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# On a stable calculation of the normal to a surface given approximately

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**Abstract.** The paper proposes a stable method for constructing a normal to a surface given approximately. The normal is calculated as the gradient of the function in the surface equation. As is known, the problem of calculating the derivative is ill-posed. In the paper, an approach is adopted to solving this problem as to the problem of calculating the values of an unbounded operator. To construct its stable solution, the principle of minimum of the smoothing functional in Morozov's formulation is used. The normal is obtained in the form of a Fourier series in the expansion in terms of eigenfunctions of the Laplace operator in a rectangle with boundary conditions of the second kind. The functional stabilizer uses the Laplacian, which makes it possible to obtain a normal in the form of a Fourier series that converges uniformly to the exact normal vector as the error in the surface definition tends to zero. The resulting approximate normal vector can be used to solve various problems of mathematical physics using surface integrals, normal derivatives, simple and double layer potentials.

**Key words and phrases:** ill-posed problem, stable derivative calculation, regularization method, discrete Fourier series

## 1. Introduction

When solving many problems of mathematical physics, which are boundary value problems for partial differential equations, there is a need to calculate the normal to the surface, in particular, when calculating the normal derivative. For example, when calculating the potentials of a simple and double layer, as well as other surface integrals.

In the case when the surface is known "exactly", that is, for example, it is given by an equation with an exactly known function

$$F(x, y, z) = 0, \tag{1}$$

then the normal (generally speaking, not unit normal) will be calculated up to the sign in the form of the gradient of the function in the equation of the surface

$$\mathbf{n}_1 = \mathrm{grad} F(x, y, z) = \nabla F. \tag{2}$$

In applied problems, a situation may arise when the surface is not known accurately. The error may be related to the measurement error, digitization, or the surface data is the result of modeling, that is, it contains the model error. That is, even in the case when the surface is given by the equation $z = F(x, y)$, where the function $F$ is given analytically, that is, “exactly”, such a surface can be considered as “model”, approximately describing “real” surface.

In applied problems, a situation may arise when the surface is not known accurately. In the case when the surface is known inaccurately, it becomes necessary to calculate the normal to the surface given approximately.

As follows from (2), the calculation of the normal is related to the calculation of the derivatives of the function in the equation of the surface. As [1] is known, such a problem is ill-posed and in the case when the surface is known approximately, the use of regularizing algorithms is required to obtain its approximate solution.

The problem of calculating the derivative of a function as an ill-posed problem has been considered in many works, for example [2–9] and others. Here, following [10], we will solve the problem of stable differentiation as a problem of calculating the values of an unbounded operator.

## 2. Problem statement

When solving the problem of constructing a normal vector to a surface, we confine ourselves to considering a surface of the form

$$S = \{(x, y, z) :\, 0 < x < l_x,\, 0 < y < l_y,\, z = F(x, y)\}, \tag{3}$$

that is, a surface given by the equation

$$z = F(x, y), \quad F \in C^2(\Pi) \tag{4}$$

on a rectangle

$$\Pi = \{(x, y) : 0 < x < l_x,\, 0 < y < l_y\}. \tag{5}$$

As follows from (2), for an exactly given function $F$, the normal vector is calculated by the formula

$$\mathbf{n}_1 = \mathrm{grad}\,(F(x, y) - z) = \nabla_{xy} F - \mathbf{k}. \tag{6}$$

Let the surface $S$ be given with an error, namely: instead of the exact function $F$ in (3), there is a function $F^\mu$ defined on a rectangle $\Pi$ of the form (5), so

$$\|F^\mu - F\|_{L_2(\Pi)} \leqslant \mu. \tag{7}$$

Let us pose the problem of constructing a uniform approximation to the normal vector to the surface that converges uniformly to the exact one as $\mu \to 0$.

Note that the normal vector is needed to calculate the normal derivative of the function $\dfrac{\partial \varphi}{\partial n} = (\mathbf{n}, \nabla \varphi)$, $\mathbf{n} = \dfrac{\mathbf{n_1}}{n_1}$, as well as when calculating surface integrals to calculate a surface element $d\sigma = n_1(x, y)dxdy$.

## 3. Constructing a stable solution to the problem

To calculate the normal vector to the surface given by the equation (2), in accordance with (6), it is necessary to calculate the gradient $\nabla_{xy} F$. To obtain a solution of the formulated problem that is stable to the error (7), we use the [11] approach, which consists in the fact that the problem of calculating the gradient $\nabla_{xy} F$ is considered as the problem of calculating the values of an unbounded operator [10]. In contrast to [11], we will consider the Laplacian instead of the gradient as an unbounded operator, which allows us to obtain a uniform approximation for the normal.

As an approximation to the function $\nabla_{xy} F$, computed from the known function $F^{\mu}$ related to the function $F$ by the condition (7), we will consider the gradient of the extremal of the functional

$$N^{\beta}[W] = \|W - F^{\mu}\|^2_{L_2(\Pi)} + \beta \|\Delta W\|^2_{L_2(\Pi)}, \quad \beta > 0 \tag{8}$$

in which the squared norm of the Laplacian of the argument of the functional is used as a stabilizer.

We assume that the surface $S$ of the form (3) satisfies the conditions

$$F'_x|_{x=0,l_x} = 0, \quad F'_y|_{y=0,l_y} = 0; \quad F'''_y|_{x=0,l_x} = 0, \quad F'''_y|_{y=0,l_y} = 0. \tag{9}$$

We obtain the Euler equation for the functional (8). To vary the functional, we have

$$\delta N^{\beta}[W] = 2\,(W - F^{\mu}, \delta W) + 2\beta(\Delta W, \Delta \delta W). \tag{10}$$

We write the second scalar product in (10) as a double integral

$$\delta N^{\beta}[W] = 2\,(W - F^{\mu}, \delta W) + 2\beta \int\limits_0^{l_x} \int\limits_0^{l_y} dxdy \Delta W(x, y) \Delta \delta W(x, y). \tag{11}$$

Separating the second derivatives in the Laplacian and changing the order of integration over the variables $x$ and $y$, we transform the double integrals by integrating by parts:

$$\int\limits_0^{l_x} \int\limits_0^{l_y} dxdy \Delta W(x, y) \Delta \delta W(x, y) =$$

$$= \int_0^{l_y} dy \int_0^{l_x} dx \Delta W(x,y) \frac{\partial^2}{\partial x^2} \delta W(x,y) + \int_0^{l_x} dx \int_0^{l_y} dy \Delta W(x,y) \frac{\partial^2}{\partial y^2} \delta W(x,y) =$$

$$= \int_0^{l_y} dy \left[ \Delta W(x,y) \delta W_x'(x,y) \Big|_{x=0}^{x=l_x} - \int_0^{l_x} dx \frac{\partial}{\partial x} \Delta W(x,y) \frac{\partial}{\partial x} \delta W(x,y) \right] +$$

$$+ \int_0^{l_x} dx \left[ \Delta W(x,y) \delta W_y'(x,y) \Big|_{y=0}^{y=l_y} - \int_0^{l_y} dy \frac{\partial}{\partial y} \Delta W(x,y) \frac{\partial}{\partial y} \delta W(x,y) \right]. \quad (12)$$

Since the extremal must satisfy the same boundary conditions, the variations of the derivatives at the boundary are equal to zero, and the one-time integrals are equal to zero. Integrating the remaining integrals by parts again, we obtain

$$\int_0^{l_x} \int_0^{l_y} dx dy \Delta W(x,y) \Delta \delta W(x,y) =$$

$$= \int_0^{l_y} dy \left[ - \int_0^{l_x} dx \frac{\partial}{\partial x} \Delta W(x,y) \frac{\partial}{\partial x} \delta W(x,y) \right] +$$

$$+ \int_0^{l_x} dx \left[ - \int_0^{l_y} dy \frac{\partial}{\partial y} \Delta W(x,y) \frac{\partial}{\partial y} \delta W(x,y) \right] =$$

$$= \int_0^{l_y} dy \left[ -\Delta W_x'(x,y) \delta W(x,y) \Big|_{x=0}^{x=l_x} + \int_0^{l_x} dx \frac{\partial^2}{\partial x^2} \Delta W(x,y) \delta W(x,y) \right] +$$

$$+ \int_0^{l_x} dx \left[ -\Delta W_y'(x,y) \delta W(x,y) \Big|_{y=0}^{y=l_y} + \int_0^{l_y} dy \frac{\partial^2}{\partial y^2} \Delta W(x,y) \delta W(x,y) \right]. \quad (13)$$

Since, in accordance with the boundary conditions

$$W_x'''|_{x=0,l_x} = 0, \quad \frac{\partial^2}{\partial y^2} W_x'|_{x=0,l_x} = 0; \quad W_y'''|_{y=0,l_y} = 0, \quad \frac{\partial^2}{\partial x^2} W_y'|_{y=0,l_y} = 0.$$

Then the one-time integrals in (13) are equal to zero and, thus, we obtain

$$\int_0^{l_x} \int_0^{l_y} dx dy \Delta W(x,y) \Delta \delta W(x,y) = \int_0^{l_y} dy \int_0^{l_x} dx \frac{\partial^2}{\partial x^2} \Delta W(x,y) \delta W(x,y) +$$

$$+\int_0^{l_x} dx \int_0^{l_y} dy \frac{\partial^2}{\partial y^2}\Delta W(x,y)\delta W(x,y) = \int_0^{l_x} dx \int_0^{l_y} dy \Delta^2 W(x,y)\delta W(x,y). \quad (14)$$

Now, for the variation of the functional (10), taking into account (14), we obtain

$$\delta N^\beta[W] = 2(W - F^\mu, \delta W) + 2\beta(\Delta W, \Delta\delta W) = \\ = 2(W - F^\mu, \delta W) + 2\beta(\Delta^2 W, \delta W). \quad (15)$$

Equating the variation to zero and adding boundary conditions (9), we obtain that the extremal of the functional (8) is a solution to the following boundary value problem for the Euler equation

$$\begin{cases} \beta\Delta^2 W + W = F^\mu, \\ W'_x|_{x=0,l_x} = 0, \\ W'_y|_{y=0,l_y} = 0, \\ W'''_x|_{x=0,l_x} = 0, \\ W'''_y|_{y=0,l_y} = 0. \end{cases}$$

We will seek the solution of this boundary value problem in the form of an expansion in the Fourier series

$$W(x,y) = \sum_{n,m=0}^{\infty} \tilde{W}_{nm} \cos\frac{\pi n x}{l_x} \cos\frac{\pi m y}{l_y} \quad (16)$$

in terms of the eigenfunctions of the Laplace operator satisfying the boundary conditions (9)

$$\left\{\cos\frac{\pi n x}{l_x} \cos\frac{\pi m y}{l_y}\right\}_{n,m=0}^{\infty}. \quad (17)$$

The solution of the boundary value problem for the Euler equation is obtained in the form

$$W^\mu_\beta(x,y) = \sum_{n,m=0}^{\infty} \frac{\tilde{F}^\mu_{nm}}{1 + \beta k^4_{nm}} \cos\frac{\pi n x}{l_x} \cos\frac{\pi m y}{l_y}, \quad (18)$$

where, for brevity, the notation is introduced

$$k_{nm} = \pi\left(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}\right)^{1/2},$$

and the Fourier coefficients $\tilde{F}^{\mu}_{nm}$ have the form

$$\tilde{F}^{\mu}_{nm} = \frac{4\varepsilon_n\varepsilon_m}{l_x l_y}\int\limits_{\Pi} F^{\mu}(x,y)\cos\frac{\pi n x}{l_x}\cos\frac{\pi m y}{l_y}dxdy,$$
$$\varepsilon_n = 1, \quad n\neq 0, \quad \varepsilon_0 = 0.5. \tag{19}$$

It is easy to see that the series (18) converges uniformly on the rectangle $\Pi$.

As an approximation to the gradient of the function $F^{\mu}$, we will consider the vector function

$$\nabla_{xy}W^{\mu}_{\beta}(x,y) =$$
$$= \sum_{n,m=0}^{\infty}\frac{\tilde{F}^{\mu}_{nm}}{1+\beta k^4_{nm}}\left(-\mathbf{i}\frac{\pi n}{l_x}\sin\frac{\pi n x}{l_x}\cos\frac{\pi m y}{l_y} - \mathbf{j}\frac{\pi m}{l_y}\cos\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y}\right). \tag{20}$$

The series (20) also converges uniformly on $\Pi$. Indeed, applying the Cauchy–Bunyakovsky inequality, in particular, for the $x$-component of the gradient, we obtain

$$\left|\frac{\partial}{\partial x}W^{\mu}_{\beta}(x,y)\right| \leqslant$$
$$\leqslant \sum_{n,m=0}^{\infty}\frac{|\tilde{F}^{\mu}_{nm}|}{1+\beta k^4_{nm}}\frac{\pi n}{l_x} \leqslant \sum_{n,m=0}^{\infty}\frac{|\tilde{F}^{\mu}_{nm}|k_{nm}}{1+\beta k^4_{nm}} \leqslant \frac{1}{\beta}\sum_{n,m=0}^{\infty}\frac{|\tilde{F}^{\mu}_{nm}|\sqrt{\varepsilon_n\varepsilon_m}}{k^3_{nm}\sqrt{\varepsilon_n\varepsilon_m}} \leqslant$$
$$\leqslant \frac{1}{\beta}\left(\frac{4}{l_x l_y}\sum_{n,m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m}{k^6_{nm}}\right)^{\frac{1}{2}}\left(\frac{l_x l_y}{4}\sum_{n,m=0}^{\infty}\frac{(\tilde{F}^{\mu}_{nm})^2}{\varepsilon_n\varepsilon_m}\right)^{\frac{1}{2}} \leqslant \frac{C}{\beta}||F^{\mu}||_{L_2(\Pi)}. \tag{21}$$

A similar estimate can be obtained for the $y$-component of the gradient. In addition, the uniform convergence of the series (18) is also proved.

Let us now prove the convergence of the series (18) and (20) to $F$ and grad$F$, respectively, as $\mu\to 0$.

Let $F^{+}-$ be an even-periodic continuation of the function $F$ with period $2l_x$ in variable $x$ and period $2l_y$ in variable $y$ from a rectangle $\Pi$ of the form (5), that is

$$F^{+}(x,y) = F(x,y), \quad (x,y)\in\Pi; \quad F^{+}(-x,y) = F(x,y), \quad (x,y)\in\Pi;$$
$$F^{+}(x,-y) = F(x,y), \quad (x,y)\in\Pi; \quad F^{+}(-x,-y) = F(x,y), \quad (x,y)\in\Pi;$$
$$F^{+}(x+2l_x n, y+2l_y m) = F^{+}(x,y), \quad (x,y)\in\mathbb{R}^2, \quad n,m=\pm1,\pm2,....$$

**Theorem 1.** *Let $F^{+}\in C^2(\mathbb{R}^2)$, $M\geqslant\|\Delta F\|_{L_2(\Pi)}$, $\beta=\beta(\mu)=\mu^2/M^2$. Then*

$$\left\|W^{\mu}_{\beta(\mu)} - F\right\|_{L_2(\Pi)} \leqslant \frac{3}{2}\mu \to 0 \quad \text{as} \quad \mu\to 0, \tag{22}$$

$$\left\|\nabla_{xy}W^{\mu}_{\beta(\mu)} - \nabla_{xy}F\right\|_{L_2(\Pi)} \leqslant 2\sqrt{\mu M} \to 0 \quad \text{as} \quad \mu\to 0. \tag{23}$$

**Proof.** Let's introduce a notation for a function of the form (18) for $\mu = 0$

$$W_\beta(x, y) = \sum_{n,m=0}^{\infty} \frac{\tilde{F}_{nm}}{1 + \beta k_{nm}^4} \cos\frac{\pi n x}{l_x} \cos\frac{\pi m y}{l_y}. \tag{24}$$

Let us prove the estimate (22) in the assertion of the theorem. Applying the triangle inequality for the norm of the difference $W_\beta^\mu - F$ we obtain

$$\left\|W_\beta^\mu - F\right\|_{L_2(\Pi)} \leqslant \left\|W_\beta^\mu - W_\beta\right\|_{L_2(\Pi)} + \left\|W_\beta - F\right\|_{L_2(\Pi)}. \tag{25}$$

Using the orthogonality of the trigonometric system, for the first norm on the right side (25) we obtain

$$\begin{aligned} &\left\|W_{\beta(\mu)}^\mu - W_{\beta(\mu)}\right\|_{L_2(\Pi)}^2 = \\ &= \sum_{n,m=0}^{\infty} \frac{\left(\tilde{F}_{nm}^\mu - \tilde{F}_{nm}\right)^2}{\left(1 + \beta k_{nm}^4\right)^2} \frac{l_x l_y}{4\varepsilon_n \varepsilon_m} \leqslant \sum_{n,m=0}^{\infty} \frac{\left(\tilde{F}_{nm}^\mu - \tilde{F}_{nm}\right)^2 l_x l_y}{4\varepsilon_n \varepsilon_m} \leqslant \\ &\leqslant \left\|F^\mu - F\right\|_{L_2(\Pi)}^2 = \mu^2. \end{aligned} \tag{26}$$

And for the second norm on the right side (25) under the conditions of the theorem, we obtain:

$$\begin{aligned} &\left\|W_{\beta(\mu)} - F\right\|_{L_2(\Pi)}^2 = \\ &= \sum_{n,m=0}^{\infty} \frac{(\beta k_{nm}^4)^2 \tilde{F}_{nm}^2}{\left(1 + \beta k_{nm}^4\right)^2} \frac{l_x l_y}{4\varepsilon_n \varepsilon_m} \leqslant \beta^2 \max_x \left(\frac{x}{1 + \beta x^2}\right)^2 \sum_{n,m=0}^{\infty} \frac{k_{nm}^4 \tilde{F}_{nm}^2 l_x l_y}{4\varepsilon_n \varepsilon_m} \leqslant \\ &\leqslant \frac{\beta}{4} \left\|\Delta F\right\|_{L_2(\Pi)}^2 \leqslant \frac{\mu^2}{4M^2} M^2 = \frac{\mu^2}{4}. \end{aligned} \tag{27}$$

Here we have used the fact that under the conditions of the theorem

$$\sum_{n,m=0}^{\infty} \frac{k_{nm}^4 \tilde{F}_{nm}^2 l_x l_y}{4\varepsilon_n \varepsilon_m} = \left\|\Delta F\right\|_{L_2(\Pi)}^2 \leqslant M^2, \tag{28}$$

as well as the value of the maximum

$$\max_x \left(\frac{x}{1 + \beta x^2}\right) = \frac{1}{2\sqrt{\beta}}.$$

For the difference norm on the left side (25) with $\beta(\mu) = \mu^2/M^2$ from (26) and (27) we obtain

$$\left\|W_{\beta(\mu)}^\mu - F\right\|_{L_2(\Pi)} \leqslant \mu + \frac{\mu}{2} = \frac{3}{2}\mu. \tag{29}$$

We now obtain the estimate (23) by applying the triangle inequality

$$\left\|\nabla_{xy}W_\beta^\mu - \nabla_{xy}F\right\|_{L_2(\Pi)} \leqslant \\ \leqslant \left\|\nabla_{xy}W_\beta^\mu - \nabla_{xy}W_\beta\right\|_{L_2(\Pi)} + \left\|\nabla_{xy}W_\beta - \nabla_{xy}F\right\|_{L_2(\Pi)}. \quad (30)$$

Estimate the first difference in on the right side (30):

$$\left\|\nabla_{xy}W_\beta^\mu - \nabla_{xy}W_\beta\right\|^2_{L_2(\Pi)} = \int\limits_\Pi \left| -\mathbf{i}\sum_{n,m=0}^{\infty} \frac{\tilde F^\mu_{nm} - \tilde F_{nm}}{1+\beta k^4_{nm}} \frac{\pi n}{l_x} \sin\frac{\pi n x}{l_x} \cos\frac{\pi m y}{l_y} - \right. \\ \left. -\mathbf{j}\sum_{n,m=0}^{\infty} \frac{\tilde F^\mu_{nm} - \tilde F_{nm}}{1+\beta k^4_{nm}} \frac{\pi m}{l_y} \sin\frac{\pi m y}{l_y} \cos\frac{\pi n x}{l_x}\right|^2 dxdy = \\ = \int\limits_\Pi \left|\sum_{n,m=0}^{\infty} \frac{\tilde F^\mu_{nm} - \tilde F_{nm}}{1+\beta k^4_{nm}} \frac{\pi n}{l_x} \sin\frac{\pi n x}{l_x} \cos\frac{\pi m y}{l_y}\right|^2 dxdy + \\ + \int\limits_\Pi \left|\sum_{n,m=0}^{\infty} \frac{\tilde F^\mu_{nm} - \tilde F_{nm}}{1+\beta k^4_{nm}} \frac{\pi m}{l_y} \sin\frac{\pi m y}{l_y} \cos\frac{\pi n x}{l_x}\right|^2 dxdy. \quad (31)$$

Using the orthogonality of the trigonometric system, we obtain:

$$\left\|\nabla_{xy}W_\beta^\mu - \nabla_{xy}W_\beta\right\|^2_{L_2(\Pi)} = \sum_{n,m=0}^{\infty} \frac{\left(\tilde F^\mu_{nm} - \tilde F_{nm}\right)^2}{(1+\beta k^4_{nm})^2} \left(\frac{\pi n}{l_x}\right)^2 \frac{l_x l_y}{4\varepsilon_n\varepsilon_m} + \\ + \sum_{n,m=0}^{\infty} \frac{\left(\tilde F^\mu_{nm} - \tilde F_{nm}\right)^2}{(1+\beta k^4_{nm})^2} \left(\frac{\pi m}{l_y}\right)^2 \frac{l_x l_y}{4\varepsilon_n\varepsilon_m} = \frac{l_x l_y}{4} \sum_{n,m=0}^{\infty} \frac{\left(\tilde F^\mu_{nm} - \tilde F_{nm}\right)^2 k^2_{nm}}{\varepsilon_n\varepsilon_m\,(1+\beta k^4_{nm})^2} \leqslant \\ \leqslant \max_x \left(\frac{x}{1+\beta x^4}\right)^2 \sum_{n,m=0}^{\infty} \frac{\left(\tilde F^\mu_{nm} - \tilde F_{nm}\right)^2 l_x l_y}{4\varepsilon_n\varepsilon_m} = \frac{1}{\sqrt{\beta}} \left\|F^\mu - F\right\|^2_{L_2(\Pi)}. \quad (32)$$

Here we have used the estimate for the maximum

$$\max_x \left(\frac{x}{1+\beta x^4}\right) = \frac{3^{3/4}}{4}\beta^{-1/4} \leqslant \beta^{-1/4}.$$

Extracting the root at (32), we obtain for the first difference at (30):

$$\left\|\nabla_{xy}W_\beta^\mu - \nabla_{xy}W_\beta\right\|_{L_2(\Pi)} \leqslant \frac{1}{\sqrt[4]{\beta}} \left\|F^\mu - F\right\|_{L_2(\Pi)} = \frac{\mu}{\sqrt[4]{\beta}}. \quad (33)$$

Similarly, to evaluate the second difference in (30), using (24), we obtain:

$$\left\|\nabla_{xy}W_\beta-\nabla_{xy}F\right\|^2_{L_2(\Pi)}=\beta^2\frac{l_xl_y}{4}\sum_{n,m=0}^{\infty}\frac{\tilde{F}^2_{nm}\left[k^4_{nm}\right]^2k^2_{nm}}{\varepsilon_n\varepsilon_m\left(1+\beta k^4_{nm}\right)^2}\leqslant$$

$$\leqslant\beta^2\max_x\left(\frac{x^3}{1+\beta x^4}\right)^2\sum_{n,m=0}^{\infty}\frac{(\tilde{F}_{nm}k^2_{nm})^2l_xl_y}{4\varepsilon_n\varepsilon_m}=\sqrt{\beta}\left\|\Delta F\right\|^2=\sqrt{\beta}M^2. \quad (34)$$

Here we have used the estimate for the maximum

$$\max_x\left(\frac{x^3}{1+\beta x^4}\right)=\frac{3^{3/4}}{4}\beta^{-3/4}\leqslant\beta^{-3/4},$$

and also by the fact that under the conditions of the theorem

$$\Delta F=\sum_{n,m=0}^{\infty}\tilde{F}_{nm}k^2_{nm}\cos\frac{\pi nx}{l_x}\cos\frac{\pi my}{l_y}.$$

Therefore, the second norm on the right side (30) after taking the square root in (34) evaluates to

$$\left\|\nabla_{xy}W_\beta-\nabla_{xy}F\right\|_{L_2(\Pi)}\leqslant\sqrt[4]{\beta}M. \quad (35)$$

Thus, using the estimates (34), (35) and the conditions of the theorem on the function $\beta(\mu)$, from (30) we obtain an error estimate in calculating the gradient of the function $F$:

$$\left\|\nabla_{xy}W^\mu_{\beta(\mu)}-\nabla_{xy}F\right\|_{L_2(\Pi)}\leqslant\frac{\mu}{\sqrt[4]{\beta}}+\sqrt[4]{\beta}M\leqslant2\sqrt{\mu M}\to0 \quad \text{as} \quad \mu\to0. \quad (36)$$

Note that for $\beta(\mu)=\mu^2/M^2$, the expression on the right represents the minimum by the parameter $\beta$.

The theorem is proved. □

Based on this theorem, we can use the formula for the approximate gradient to construct an approximate normal to the surface $S$ by the formula (6)

$$\mathbf{n}^\mu_1=\nabla_{xy}W^\mu_{\beta(\mu)}-\mathbf{k}. \quad (37)$$

then from (37) and (36) follows an estimate of the deviation of the approximate normal $\mathbf{n}^\mu_1$ from the exact:

$$\left\|\mathbf{n}^\mu_1-\mathbf{n}_1\right\|_{L_2(\Pi)}=\left\|\nabla_{xy}W^\mu_\beta-\nabla_{xy}F\right\|_{L_2(\Pi)}\leqslant2\sqrt{\mu M}.$$

The surface defined by the equation $z=W^\mu_{\beta(\mu)}(x,y)$, where $W^\mu_{\beta(\mu)}$ has the form (18), denote

$$S^\mu=\left\{(x,y,z):0<x<l_x,\,0<y<l_y,\,z=W^\mu_{\beta(\mu)}(x,y)\right\}. \quad (38)$$

Since the series (18) converges uniformly, the surface $S^\mu$ is given by a continuous function.

When solving various problems of mathematical physics that use surface integrals and a normal derivative on a surface given approximately by the condition (7), an approximately given surface $z = F^\mu(x, y)$ can be replaced by the surface $S^\mu$, and the normal to the surface can be calculated according to the formula (37).

## 4. Application of the problem of calculating the normal to the inverse problem of thermography

Calculation of the normal to the surface may be necessary, in particular, when solving the inverse problem of thermography. In this case, we consider the problem of correcting the thermogram $f$, which is a digitized temperature distribution on the surface of the investigated heat-conducting body containing heat sources. The image of body sources on a thermogram is, as a rule, distorted due to the process of heat conduction, heat transfer, and the relative remoteness of heat sources from the body surface. In order to refine the image in a cylindrical area of rectangular cross section

$$D(F,H) = \{(x,y,z) : \ 0 < x < l_x,\ 0 < y < l_y,\ F(x,y) < z < H\}. \tag{39}$$

a boundary value problem for the Laplace equation is considered (we assume that the support of the heat source density function $\rho$ is located in the region $z > H$)

$$\begin{cases} \Delta u(M) = 0, \quad M \in D(F,H), \\ u\big|_S = f, \\ \dfrac{\partial u}{\partial n}\Big|_S = h(U_0 - f)\Big|_S, \\ u|_{\Gamma_H} = 0. \end{cases} \tag{40}$$

The set of side faces of $D(F,H)$ is denoted as $\Gamma_H$.

Note that in the problem (40) on the surface $S$ of the form (3), the Cauchy conditions are specified, that is, the boundary values $f$ of the desired function $u$ and the values of its normal derivative are given, so the problem (40) has a unique solution. The boundary $z = H$ of the domain $D(F,H)$ is free and thus the problem (40) is not robust against data errors, i.e. ill-posed.

The function $u\big|_{z=H}$ will be considered as an adjusted thermogram. Since the plane $z = H$ is located closer to the density carrier $\rho$ than the surface $S$ from which the original thermogram is taken, it should be expected that the corrected thermogram more accurately conveys information about the distribution of heat sources than the original thermogram.

We will assume that the function $f$ in the problem (40) is given with an error, that is, instead of $f$, the function $f^\delta$ is given, such that

$$\left\|f^\delta - f\right\|_{L_2(\Pi)} \leqslant \delta. \tag{41}$$

In [12], an approximate solution to an ill-posed problem (40) is constructed as

$$u_\alpha^\delta(M) = v_\alpha^\delta(M) + \Phi^\delta(M), \quad M \in D(F,H), \tag{42}$$

where the function

$$\Phi^\delta(M) = \int\limits_\Pi \Big[h(U_0 - f^\delta(x_P, y_P))\varphi(M,P)\big|_{P\in S} n_1(x_P, y_P) - \\ - f^\delta(x_P, y_P)(\mathbf{n}_1, \nabla_P \varphi(M,P))\big|_{P\in S}\Big] dx_P dy_P \tag{43}$$

is calculated using the problem data (40) and the Dirichlet problem source function

$$\varphi(M,P) = \\ = \frac{2}{l_x l_y} \sum_{n,m=1}^{\infty} \frac{e^{-k_{nm}|z_M - z_P|}}{k_{nm}} \times \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y} \sin\frac{\pi n x_P}{l_x} \sin\frac{\pi m y_P}{l_y}, \tag{44}$$

$$k_{nm} = \pi\left(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}\right)^{1/2}$$

in the infinite cylinder

$$D^\infty = \{(x,y,z) : 0 < x < l_x,\ 0 < y < l_y,\ -\infty < z < \infty\} \subset \mathbb{R}^3.$$

The function $v_\alpha^\delta$, which is an approximation to the density potential $\rho$, in [12] is obtained using the Tikhonov regularization method [1]

$$v_\alpha^\delta(M) = \\ = -\sum_{n,m=1}^{\infty} \frac{\tilde{\Phi}_{nm}^\delta(a)\exp\{k_{nm}(z_M - a)\}}{1 + \alpha\exp\{2k_{nm}(H-a)\}} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y}, \quad \alpha > 0, \tag{45}$$

$\tilde{\Phi}_{nm}^\delta(a)$ are Fourier coefficients of the function $\Phi^\delta(M)$ of the form (43)

$$\tilde{\Phi}_{nm}^\delta(a) = \frac{4}{l_x l_y} \int\limits_\Pi \Phi^\delta(x,y,a) \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y} dx dy. \tag{46}$$

As follows from the formula (43) when calculating the value of $\Phi$, the normal to the surface is used. Estimates of the error in calculating the function $\Phi$ and the approximate solution $u$ that arise when replacing the exact normal $\mathbf{n}_1$ with an approximate normal are obtained in [13].

## 5. Conclusion and discussion

Formulas (37), (20) for approximate calculation of the normal to an approximately given surface can be used in the calculation of surface integrals and potentials of a simple and double layer and in other problems [14] using the normal to the surface. For numerical summation of Fourier series (45) and calculation of Fourier coefficients (46) algorithms for summing discrete Hamming series [15, 16] can be used. Discretization of formulas (45), (46) can be done in accordance with [17].

## References


[1] A. N. Tikhonov and V. J. Arsenin, *Methods for solving ill-posed problems [Metody resheniya nekorrektnyh zadach]*. Moscow: Nauka, 1979, in Russian.

[2] T. F. Dolgopolova and V. K. Ivanov, "On numerical differentiation," *USSR Computational Mathematics and Mathematical Physics*, vol. 6, no. 3, pp. 223–232, 1966. DOI: 10.1016/0041-5553(66)90145-5.

[3] V. V. Vasin, "The stable evaluation of a derivative in space $C(-\infty, \infty)$," *USSR Computational Mathematics and Mathematical Physics*, vol. 13, no. 6, pp. 16–24, 1973, english. DOI: 10.1016/0041-5553(73)90002-5.

[4] R. S. Anderssen and P. Bloomfield, "Numerical differentiation procedures for non-exact data," *Numerische Mathematik*, vol. 22, pp. 157–182, 1974. DOI: 10.1007/BF01436965.

[5] C. W. Groetsch, "Optimal order of accuracy in Vasin's method for differentiation of noisy functions," *Journal of Optimization Theory and Applications*, vol. 74, pp. 373–378, 1992. DOI: 10.1007/BF00940901.

[6] M. Hanke and O. Scherzer, "Inverse problems light: Numerical differentiation," *American Mathematical Monthly*, vol. 108, no. 6, pp. 512–521, 2001, english. DOI: 10.1080/00029890.2001.11919778.

[7] S. Ahn, U. J. Choi, and A. G. Ramm, "A scheme for stable numerical differentiation," *Journal of Computational and Applied Mathematics*, vol. 186, no. 6, pp. 325–334, 2006, english. DOI: 10.1016/j.cam.2005.02.002.

[8] Z. Meng, Z. Zhao, D. Mei, and Y. Zhou, "Numerical differentiation for two-dimensional functions by a Fourier extension method," *Inverse Problems in Science and Engineering*, vol. 28, no. 1, pp. 126–143, 2020, english. DOI: 10.1080/17415977.2019.1661410.

[9] E. V. Semenova, S. G. Solodky, and S. A. Stasyuk, "Application of Fourier truncation method to numerical differentiation for bivariate functions," *Computational Methods in Applied Mathematics*, vol. 22, no. 2, pp. 477–491, 2022, english. DOI: 10.1515/cmam-2020-0138.

[10] V. A. Morozov, "On a stable method for computing the values of unbounded operators [Ob odnom ustoychivom metode vychisleniya znacheniy neogranichennykh operatorov]," *Dokladi AN SSSR*, vol. 185, no. 2, pp. 267–270, 1969, in Russian.

[11] E. B. Laneev and M. N. Muratov, “On the stable solution of a mixed boundary value problem for the Laplace equation with an approximately given boundary [Ob ustoychivom reshenii smeshannoy krayevoy zadachi dlya uravneniya Laplasa s priblizhenno zadannoy granitsey],” *Vestnik RUDN. Matematika*, vol. 9, no. 1, pp. 102–111, 2002, in Russian.

[12] E. B. Laneev, N. Y. Chernikova, and O. Baaj, “Application of the minimum principle of a Tikhonov smoothing functional in the problem of processing thermographic data,” *Advances in Systems Science and Applications*, no. 1, pp. 139–149, 2021. DOI: 10.25728/assa.2021.21.1.1055.

[13] O. Baaj, N. Y. Chernikova, and E. B. Laneev, “Correction of thermographic images based on the minimization method of Tikhonov functional,” *Yugoslav Journal of Operations Research*, vol. 32, no. 3, pp. 407–424, 2022. DOI: 10.2298/YJOR211015026B.

[14] E. B. Laneev and E. Y. Ponomarenko, “On a linear inverse potential problem with approximate data on the potential field on an approximately given surface,” *Eurasian mathematical journal*, vol. 14, no. 1, pp. 57–70, 2023. DOI: 10.32523/2077-9879-2023-14-1-55-70.

[15] R. W. Hamming, *Numerical methods for scientists and engineers*. New York: McGraw-Hill Book Company, 1962.

[16] E. B. Laneev and O. Baaj, “On a modification of the Hamming method for summing discrete Fourier series and its application to solve the problem of correction of thermographic images,” *Discrete and Continuous Models and Applied Computational Science*, vol. 30, no. 4, pp. 342–356, 2022. DOI: 10.22363/2658-4670-2022-30-4-342-356.

[17] E. B. Laneev, M. N. Mouratov, and E. P. Zhidkov, “Discretization and its proof for numerical solution of a Cauchy problem for Laplace equation with inaccurately given Cauchy conditions on an inaccurately defined arbitrary surface,” *Physics of Particles and Nuclei Letters*, vol. 5, no. 3, pp. 164–167, 2002. DOI: 10.1134/S1547477108030059.

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# Об устойчивом вычислении нормали к поверхности, заданной приближённо

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**Аннотация.** В работе предлагается устойчивый метод построения нормали к поверхности, заданной приближённо. Нормаль вычисляется как градиент функции в уравнении поверхности. Как известно, задача вычисления производной является некорректно поставленной. В работе принят подход к решению этой задачи как к задаче вычисления значений неограниченного оператора. Для построения её устойчивого решения используется принцип минимума сглаживающего функционала в формулировке Морозова. Нормаль получена в виде ряда Фурье в разложении по собственным функциям оператора Лапласа в прямоугольнике с краевыми условиями второго рода. В стабилизаторе функционала используется лапласиан, что позволяет получить нормаль в виде ряда Фурье, равномерно сходящегося к точному вектору нормали при стремлении к нулю погрешности в задании поверхности. Полученный приближенный вектор нормали может использоваться при решении различных задач математической физики, использующих поверхностные интегралы, нормальные производные, потенциалы простого и двойного слоя.

**Ключевые слова:** некорректная задача, устойчивое вычисление производной, метод регуляризации, дискретный ряд Фурье