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# Constitutive tensor in the geometrized Maxwell theory

**Anna V. Korolkova**

*Peoples' Friendship University of Russia (RUDN University),*
*6, Miklukho-Maklaya St., Moscow, 117198, Russian Federation*

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**Abstract.** It is generally accepted that the main obstacle to the application of Riemannian geometrization of Maxwell's equations is an insufficient number of parameters defining a geometrized medium. In the classical description of the equations of electrodynamics in the medium, a constitutive tensor with 20 components is used. With Riemannian geometrization, the constitutive tensor is constructed from a Riemannian metric tensor having 10 components. It is assumed that this discrepancy prevents the application of Riemannian geometrization of Maxwell's equations. It is necessary to study the scope of applicability of the Riemannian geometrization of Maxwell's equations. To determine whether the lack of components is really critical for the application of Riemannian geometrization. To determine the applicability of Riemannian geometrization, the most common variants of electromagnetic media are considered. The structure of the dielectric and magnetic permittivity is written out for them, the number of significant components for these tensors is determined. Practically all the considered types of electromagnetic media require less than ten parameters to describe the constitutive tensor. In the Riemannian geometrization of Maxwell's equations, the requirement of a single impedance of the medium is critical. It is possible to circumvent this limitation by moving from the complete Maxwell's equations to the approximation of geometric optics. The Riemannian geometrization of Maxwell's equations is applicable to a wide variety of media types, but only for approximating geometric optics.

**Key words and phrases:** geometrization of Maxwell's equations, permeability tensor, dielectric constant, magnetic permeability, geometric optics

## 1. Introduction

With the advent of the model Cayley–Klein [1, 2] the formalism of non-Euclidean spaces became used to describe physical models. This approach received popularity after the creation Einstein's general theory of relativity [3]. At the same time, there were attempts to geometrize Maxwell's electrodynamics [4–6].

However, this approach remained quite marginal until the golden age of theory of relativity [7, 8].

This direction became popular again in the new century and gave rise to the development of transformational optics [9–12]. However, it became visible that Riemannian geometry is insufficient for geometrization of Maxwell's equations [13, 14].

In this paper, the author expects to figure out what could hinder the application of Riemannian geometrization of Maxwell's equations and what is the scope of its applicability. To do this, we consider different electromagnetic media options and the limitations imposed by them are studied for possible geometrizations.

### 1.1. Article structure

In paragraph 1.2 we provide basic notation and conventions used in the article. In the section 1.3 we consider the limitation only for the case of a local linear medium. In the section 2 the constitutive tensor is formulated in a six-dimensional space. This is being done for clarity, to represent it as a matrix $6 \times 6$. In the section 3 the reader is reminded of Riemannian geometrization of Maxwell's equations.

### 1.2. Notations and conventions

1. Greek indexes ($\alpha$, $\beta$) will relate to a four-dimensional space and in a component form will have the following values: $\alpha = \overline{0,3}$.
2. Latin indexes from the middle of the alphabet ($i$, $j$, $k$) will refer to three-dimensional space and in component form will have the following values: $i = \overline{1,3}$.
3. In uppercase Latin letters denote the indices of the six-dimensional spaces: $I = \overline{1,6}$.
4. To write the equations of electrodynamics in the work is used symmetrical CGS system [15].

### 1.3. The variations of physical environment

It is possible to consider several options for setting the constitutive laws depending on the medium (see Table 1).

Table 1

Constitutive laws depending on the medium

| Medium type | Local case | Non-local case |
|---|---|---|
| Linear medium | $G^{\alpha\beta} = \lambda^{\alpha\beta\gamma\delta} F_{\gamma\delta}$ | $G(x) = \int \lambda(x, s) \wedge F(s) \, \mathrm{d}s$ |
| Non-linear medium | $G^{\alpha\beta} = \lambda(F_{\gamma\delta})$ | $G(x) = \int \lambda(x, F(s)) \, \mathrm{d}s$ |

Tensors $F_{\alpha\beta}$ and $G^{\alpha\beta}$ have a sense of curvature in cotangent ($T^*X$) and tangent ($TX$) bundles.

In the linear local case, the tensors $F_{\alpha\beta}$ and $G^{\alpha\beta}$ are connected using fourth rang tensors.

In the linear non-local case, the connection is carried out using an integral kernel. However, in the presence of translational symmetry, the linear nonlocal case is reduced to the linear local case using the Fourier transform [16]. The non-local linear relationship between $F$ and $G$ looks in this case as follows:

$$G(x) = \int \lambda(x, s) \wedge F(s)\, \mathrm{d}s\,. \tag{1}$$

In the case of translational invariance $\lambda(x, s) = \lambda(x - s)$ the relationship between $F$ and $G$ will have the form:

$$G^{\alpha\beta}(\omega, k_i) = \lambda^{\alpha\beta\gamma\delta}(\omega, k_i) F_{\gamma\delta}(\omega, k_i). \tag{2}$$

In the case of a nonlinear medium, it is assumed that through the linearization procedure, a tensor term similar to the local linear case can be distinguished in it.

Thus, it seems sufficient to consider only the local linear case.

## 2. Structure of the constitutive tensor

### 2.1. Representation of the constitutive tensor in space $\mathbb{R}^4$

The constitutive tensor $\lambda^{\alpha\beta}_{\gamma\delta}$ is a 4-tensor. We assume that the mapping $\lambda : \Lambda^2 M \to \Lambda_2 M$ is linear and local. Then it can be represented in the following form:

$$G^{\alpha\beta} = \lambda^{\alpha\beta\gamma\delta} F_{\gamma\delta}. \tag{3}$$

Here $\lambda^{\alpha\beta\gamma\delta}$ is a constitutive tensor containing information about both permeability and permittivity and electromagnetic connection [4–6, 17]. It can be seen that $\lambda^{\alpha\beta\gamma\delta}$ has the following symmetry:

$$\lambda^{\alpha\beta\gamma\delta} = \lambda^{[\alpha\beta][\gamma\delta]}.$$

To clarify the symmetry, the tensor $\lambda^{\alpha\beta\gamma\delta}$ can be represented as follows:

$$\lambda^{\alpha\beta\gamma\delta} = {}^{(1)}\lambda^{\alpha\beta\gamma\delta} + {}^{(2)}\lambda^{\alpha\beta\gamma\delta} + {}^{(3)}\lambda^{\alpha\beta\gamma\delta}.$$

The components of the tensor have the following symmetry:

$${}^{(1)}\lambda^{\alpha\beta\gamma\delta} = \lambda^{([\alpha\beta][\gamma\delta])}, \quad {}^{(2)}\lambda^{\alpha\beta\gamma\delta} = \lambda^{[[\alpha\beta][\gamma\delta]]}, \quad {}^{(3)}\lambda^{\alpha\beta\gamma\delta} = \lambda^{[\alpha\beta\gamma\delta]}.$$

Obviously, in this case $\lambda^{\alpha\beta\gamma\delta}$ has 36 independent components, ${}^{(1)}\lambda^{\alpha\beta\gamma\delta}$ has 20 independent components (*principal part*), ${}^{(2)}\lambda^{\alpha\beta\gamma\delta}$ has 15 independent components (*skewon*), ${}^{(3)}\lambda^{\alpha\beta\gamma\delta}$ has one independent component (*axion*).

We will consider only part of ${}^{(1)}\lambda^{\alpha\beta\gamma\delta}$. For this case, we write down the material equations:

$$\begin{cases} D^i = \varepsilon^{ij} E_j + {}^{(1)}\gamma^i_j B^j, \\ H_i = \left(\mu^{-1}\right)_{ij} B^j + {}^{(2)}\gamma^j_i E_j, \end{cases} \tag{4}$$

where $\varepsilon^{ij}$ and $\mu^{ij}$ are the tensors of dielectric and magnetic permeability, ${}^{(1)}\gamma^i_j$ and ${}^{(2)}\gamma^i_j$ are cross terms.

Taking into account the structure of the tensors $F_{\alpha\beta}$ and $G^{\alpha\beta}$, as well as the constraints equations, we write:

$$\begin{gathered} F_{0i} = E_i, \quad G^{0i} = -D^i, \\ G^{ij} = -e^{ijk} H_k, \quad F_{ij} = -e_{ijk} B^k. \end{gathered} \tag{5}$$

Here the alternating tensor is denoted by $e_{ijk}$.

### 2.2. Representation of constitutive tensors in $A_2(\mathbb{R}^4)$ and $A^2(\mathbb{R}^{4*})$ spaces

Consider vector spaces $A^2(\mathbb{R}^{4*})$ and $A_2(\mathbb{R}^4)$ as typical layers of bundles $\Lambda^2 M$ and $\Lambda_2 M$ and we will make the transition to a six-dimensional space. Basis $A_2(\mathbb{R}^4)$ in this case has the form $\zeta_I$, $I = 1, \dots, 6$, and the basis $A^2(\mathbb{R}^{4*})$ consists of components $\zeta^I$, $I = 1, \dots, 6$. Let $\delta_\mu$, $\mu = 0, \dots, 3$ be the basis in $\mathbb{R}^4$, and $\delta^\mu$, $\mu = 0, \dots, 3$ — the basis in $\mathbb{R}^{4*}$. Define the basis $\zeta_I$ in $A_2(\mathbb{R}^4)$ as follows:

$$\zeta_i = \delta_0 \wedge \delta_i, \quad \zeta_{i+3} = \frac{1}{2} \varepsilon_{ijk} \delta_j \wedge \delta_k, \quad i, j, k = 1, \dots, 3, \tag{6}$$

and basis $\zeta^I$ in $A^2(\mathbb{R}^{4*})$ in form

$$\zeta^i = \delta^0 \wedge \delta^i, \quad \zeta^{i+3} = \frac{1}{2} \varepsilon^{ijk} \delta^j \wedge \delta^k, \quad i, j, k = 1, \dots, 3. \tag{7}$$

Then the intensity of the electromagnetic field $F$ can be represented as follows:

$$F = E_i \zeta^i + B_i \zeta^{i+3}. \tag{8}$$

We split the tensor $\lambda^{IJ}$ such as

$$\lambda^{IJ} = {}^{(1)}\lambda^{IJ} + {}^{(2)}\lambda^{IJ} + {}^{(3)}\lambda^{IJ}.$$

The components of the tensor $\lambda^{IJ}$ have the following symmetry:

$${}^{(1)}\lambda^{IJ} = \lambda^{(IJ)} - \lambda^K_K \tilde{I}^{IJ}, \quad {}^{(2)}\lambda^{IJ} = \lambda^{[IJ]}, \quad {}^{(3)}\lambda^{IJ} = \lambda^K_K \tilde{I}^{IJ},$$

$$\tilde{I} := \begin{pmatrix} 0 & I^{ij} \\ I^{ij} & 0 \end{pmatrix}.$$

We indicate the number of components in these tensors:
— ${}^{(1)}\lambda^{IJ}$ has 20 components;
— ${}^{(2)}\lambda^{IJ}$ has 15 components;
— ${}^{(3)}\lambda^{IJ}$ has 1 component.

Let's write out the main part of the constitutive tensor:

$$
{}^{(1)}\lambda^{IJ} = \begin{pmatrix} -\varepsilon^{ij} & {}^{(1)}\gamma^{i}_{j} \\ {}^{(2)}\gamma^{j}_{i} & \tilde{\mu}_{ij} \end{pmatrix}, \quad \tilde{\mu}_{ij} := \left(\mu^{-1}\right)_{ij}. \tag{9}
$$

Later in this article we will omit the left index of the main part of the constitutive tensor.

## 3. Riemannian geometrization of Maxwell's equations

We assume that the bundle has the structure of a Riemannian manifold. In this case, we can introduce a Riemannian metric on the manifold, which:
— is symmetric: $g_{\alpha\beta} := g_{(\alpha\beta)}$;
— is consistent with connection: $\nabla_{\alpha} g^{\alpha\beta} := 0$.

This statement is equivalent to the fact that we use connection Levi–Civitas.

We introduce an effective metric based on the bundle $g_{\alpha\beta}$. Then the metric is induced into layers and the Lagrangian of the electromagnetic field can be written in the form of the Yang–Mills Lagrangian:

$$
L = -\frac{1}{16\pi c} G^{\alpha\beta} F_{\alpha\beta} - \frac{1}{c^2} A_{\alpha} j^{\alpha} \sqrt{-g},
$$

which is equivalent to the following entry

$$
L = -\frac{1}{16\pi c} g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} \sqrt{-g} - \frac{1}{c^2} A_{\alpha} j^{\alpha} \sqrt{-g}.
$$

Let's construct the tensor $\lambda^{\alpha\beta\gamma\delta}$ as follows:

$$
\lambda^{\alpha\beta\gamma\delta} = \sqrt{-g} g^{\alpha\beta} g^{\gamma\delta} = \frac{\sqrt{-g}}{2} \left( g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\gamma} \right) + \frac{\sqrt{-g}}{2} \left( g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma} \right).
$$

Then the material equations will take the following form (for symmetry reasons):

$$
G^{\alpha\beta} = \frac{\sqrt{-g}}{2} \left( g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma} \right) F_{\gamma\delta}.
$$

In the case of writing by components, we get the following expressions:

$$
\begin{aligned}
G^{0i} &= \frac{\sqrt{-g}}{2} (g^{00} g^{ij} - g^{0i} g^{0j}) F_{0j} + \frac{\sqrt{-g}}{2} (g^{0j} g^{ik} - g^{0k} g^{ij}) F_{jk}, \\
G^{ij} &= \frac{\sqrt{-g}}{2} (g^{i0} g^{jk} - g^{0j} g^{ik}) F_{0k} + \frac{\sqrt{-g}}{2} (g^{ik} g^{jl} - g^{il} g^{jk}) F_{kl}.
\end{aligned} \tag{10}
$$

Formally, it is possible to write out an expression for the permittivity:

$$\varepsilon^{ij} = -\sqrt{-g}(g^{00} g^{ij} - g^{0i} g^{0j}) \tag{11}$$

and the expression for magnetic permeability:

$$(\mu^{-1})_{ij} = \sqrt{-g}\varepsilon_{mni}\varepsilon_{klj} g^{nk} g^{ml}. \tag{12}$$

Thus the geometrized connection equations in coordinates have the following form:

$$\begin{aligned} D^i &= \varepsilon^{ij} E_j + {}^{(1)}\gamma^i_j B^j, \\ H_i &= (\mu^{-1})_{ij} B^j + {}^{(2)}\gamma_i^j E_j, \\ \varepsilon^{ij} &= -\sqrt{-g}(g^{00} g^{ij} - g^{0i} g^{0j}), \\ (\mu^{-1})_{ij} &= \sqrt{-g}\varepsilon_{mni}\varepsilon_{klj} g^{nk} g^{ml}, \\ {}^{(1)}\gamma^i_j &= {}^{(2)}\gamma^i_j = \sqrt{-g}\varepsilon_{klj} g^{0k} g^{il}. \end{aligned} \tag{13}$$

**Statement.** Let the space be represented as $\mathbb{R}^4 = \mathbb{R}^1 \times \mathbb{R}^3$. Then in Riemannian geometrization, under the condition $g^{0i} = 0$, the equality holds

$$\varepsilon^{ij} = \mu^{ij}. \tag{14}$$

**Proof.** Note that $\Delta_{ij} = \varepsilon_{mni}\varepsilon_{klj} g^{nk} g^{ml}$ is an algebraic complement for $g^{ij}$. Then

$$\varepsilon^{ij}(\mu^{-1})_{ip} = -\sqrt{-g} g^{00} g^{ij} \sqrt{-g}\varepsilon_{mni}\varepsilon_{klp} g^{nk} g^{ml} = g g^{00} \det\{g^{kl}\}\delta^j_p = \delta^j_p.$$

It follows that $\varepsilon^{ij} = \mu^{ij}$. □

Then the geometrized constitutive tensor has the following form:

$$\lambda^{IJ} = \begin{pmatrix} -\varepsilon^{ij} & {}^{(1)}\gamma^i_j \\ {}^{(2)}\gamma_i^j & (\varepsilon^{-1})_{ij} \end{pmatrix}. \tag{15}$$

Consider the limitations of this approach:

1. Since the metric tensor $g_{ij}$ has 10 components, the geometrized constitutive tensor cannot have more than 10 independent components.
2. Given the constitutive equations, only media with a single impedance can be considered.

However, the geometrized version can be used to approximate geometric optics when the dielectric $\varepsilon^{ij}$ and magnetic $\mu_{ij}$ permeability are not used separately. Instead, in the approximation of geometric optics, the refractive index of the medium is used:

$$n^i_j = \sqrt{\varepsilon^i_k \mu^k_j}. \tag{16}$$

In this case, the geometrized constitutive tensor has the following form:

$$\lambda^{IJ} = \begin{pmatrix} -(\sqrt{n})^{ij} & {}^{(1)}\gamma^{i}_{j} \\ {}^{(2)}\gamma^{j}_{i} & \left(\frac{1}{\sqrt{n}}\right)_{ij} \end{pmatrix}. \tag{17}$$

## 4. Examples of media

### 4.1. Linear isotropic media

The most elementary electromagnetic media are linear isotropic media, such as classical vacuum. The term *isotropic* refers to invariance with respect to spatial rotations in the selected frame of reference. The rotation of any closed system as a whole does not change its physical properties. There is no particular direction in space with respect to which there is any special symmetry. All directions are equal. The electromagnetic properties of the medium do not depend on the direction. In this case, the elements of the tensor $\lambda^{IJ}$ are represented as:

$$\varepsilon^{ij} = \varepsilon(x^i)\delta^{ij}, \quad \tilde{\mu}_{ij} := (\mu^{-1}(x^i))\delta_{ij}, \quad {}^{(1)}\gamma^{i}_{j} = 0, \quad {}^{(2)}\gamma^{j}_{i} = 0$$

or in matrix form:

$$\lambda^{IJ} = \begin{pmatrix} -\varepsilon(x^i)\delta^{ij} & 0 \\ 0 & \mu^{-1}(x^i)\delta_{ij} \end{pmatrix}. \tag{18}$$

In this case, the permeability matrix contains only two independent components in the laboratory reference frame. Function $\varepsilon(x^i)$ is called the dielectric constant of the medium. Function $\mu(x^i)$ is called the magnetic permeability of the medium. When these functions are constant in the selected frame of reference, the medium is called *homogeneous.*

The classical electromagnetic vacuum is assumed to be linear, isotropic and homogeneous. Its dielectric constant (in the SI system) is denoted by $\varepsilon_0$, and the magnetic permeability is denoted by $\mu_0$.

The application of a geometrized constitutive tensor is possible in the approximation of geometric optics. In this case $\lambda^{IJ}$ will have the form:

$$\lambda^{IJ} = \begin{pmatrix} -\sqrt{n(x^i)}\delta^{ij} & 0 \\ 0 & \frac{1}{\sqrt{n(x^i)}}\delta_{ij} \end{pmatrix}. \tag{19}$$

In this case, the permeability matrix contains only one independent component.

### 4.2. Linear optical medium

It is assumed that the permittivity $\varepsilon^{ij}$ can be inhomogeneous and (or) anisotropic. Heterogeneity is most common when matrix components are piecewise constant and undergo discontinuities at the interface of heterogeneous media. Since the magnetic permeability of optical media is neglected, they are considered to be dielectric media:

$$\left(\mu^{-1}\right)_{ij} = \delta_{ij}. \tag{20}$$

If the permittivity $\varepsilon^{ij}$ is anisotropic, but at the same time symmetrical, then it is possible to determine the main reference point in which it takes the form of a diagonal matrix:

$$\varepsilon^{ij} := \operatorname{diag}(\varepsilon_x, \varepsilon_y, \varepsilon_z). \tag{21}$$

The diagonal elements represented by the main permittivity are in this case the eigenvalues of the matrix

$$\varepsilon^i_j = \varepsilon^{ik} g_{jk}. \tag{22}$$

Since the matrix $\varepsilon^{ij}$ is symmetric, then the eigenvalues exist and are valid, and the eigenvectors are orthogonal.

Media variants:

- — when all eigenvalues are equal, the medium is called isotropic;
- — when two eigenvalues are equal, and the third is different from them, the medium is called uniaxial anisotropic;
- — when all three eigenvalues are unequal, the medium is called biaxial anisotropic.

More generally, the magnetic permeability is not singular:

$$\left(\mu^{-1}\right)_{ij} = \operatorname{diag}((\mu^{-1})_x, (\mu^{-1})_y, (\mu^{-1})_z), \tag{23}$$

and $\lambda^{IJ}$ can be represented as a matrix:

$$\lambda^{IJ} = \operatorname{diag}\left(-\varepsilon_x(x^i), -\varepsilon_y(x^i), -\varepsilon_z(x^i), \mu_x^{-1}(x^i), \mu_y^{-1}(x^i), \mu_z^{-1}(x^i)\right). \tag{24}$$

The application of a geometrized constitutive tensor is possible in the approximation of geometric optics. Then the tensor $\lambda^{IJ}$ will take the form:

$$\lambda^{IJ} = \operatorname{diag}\left(-\sqrt{n_x(x^i)}, -\sqrt{n_y(x^i)}, -\sqrt{n_z(x^i)}, \frac{1}{\sqrt{n_x(x^i)}}, \frac{1}{\sqrt{n_y(x^i)}}, \frac{1}{\sqrt{n_z(x^i)}}\right).$$

In this case, the permeability matrix contains three independent components.

### 4.3. Bi-isotropic media

The special properties of these media are due to the connection between electric and magnetic fields, which can be described by some defining relations. Bi-isotropic media can change the polarization of light either by refraction or by transmission [18]. These media are similar to isotropic media, but the cross terms are not zero.

The coupling equations in the case of an isotropic medium have the following form:

$$
\begin{aligned}
D^i &= \varepsilon g^{ij} E_j + \gamma g^i_j B^j, \\
H_i &= (\mu^{-1}) g_{ij} B^j + \gamma g^j_i E_j.
\end{aligned}
\tag{25}
$$

The elements of the tensor $\lambda^{IJ}$ have the form:

$$
\varepsilon^{ij} = \varepsilon(x^i) g^{ij}, \quad (\mu^{-1})_{ij} := (\mu^{-1}(x^i)) g_{ij}, \quad {}^{(1)}\gamma^i_j = \gamma(x^i) g^i_j, \quad {}^{(2)}\gamma^j_i = \gamma(x^i) g^j_i.
$$

In matrix form, the tensor $\lambda^{IJ}$ for a bi-isotropic medium will have the form:

$$
\lambda^{IJ} = \begin{pmatrix} -\varepsilon(x^i) g^{ij} & \gamma(x^i) g^i_j \\ \gamma(x^i) g^j_i & \mu^{-1}(x^i) g_{ij} \end{pmatrix}.
\tag{26}
$$

For the case of geometric optics, the tensor $\lambda^{IJ}$ for a bi-isotropic medium will take the form:

$$
\lambda^{IJ} = \begin{pmatrix} -\sqrt{n(x^i)} g^{ij} & \gamma(x^i) g^i_j \\ \gamma(x^i) g^j_i & n^{-1/2}(x^i) g_{ij} \end{pmatrix}.
\tag{27}
$$

The permeability matrix in this case contains two independent components.

### 4.4. Bi-anisotropic media

In bi-anisotropic media, the dielectric constant, magnetic permeability, and coupling coefficient are complete tensors. In this case , the coupling equations have the following form:

$$
\begin{aligned}
D^i &= \varepsilon^{ij} E_j + {}^{(1)}\gamma^i_j B^j, \\
H_i &= (\mu^{-1})_{ij} B^j + {}^{(2)}\gamma^j_i E_j.
\end{aligned}
\tag{28}
$$

The elements of the tensor $\lambda^{IJ}$ take the form

$$
\varepsilon^{ij} := \varepsilon^{ij}(x^i), \quad (\mu^{-1})_{ij} := (\mu^{-1}(x^i))_{ij}, \quad {}^{(1)}\gamma^i_j = \gamma^i_j(x^i), \quad {}^{(2)}\gamma^j_i = \gamma^j_i(x^i).
$$

In matrix form, the tensor $\lambda^{IJ}$ for a bi-isotropic medium has the form:

$$
\lambda^{IJ} = \begin{pmatrix} -\varepsilon^{ij}(x^i) & \gamma^i_j(x^i) \\ \gamma^j_i(x^i) & (\mu^{-1})_{ij}(x^i) \end{pmatrix}.
\tag{29}
$$

For the case of geometric optics, the tensor $\lambda^{IJ}$ for a bi-anisotropic medium is represented as:

$$
\lambda^{IJ} = \begin{pmatrix} -(\sqrt{n})^{ij}(x^i) & \gamma^i_j(x^i) \\ \gamma^j_i(x^i) & \left(n^{-1/2}\right)_{ij}(x^i) \end{pmatrix}.
\tag{30}
$$

In this case, the permeability matrix contains twenty independent components.

Thus, the following conclusions can be drawn:

— in most practical cases, it is necessary to take into account less than ten components of the tensor $\lambda^{IJ}$;

— for the case of an anisotropic medium, more than ten components of the constitutive tensor should be taken into account.

## 5. Conclusion

When applying the Riemannian geometrization of Maxwell's equations, two significant obstacles arise.

It is usually considered that the main obstacle to the application of Riemannian geometrization of Maxwell's equations is an insufficient number of parameters defining a geometrized medium. It is known that in the classical description of the equations of electrodynamics in the medium, a constitutive tensor with 20 components is used, and in Riemannian geometrization, the constitutive tensor is constructed from a Riemannian metric tensor with only 10 components. Thus, most authors point out that the main limitation of the application of Maxwell's geometrized theory is the number of free components in the constitutive tensor. However, this is not the case. It is enough to consider the most popular variants of electromagnetic media to make sure that in practically used cases the number of components is significantly less than ten.

Another limitation is that Maxwell's geometrized theory in the case of Riemannian geometrization requires that the medium has a unit impedance. This restriction is too strong in the general case. It seems that the geometrized Maxwell theory in the case of Riemannian geometrization is not applicable in the case of the complete Maxwell theory and in the case of the approximation of the wave equation. But this limitation can be circumvented by switching to the approximation of geometric optics, since in this case the impedance of the medium is not taken into account.

Thus, we can conclude that Maxwell's geometrized theory in the case of Riemannian geometrization is applicable to the description of Maxwell's theory, but mainly for the case of approximation of geometric optics.

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**Information about the authors**:


**Korolkova, Anna V.** — Docent, Candidate of Sciences in Physics and Mathematics, Associate Professor of Department of Applied Probability and Informatics of Peoples' Friendship University of Russia (RUDN University) (e-mail: korolkova-av@rudn.ru, phone: +7(495) 952-02-50, ORCID: https://orcid.org/0000-0001-7141-7610)

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# Тензор проницаемостей в геометризованной теории Максвелла

**А. В. Королькова**

*Российский университет дружбы народов,*
*ул. Миклухо-Маклая, д. 6, Москва, 117198, Россия*

**Аннотация.** Считается, что основным препятствием к применению римановой геометризации уравнений Максвелла является недостаточное количество параметров, задающих геометризованную среду. При классическом описании уравнений электродинамики в среде используется тензор проницаемостей, имеющий 20 компонент. При римановой геометризации тензор проницаемостей строится из риманового метрического тензора, имеющего только 10 компонент. Предполагается, что данное несоответствие мешает применению римановой геометризации уравнений Максвелла. В статье предложено определить, действительно ли недостаток компонент является критическим для применения римановой геометризации уравнений Максвелла. Для определения области применимости римановой геометризации рассмотрены наиболее распространённые варианты электромагнитных сред. Для них выписана структура диэлектрической и магнитной проницаемостей, определено количество значащих компонент для этих тензоров. Показано, что практически все рассмотренные типы электромагнитных сред требуют менее десяти параметров для описания тензора проницаемостей. При римановой геометризации уравнений Максвелла критическим является требование единичного импеданса среды. Обойти данное ограничение возможно путём перехода от полных уравнений Максвелла к приближению геометрической оптики. Показано, что риманова геометризация уравнений Максвелла применима для большого разнообразия типов среды, но только для приближения геометрической оптики.

**Ключевые слова:** геометризация уравнений Максвелла, тензор проницаемостей, диэлектрическая проницаемость, магнитная проницаемость, геометрическая оптика