\partial f(x,t,u(x,t),u_{x_j}(x,t),u_t(x,t))}{\partial u_t}\right)\right)\mathrm{d}t\mathrm{d}x. \quad (12)$$

$$I_5 = \sum_{\lambda=-1}^{1}\int_{\Omega}\int_{t_0}^{t_1} h(x,t)\frac{\partial f(x,t,u(x,t),u_{x_j}(x,t),u_t(x,t))}{\partial u}g(x,t+\lambda\tau)\mathrm{d}t\mathrm{d}x. \quad (13)$$

Thus from (7) and (9)–(13) we obtain

$$\langle N'_u h, g\rangle = \sum_{\lambda=-1}^{1}\int_{\Omega}\int_{t_0}^{t_1} h(x,t)\left\{\left(\frac{\partial^2 a_{-\lambda}(x,t+\lambda\tau)}{\partial t^2} - \frac{\partial^2 b^{ij}_{-\lambda}(x,t+\lambda\tau)}{\partial x_i\partial x_j} - \right.\right.$$
$$-D_{x_j}\left(\frac{\partial f(x,t,u(x,t),u_{x_j}(x,t),u_t(x,t))}{\partial u_{x_j}}\right) - D_t\left(\frac{\partial f(x,t,u(x,t),u_x(x,t),u_t(x,t))}{\partial u_t}\right)+$$

$$+\frac{\partial f(x,t,u(x,t),u_x(x,t),u_t(x,t))}{\partial u}\Bigg)g(x,t+\lambda\tau)+$$

$$+a_{-\lambda}(x,t+\lambda\tau)\frac{\partial^2 g(x,t+\lambda\tau)}{\partial t^2}-b^{ij}_{-\lambda}(x,t+\lambda\tau)\frac{\partial^2 g(x,t+\lambda\tau)}{\partial x_i\partial x_j}-$$

$$-\left(2\frac{\partial b^{ij}_{-\lambda}(x,t+\lambda\tau)}{\partial x_i}+\frac{\partial f(x,t,u(x,t),u_{x_j}(x,t),u_t(x,t))}{\partial u_{x_j}}\right)g_{x_j}(x,t+\lambda\tau)+$$

$$+\left(2\frac{\partial a_{-\lambda}(x,t+\lambda\tau)}{\partial t}-\frac{\partial f(x,t,u(x,t),u_x(x,t),u_t(x,t))}{\partial u_t}\right)g_t(x,t+\lambda\tau)\Bigg\}\mathrm{d}t\mathrm{d}x$$

$$\forall u\in D(N),\ \forall h,g\in D(N'_u),\quad \forall i,j=\overline{1,n}. \quad (14)$$

Using the Gâteaux derivative of the operator (3) , we get

$$\langle N'_u g,h\rangle = \sum_{\lambda=-1}^{1}\int_{\Omega}\int_{t_0}^{t_1}\Bigg\{a_\lambda(x,t)\frac{\partial^2 g(x,t+\lambda\tau)}{\partial t^2}-b^{ij}_\lambda(x,t)\frac{\partial^2 g(x,t+\lambda\tau)}{\partial x_i\partial x_j}+$$

$$+\left\{\frac{\partial f}{\partial u}g+\frac{\partial f}{\partial u_{x_j}}g_{x_j}+\frac{\partial f}{\partial u_t}g_t\right\}(x,t+\lambda\tau)\Bigg\}h(x,t)\mathrm{d}t\mathrm{d}x$$

$$\forall u\in D(N),\ \forall h,g\in D(N'_u),\quad \forall i,j=\overline{1,n}. \quad (15)$$

We mean $I_\lambda f(x,t,u,u_x,u_t)=f(x,t+\lambda\tau,u(x,t+\lambda\tau),u_x(x,t+\lambda\tau),u_t(x,t+\lambda\tau))$.

The equations (14) and (15) are equal if and only if

$$I_\lambda f(x,t,u,u_x,u_t)=f(x,t,u,u_x,u_t), \quad (16)$$

$$a_\lambda(x,t)=a_{-\lambda}(x,t+\lambda\tau),\quad b^{ij}_\lambda(x,t)=b^{ij}_{-\lambda}(x,t+\lambda\tau)\quad \forall\lambda=-1,0,1,$$

$$\sum_{\lambda=-1}^{1}\int_{\Omega}\int_{t_0}^{t_1}h(x,t)\Bigg\{\Bigg(\frac{\partial^2 a_\lambda(x,t)}{\partial t^2}-\frac{\partial^2 b^{ij}_\lambda(x,t)}{\partial x_i\partial x_j}-$$

$$-D_{x_j}\left(\frac{\partial f(x,t,u(x,t),u_{x_j}(x,t),u_t(x,t))}{\partial u_{x_j}}\right)-$$

$$-D_t\left(\frac{\partial f(x,t,u(x,t),u_{x_j}(x,t),u_t(x,t))}{\partial u_t}\right)\Bigg)g(x,t+\lambda\tau)-$$

$$-2\left(\frac{\partial b^{ij}_\lambda(x,t)}{\partial x_i}+\frac{\partial f}{\partial u_{x_j}}\right)g_{x_j}(x,t+\lambda\tau)+2\left(\frac{\partial a_\lambda(x,t)}{\partial t}-\frac{\partial f}{\partial u_t}\right)g_t(x,t+\lambda\tau)\Bigg\}\mathrm{d}t\mathrm{d}x=0$$

$$\forall u\in D(N),\quad \forall h,g\in D(N'_u),\quad \forall i,j=\overline{1,n}.$$

The condition (16) is fulfiled for some periodic functions, for example. For the potentiality of the operator N of the equation (3) on the set $D(N)$ (4) relatively to some bilinear form (1), it is necessary and sufficient that the following conditions hold for all h and g

$$\begin{cases} \dfrac{\partial b_\lambda^{ij}(x,t)}{\partial x_j + \frac{\partial f}{\partial u_{x_j}} = 0,} \\ \dfrac{\partial a_\lambda(x,t)}{\partial t} - \dfrac{\partial f}{\partial u_t} = 0, \\ D_t\left(\dfrac{\partial a_\lambda(x,t)}{\partial t} - \dfrac{\partial f}{\partial u_t}\right) - D_{x_j}\left(\dfrac{\partial b_\lambda^{ij}(x,t)}{\partial x_i} + \dfrac{\partial f}{\partial u_{x_j}}\right) = 0. \end{cases} \tag{17}$$

The third condition is a consequence of the first and the second conditions.

$$I_\lambda f(x,t,u,u_x,u_t) = \frac{\partial a_\lambda(x,t)}{\partial t} u_t(x,t+\lambda\tau) + I_\lambda \varphi(x,t,u,u_x), \tag{18}$$

where φ is sufficiently smooth function

Substituting (18) for f in the second condition of the sistem (17), we get

$$I_\lambda \varphi(x,t,u,u_x) = -\frac{\partial b_\lambda^{ij}(x,t)}{\partial x} u_x(x,t+\lambda\tau) + I_\lambda \psi(x,t,u), \tag{19}$$

where ψ is sufficiently smooth function.

Thus from equalities (18) and (19), we obtain

$$I_\lambda f(x,t,u,u_x,u_t) = \frac{\partial a_\lambda(x,t)}{\partial t} u_t(x,t+\lambda\tau) - \frac{\partial b_\lambda^{ij}(x,t)}{\partial x} u_x(x,t+\lambda\tau) + I_\lambda \psi(x,t,u), \tag{20}$$

The proof of the theorem is completed. □

The construction of the functional $F_N[u]$ for the operator N of the equation (3)

We can write (3), making use of (20)

$$N(u) \equiv \sum_{\lambda=-1}^{1} a_\lambda(x,t)\frac{\partial^2 u}{\partial t^2}(x,t+\lambda\tau) - b_\lambda^{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t+\lambda\tau) + \\ + \frac{\partial a_\lambda(x,t)}{\partial t} u_t(x,t+\lambda\tau) - \frac{\partial b_\lambda^{ij}(x,t)}{\partial x} u_x(x,t+\lambda\tau) + I_\lambda \psi(x,t,u) = 0. \tag{21}$$

We can find the functional $F_N[u]$

$$F_N[u] = \int_0^1 \langle N(u_0 + \mu(u-u_0)), u - u_0 \rangle \, d\mu = \\ = \sum_{\lambda=-1}^{1} \int_\Omega \int_{t_0}^{t_1} \int_0^1 (a_\lambda(x,t)[u_{0tt} + \mu(u_{tt} - u_{0tt})] - b_\lambda^{ij}(x,t)[u_{0x_ix_j} + \mu(u_{x_ix_j} - u_{0x_ix_j})] + \\ + a_{\lambda t}(x,t)[u_{0t} + \mu(u_t - u_{0t})] - b_{\lambda x_i}^{ij}(x,t)[u_{0x_j} + \mu(u_{x_j} - u_{0x_j})])(u-u_0)\mathrm{d}\mu\mathrm{d}t\mathrm{d}x + \\ + \sum_{\lambda=-1}^{1} \int_\Omega \int_{t_0}^{t_1} \int_0^1 I_\lambda \psi(x,t,\tilde{u})(u-u_0)\mathrm{d}\mu\mathrm{d}t\mathrm{d}x, \quad \tilde{u} = u_0 + \mu(u-u_0).$$

We integrate the first integral (21) in the variable μ and get

$$F_N[u] = \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0}^{t_1} \int_{0}^{1} \Bigg(a_\lambda(x,t) \left[u_{0tt} + \frac{1}{2}(u_{tt} - u_{0tt}) \right] - $$

$$- b_\lambda^{ij}(x,t) \left[u_{0x_ix_j} + \frac{1}{2}(u_{x_ix_j} - u_{0x_ix_j}) \right] + a_{\lambda t}(x,t) \left[u_{0t} + \frac{1}{2}(u_t - u_{0t}) \right] -$$

$$- b_{\lambda x_i}^{ij}(x,t) \left[u_{0x_j} + \frac{1}{2}(u_{x_j} - u_{0x_j}) \right] \Bigg) (u - u_0) \mathrm{d}\mu \mathrm{d}t \mathrm{d}x +$$

$$+ \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0}^{t_1} \int_{0}^{1} I_\lambda \psi(x,t,\tilde{u})(u - u_0) \mathrm{d}\mu \mathrm{d}t \mathrm{d}x$$

or

$$F_N[u] = \frac{1}{2} \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0}^{t_1} (a_\lambda(x,t)[u_{0tt} + u_{tt}] - b_\lambda^{ij}(x,t)[u_{0x_ix_j} + u_{x_ix_j}] +$$

$$+ a_{\lambda t}(x,t)[u_{0t} + u_t] - b_{\lambda x_i}^{ij}(x,t)[u_{0x_j} + u_{x_j}])(u - u_0) \mathrm{d}t \mathrm{d}x +$$

$$+ \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0}^{t_1} \int_{0}^{1} I_\lambda \psi(x,t,\tilde{u})(u - u_0) \mathrm{d}\mu \mathrm{d}t \mathrm{d}x = \frac{1}{2} \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0}^{t_1} D_t \Big(a_\lambda(x,t)(u_{0t} + u_t)(u - u_0) \Big) -$$

$$- D_{x_i} \Big(b_\lambda^{ij}(x,t)(u_{0x_j} + u_{x_j})(u - u_0) \Big) - a_\lambda(x,t)(u_{0t} + u_t)(u_t - u_{0t}) +$$

$$+ b_\lambda^{ij}(x,t)(u_{0x_j} + u_{x_j})(u_{x_i} - u_{0x_i}) \mathrm{d}t \mathrm{d}x + \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0}^{t_1} \int_{0}^{1} I_\lambda \psi(x,t,\tilde{u})(u - u_0) \mathrm{d}\mu \mathrm{d}t \mathrm{d}x. \quad (22)$$

Thus, the unknown functional for the operator (21) (from the equality (22)) is obtained in the following form

$$F_N[u] = -\frac{1}{2} \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0}^{t_1} (a_\lambda(x,t) u_t^2 - b_\lambda^{ij}(x,t) u_{x_i} u_{x_j}) \mathrm{d}t \mathrm{d}x +$$

$$+ \sum_{\lambda=-1}^{1} \int_{\Omega} \int_{t_0}^{t_1} \int_{0}^{1} I_\lambda \psi(x,t,\tilde{u})(u - u_0) \mathrm{d}\mu \mathrm{d}t \mathrm{d}x.$$

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Структура дифференциально-разностного оператора второго порядка, допускающего вариационный принцип

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В статье исследуется на потенциальность оператор на заданной области определения и относительно некоторой билинейной формы. В случае потенциальности строится соответствующий функционал.

Ключевые слова: дифференциально-разностные уравнения, функционально-дифференциальные уравнения, обратная задача вариационного исчисления, вариационный принцип, уравнения с отклоняющимися аргументами.