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Solving Differential Equations of Motion for Constrained Mechanical Systems

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This paper presents an investigation of modeling and solving system of differential equations in the study of mechanical systems with holonomic constraints. A method is developed for constracting equation of motion for mechanical system with constraints. A technique is developed how to approximate the solution of the problem that is obtained from modeling of kinematic constraint equation which is stable. A perturbation analysis shows that velocity stabilization is the most efficient projection with regard to improvement of the numerical integration. How frequently the numerical solution of the ordinary differential equation should be stabilized is discussed. A procedure is indicated to get approximate solution when the systems of differential equations can't be solved analytically. A new approach is applied for constructing and stabilyzing Runge-Kutta numerical methods. The Runge-Kutta numerical methods are reformulated in a new approach. Not only the technique of formulation but also the test developed for its stability is new.Finally an example is presented not only to demonstrate how the stability of the solution depends on the variation of the factor but also how to find an approximate solution of the problem using numerical integration.

Key words and phrases: numerical integration, kinematic constraint, stable solution, Taylor expansion, row decomposition.

1. Introduction

The theory of mechanical systems with kinematic constraints goes back to the last century, with important contributions by Herzt (1894), Ferrers (1871), Neumann (1888), Vierkandt (1892), Chaplygin (1897). Several recent papers [1] show a strong renewal of interest in that theory, in relation with new developments in control theory.

In the construction of equations of motion, one can treat the constraints imposed on a mechanical system as servo-constraints, and the constraint reactions can be treated as the corresponding controls. Then the construction of equations of motion can be reduced to finding expressions for the constraint reactions on the right-hand sides guaranteeing that the solutions of the system satisfy the constraint equations with the desired accuracy [2]. This is, in a sense, the constraint stabilization problem, and to solve it, one should take account of not only the deviations from the constraint equations but also their derivatives [2]. It turns out that the constraint stabilization is possible only if the constraint equations are particular integrals of the equations of motion of the mechanical system and the differential equations for the deviations from the constraint equations have an asymptotically stable trivial solution [2]. Still this does not guarantee that the deviations from the constraint equations remain bounded in the numerical solution of the differential-algebraic equations comprised by the equations of motion and the constraint equations. In the present paper, to estimate the deviations from the constraint equations, we introduce the equations of constraints and the equations of constraint perturbations and define the notions of stable and asymptotically stable constraints.

2. Modeling of Kinematic Mechanical System

Kinematic state of mechanical system can be described by ordinary differential equations:

$$\dot{y} = v(y, t), \quad \text{where} \quad y(t_0) = y^0, \quad y \in R^n. \tag{1}$$

Assume the constraints to be imposed:

$$f(y, t) = 0, \quad \text{where} \quad f \in R^m, \quad m \leqslant n, \quad f(y^0, t_0) = 0. \tag{2}$$

The vector on the right hand side of equation (1) should be taken so that for all $t \geqslant t_0$ and $q = q(t)$ satisfying equation (2) it also satisfies the following equality:

$$f_y v + f_t = 0, \tag{3}$$

where $f_y = (f_{ij}),\ f_t = (f_{it}),\ f_{ij} = \frac{\partial f_i}{\partial y_j}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. The solution of equation (3) can be constructed directly if some unknowns $v^2 = (v_{m+1}, \ldots, v_n)$ given arbitrary and determine the rest $v^1 = (v_1, \ldots, v_m)$ using

$$f_{y1} v^1 = -(f_{y2} v^2 + f_t),$$

$$\text{where}, \quad f_{y1} = (f_{ik}), \quad f_{y2} = (f_{il}), \quad i, k = 1, \ldots, m, \quad l = m + 1, \ldots, n.$$

Then the system in equation (1) will have the form [2]:

$$\dot{y}^1 = -f_{y1}^{-1}(f_{y2} v^2 + f_t), \quad \dot{y}^2 = v^2(y, t), \tag{4}$$

To use the above system (4), it is important to require the validity of initial condition (2) and the relation $\det(f_{y1}) \neq 0$. In some cases, this condition is satisfied. Unfortunately, the requirement $\det(f_{y1}) \neq 0$ is not the only requirement to use the method mentioned above for the construction of differential equations. In the application of this method one should have in mind the possibility of integration error accumulation for equation (4), which in the course of time, leads to the destruction of the constraint equation (2). Consequently, equation (1) should be constructed so as to ensure that the solution $y = y(t)$ satisfying the initial conditions $y(t_0) = y^0$ with

$$\left\| f(y^0, t_0) \right\| \leqslant \varepsilon \tag{5}$$

deviates from constraint equation (2) in the course of numerical integration by a quantity of the order of ε:

$$\| f(y, t) \| \leqslant \varepsilon. \tag{6}$$

To define the right-hand side $v(y, t)$ of equation (1) instead of equation (3), one uses the relation [2]

$$f_y v + f_t = a, \tag{7}$$

where $a = a(f, y, t)$, $a(0, y, t) = 0$. It follows from (7) that the vector v should be determined as a solution of m linear equations with n unknowns:

$$Av = b, \tag{8}$$

where $A = f_y$, $b = a - f_t$.The structure of the general solution of equation (8) is described by the following theorem [2]:

Theorem 1. *The set of all solutions of the linear systems* (6)*, in which the matrix A has rank r is determined by the relation*

$$v = c\,[AC] + A^{+} b = c v^{\tau} + v^{\nu}, \tag{9}$$

where c is an arbitrary scalar quantity, $[AC] = [A_1, \ldots, A_m C_{m+1}, \ldots, C_{n-1}]$ *is vector product,*$A_i = (A_{ij})$, *and arbitrary* $C_\tau = (C_{\tau j})$, $\tau = m+1, \ldots, n-1$, $j = 1, \ldots, n$, $A^+ = A^T(AA^T)^{-1}$ *and the component* $[AC]_j$ *of a determinant is*

$$[AC]_j = \begin{bmatrix} \delta_{j1} & \delta_{j2} \ldots & \delta_{jn} \\ A_{11} & A_{12} \ldots & A_{1n} \\ \cdot & \cdot \ldots & \cdot \\ A_{m1} & A_{m2} \ldots & A_{mn} \\ C_{m+1,1} & C_{m+1,2} \ldots & C_{m+1,n} \\ \cdot & \cdot \ldots & \cdot \\ C_{n-1,1} & C_{n-1,2} \ldots & C_{n-1,n} \end{bmatrix}, \quad \text{where} \quad \delta_{jk} = \begin{cases} 0, & \text{if } j \neq k; \\ 1, & \text{if } j = k. \end{cases}$$

Example 1. A slider crank mechanism is given in the figure below.

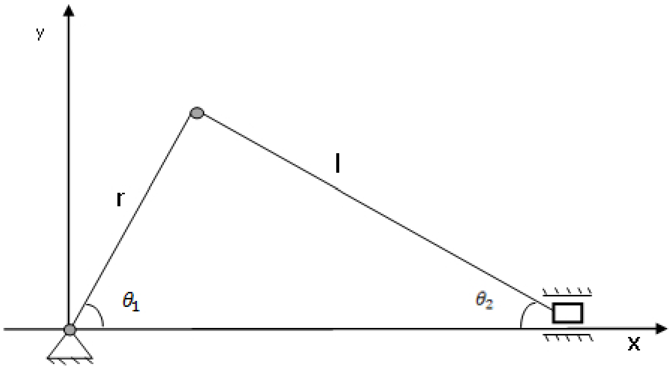

Figure 1. Slider crank mechanism

The translational displacement x of a unit mass and the angular displacement θ of crank are joined by the rod l. Then the constraint equation will be

$$f(y) = r\sin\theta_1 - l\sin\theta_2 = 0$$

and $Av = b$, where $A = f_y$, $b = a = kf$, as $f_t = 0$, the solution of this can be put as

$$v = cv^\tau + v^\nu, \quad v^\tau = [f_y], \quad v^\nu = f_y^+ a$$

where

$$v_1^\tau = -l\cos\theta_2, \quad v_2^\tau = -r\cos\theta_1,$$

$$v_1^\nu = \frac{kr\cos\theta_1(r\sin\theta_1 - l\sin\theta_2)}{(r\cos\theta_1)^2 + (l\cos\theta_2)^2}, \quad v_2^\nu = \frac{-kl\cos\theta_2(r\sin\theta_1 - l\sin\theta_2)}{(r\cos\theta_1)^2 + (l\cos\theta_2)^2},$$

where $k > 0$. Then

$$v_1 = cv_1^\tau + v_1^\nu = -cl\cos\theta_2 + \frac{kr\cos\theta_1(r\sin\theta_1 - l\sin\theta_2)}{(r\cos\theta_1)^2 + (l\cos\theta_2)^2},$$

$$v_2 = cv_2^\tau + v_2^\nu = -cr\cos\theta_1 - \frac{kl\cos\theta_2(r\sin\theta_1 - l\sin\theta_2)}{(r\cos\theta_1)^2 + (l\cos\theta_2)^2}.$$

Taking $c = 1,\ r = 1, l = 2$ and the value of k to be choosen depending on the stability of the solution.

$$v_1 = -2\cos\theta_2 + \frac{k\cos\theta_1(\sin\theta_1 - 2\sin\theta_2)}{(\cos\theta_1)^2 + (2\cos\theta_2)^2}, \quad v_2 = -\cos\theta_1 - \frac{2k\cos\theta_2(\sin\theta_1 - 2\sin\theta_2)}{(\cos\theta_1)^2 + (2\cos\theta_2)^2}.$$

3. An Approximate Solution of Differential Equations

Many systems involving differential equations are so complex, or the systems that they describe are so large, that a purely mathematical analysis is not possible. It is in these complex systems where computer simulations and numerical approximations are useful [3].

Most often, systems of differential equations can not be solved analytically. Algorithms based on numerical methods are therefore needed. By numerical integration, we mean to compute, from y^0 (the initial condition), each successive result y^1, y^2, y^3, ... that satisfy equation

$$\frac{dy}{dt} = v(y, t). \tag{10}$$

An algorithm is thus a program that computes as precisely as possible y^{n+1} from y^n,where, $y^n = y(t_n)$, $t_{n+1} = t_n + \tau$, and $\dot{y} = c\,[f_y C] + f_y^+(Kf - f_t)$.

Of course, y can be a vector and the equations can be non-linear differential equations [3].

3.1. Euler's method

Let us assume that the initial conditions t_0 and $y(t_0)$ be known for the solution of the equation (10). Setting $a = Kf$ and using the right side of (10) construct the equation

$$y^{n+1} = y^n + \tau \dot{y}^n. \tag{11}$$

For a differential equation the following assertion holds [2]:

Theorem 2. *There are constants $\alpha, \tau_1, \varepsilon$ and matrix $K(y,t)$, which if*
1) $\left\|f(y^0, t_0)\right\| \leqslant \varepsilon$,
2) $\tau \leqslant \tau_1$, *and for all* $y = y^n, t = t_n, n = 0, 1, \dots$ *the inequality is fulfilled*
3) $\|I + \tau K(y,t)\| \leqslant \alpha < 1$,
4) $\frac{\tau_1^2}{2}\left\|f^{(2)}\right\| \leqslant (1-\alpha)\varepsilon$,
where $f^{(2)} = v^T f_{y^T y} v + 2 f_{yt} + f_{tt}$, *then the solution of the difference equation* (11) *will satisfy the condition* $\|f(y^n, t_n)\| \leqslant \varepsilon$, *for any* $n = 1, 2, \dots$.

The above theorem helps us to estimate the value of the constant matrix K in our example1, so that the solution is stable.

Example 2. Consider the problem what we have constructed in Ex. 1, that is,

$$\dot{\theta}_1 = v_1(\theta_1, \theta_2), \quad \dot{\theta}_2 = v_2(\theta_1, \theta_2),$$

$$v_1 = c v_1^\tau + v_1^\nu = -cl\cos\theta_2 + \frac{kr\cos\theta_1(r\sin\theta_1 - l\sin\theta_2)}{(r\cos\theta_1)^2 + (l\cos\theta_2)^2},$$

$$v_2 = c v_2^\tau + v_2^\nu = -cr\cos\theta_1 - \frac{kl\cos\theta_2(r\sin\theta_1 - l\sin\theta_2)}{(r\cos\theta_1)^2 + (l\cos\theta_2)^2} \tag{12}$$

System (12) has a particular solution

$$f(\theta_1, \theta_2) = r\sin\theta_1 - l\sin\theta_2 = 0 \tag{13}$$

θ_1^{n+1} and θ_2^{n+1} be calculated as $\theta_1^{n+1} = \theta_1^n + \tau v_1^n$, $\theta_2^{n+1} = \theta_2^n + \tau v_2^n$, $\theta_1^0 = 0$, and $\theta_2^0 = 0$. From (13) it follows that condition 1. holds clearly: $|f(0,0)| = 0 < \varepsilon$. Condition 3 imposes a limit and k: $|1 + \tau k| \leqslant \alpha < 1$, that is,

$$\frac{1-\alpha}{\tau} \leqslant k \leqslant \frac{\alpha+1}{\tau} \tag{14}$$

To check condition 4 for this problem, consider the right hand side of (12) and rewrite it as $|r\sin\theta_1 - l\sin\theta_2| \leqslant \varepsilon$, $(r\cos\theta_1)^2+(l\cos\theta_2)^2 \geqslant l^2\cos^2\theta_2 \geqslant l^2-r^2\sin^2\theta_1-2r\,|\sin\theta_1|-\varepsilon^2 \geqslant l^2-r^2-2r\varepsilon_1-\varepsilon_1^2 = l^2-(r+\varepsilon_1)^2$. Then it follows:

$$|v_1| \leqslant cl + \frac{kr\varepsilon}{l^2-(r+\varepsilon_1)^2} = W_1, \quad |v_2| \leqslant cr + \frac{kl\varepsilon}{l^2-(r+\varepsilon_1)^2} = W_2.$$

So that $f^{(2)} = \frac{\partial^2 f}{\partial\theta_1^2}v_1^2 + \frac{\partial^2 f}{\partial\theta_2^2}v_2^2$,

$$\left|f^{(2)}\right| = \left|\frac{\partial^2 f}{\partial\theta_1^2}v_1^2 + \frac{\partial^2 f}{\partial\theta_2^2}v_2^2\right| \leqslant \left|\frac{\partial^2 f}{\partial\theta_1^2}v_1^2\right| + \left|\frac{\partial^2 f}{\partial\theta_2^2}v_2^2\right| =$$
$$= \left|\frac{\partial^2 f}{\partial\theta_1^2}\right|\left|v_1^2\right| + \left|\frac{\partial^2 f}{\partial\theta_2^2}\right|\left|v_2^2\right| \leqslant rW_1 + lW_2 = F.$$

Conditions 2 and 4 take the form:$\tau \leqslant \tau_1,\ \tau_1^2F \leqslant 2(1-a)\varepsilon$. Taking $r=1, l=2, c=1$ in equation (12) and $\varepsilon=\varepsilon_1=10^{-4}$,setting (approximate values):$W_1=3, W_2=2$.Then $F=17$, correspondingly condition 4 takes the form $17\times10^4\tau_1^2 \leqslant 2(1-\alpha)$ and again let us assume $\tau_1=10^{-3}, \alpha \leqslant 0.9$. Taking $\alpha=0.9$, we obtain a restriction for k: $100 \leqslant k \leqslant 1900$.

3.2. Runge-Kutta Method

Runge-Kutta methods introduce values between t_n and t_{n+1}, and evaluate v at these intermediate points [4]. The general Runge-Kutta method is defined by

$$y^{n+1} = y^n + \tau h(t_n, y^n, \tau), \tag{15}$$

where $h(t_n,y^n,\tau) = \sum_r^R c_r k_r$, with

$$k_1 = v(t_n, y^n), \quad k_r = v\left(t_n + \tau a_r, y^n + \tau\sum_{s=1}^{r-1} b_{rs}k_s\right), \tag{16}$$

and $a_r = \sum_{s=1}^{r-1} b_{rs}$, for $r=2,3,\ldots,R$. These constants, c_r, a_r and b_{rs}, are determined to ensure the highest order accuracy for the method. To establish the order of accuracy, consider the Taylor series expansion of $y(t_{n+1})$

$$y^{n+1} = y^n + \tau\dot{y}^n + \frac{\tau^2}{2}\ddot{y}^n + \ldots = y^n + \tau\left(\dot{y}^n + \frac{\tau}{2}\ddot{y}^n + \ldots\right).$$

From this we may recall that the Taylor's series method of order p for h in (15) can be written with

$$h(t_n,y^n,\tau) = h_T(t_n,y^n,\tau) = v(t_n,y^n)+\frac{\tau}{2!}v^{(1)}(t_n,y^n)+\ldots+\frac{\tau^{p-1}}{p!}v^{(p-1)}(t_n,y^n), \tag{17}$$

where $v^{(i)} = \frac{d^i}{dt^i}v(t_n,y^n),\ i=1,2,\ldots,(p-1)$. We further expand k_2, k_3 and k_4 upto order 3 and represnt the summary as follows:

$$k_2 = v(t_n + \tau a_2, y^n + \tau a_2 k_1),$$

$$k_2 = v^n + \tau a_2 F^n + \frac{\tau^2}{2}a_2^2G^n + \frac{\tau^3}{6}a_2^3H^n + \tau^4 R_v^{k_4}, \tag{18}$$

where $v^n = v(t_n, y^n)$, $F^n = f_t^n + v^n f_y^n$ and

$$G^n = f_{tt}^n + 2v^n f_{ty}^n + (v^n)^2 f_{yy}^n, \quad H^n = (v^n)^3 f_{yyy}^n + 3(v^n)^2 f_{yyt}^n + 3v^n f_{ytt}^n + f_{ttt}^n,$$

$$k_3 = v(t_n + \tau a_3, y^n + \tau(b_{31}k_1 + b_{32}k_2)), \quad a_3 = b_{31} + b_{32},$$

$$k_3 = v^n + \tau a_3 F^n + \frac{\tau^2}{2}(a_3^2 G^n + 2a_2 b_{32} f_y^n F^n) + \\ + \frac{\tau^3}{6}(a_3^3 H^n + 6a_2 b_{32} a_3 I^n F^n + 3a_2^3 b_{32} f_y^n G^n) + \tau^4 R_v^{k_4}, \quad (19)$$

where $I^n = v^n f_{yy}^n + f_{yt}^n$.

$$k_4 = v(t_n + \tau a_4, y^n + \tau(b_{41}k_1 + b_{42}k_2 + b_{43}k_3)), \quad a_4 = b_{41} + b_{42} + b_{43},$$

$$k_4 = v^n + \tau a_4 F^n + \frac{\tau^2}{2} a_4^2 G^n + \tau^2(a_2 b_{42} + a_3 b_{43}) f_y^n F^n + \frac{\tau^3}{2}(a_2^2 b_{42} + a_3^2 b_{43}) f_y^n G^n + \\ + \tau^2 a_2 b_{32} (f_y^n)^2 F^n + \tau^3(a_2 b_{42} a_4 + a_3 b_{43} a_4) I^n F^n + \frac{\tau^2}{6} a_4^2 H^n + \tau^4 R_f^{k_4}. \quad (20)$$

using equation (15) and Regrouping similar terms we get

$$h(t_n, y^n, \tau) = (c_1 + c_2 + c_3 + c_4)v^n + \tau(c_2 a_2 + c_3 a_3 + c_4 a_4)F^n + \\ + \frac{\tau^2}{2}(a_2^2 c_2 + a_3^2 c_3 + a_4^2 c_4)G^n + \tau^2(c_3 a_2 b_{32} + c_4(a_2 b_{42} + a_3 b_{32}))(f_y F)^n + \\ + \frac{\tau^3}{6}((c_2 a_2^3 + c_3 a_3^3 + c_4 a_4^3)H^n + 2(c_3 a_2 b_{32} a_3 + c_4 a_2 b_{42} a_4 + c_4 a_3 b_{43} a_4)(FI)^n + \\ + 3(c_3 a_2^2 b_{32} + c_4 a_2^2 b_{42} + c_4 a_3^2 b_{43})(f_y G)^n + 6c_4 a_2 b_{32} (f_y^n)^2 F^n) + \tau^4 R_v^{k_4}. \quad (21)$$

Now we have to match equation (17) with (21) to find unkown parameters.

3.2.1. When $R = 2$

As $c_3 = 0$, equation (21) reduces to

$$h(t_n, y^n, \tau) = (c_1 + c_2)v^n + \tau a_2 c_2 F^n + \frac{\tau^2}{2} c_2 a_2^2 G^n + \tau^3 R^{k3}, \quad (22)$$

and comparing this with (17), we have

$$c_1 + c_2 = 1; \;\; a_2 c_2 = \frac{1}{2}. \quad (23)$$

This gives a set of two equations in three unknowns and there exists a one parameter family of solutions.
Combining (22) with (15), we can rewrite Δy^n as

$$y^{n+1} - y^n = \tau h(t_n, y^n, \tau) = \tau(c_1 + c_2)v^n + \tau^2 a_2 c_2 F^n + \tau^3 R^{k3},$$

$$\Delta y^n = \tau v^n + \frac{\tau^2}{2} F^n + \tau^3 R^{k3}. \quad (24)$$

Consider the row decomposition of vector $f^{n+1} = f(y^{n+1}, t_{n+1})$ [2]:

$$f^{n+1} = f^n + f_y^n \Delta y^n + \tau f_t^n + \frac{1}{2}((\Delta y^n)^T f^n_{y^T y} \Delta y^n + 2\tau f^n_{ty} \Delta y^n + \tau^2 f^n_{tt}) + \tau^3 R_f^{k_3}, \quad (25)$$

where $R_f^{k_3} = (R_{f_1}^{k_3}, \ldots, R_{f_m}^{k_3})$, $R_{f_i}^{k_3} = \frac{1}{3!\tau^3}(\sum_{p,q,r} \hat{f}^n_{i,pqr} \Delta y_p^n \Delta y_q^n \Delta y_r^n$ $+ \tau \sum_{p,q} \hat{f}^n_{i,pqt} \Delta y_p^n \Delta y_q^n + 3\tau^2 \sum_p \hat{f}^n_{i,ptt} \Delta y_p^n + \tau^3 \hat{f}^n_{i,ttt})$, and $\hat{f}^n_{i,pqr}$, $\hat{f}^n_{i,pqt}$, $\hat{f}^n_{i,ptt}$, $\hat{f}^n_{i,ttt}$ are the values of third partial derivative, to be determined for $y = y^n + B\Delta y^n, t = t_n + \theta\tau$. The matrix B and the scalar θ admit the corresponding intermediate values of derivatives.

Theorem 3. *If a solution of* (25) *was obtained using a difference scheme of the second order of accuracy* (24) *and for all values of the variables* $y = y^n, t = t_n$, $n = 0, 1, \ldots, N$, *the values* $\tau, q > 0, c_2, a_2$, *the vector* f^0, *the matrix* $K(y,t)$ *and the remainder in Taylor series expansion* $\tau^3 R^{k_3}$ *satisfy the inequalities:*
$\|f^0\| \leqslant \varepsilon$, $2a_2c_2 = 1$, $\tau^3 \|R^{k_3}\| \leqslant (1-q)\varepsilon$, $\left\|I + \tau K^n + \frac{\tau^2}{2}(\dot{K}^n + (K^n)^2)\right\| \leqslant q < 1$, *then* $\|f^n\| \leqslant \varepsilon$ *for all* $n = 1, 2, \ldots, N$.

Proof. Assume that the inequality $\|f^0\| \leqslant \varepsilon$ holds for the initial condition $y(t_0) = y^0$. The equality $\Delta y^n = \tau(1-c_2)v^n + \tau c_2 v(t_n + \tau a_2, y^n + \tau a_2 v^n)$, which is obtained from (24) can be written in the form

$$\Delta y^n = \tau(v^n + \tau c_2 a_2 F^n) + \tau^3 R_v^{k_3}, \quad (26)$$

where $R_v^{k_3} = (R_{v_1}^{k_3}, \ldots, R_{v_m}^{k_3})$, $R_{v_i}^{k_3} = \frac{1}{2!} a_2^2 c_2 (\sum_{p,q} \hat{f}^n_{i,pq} v_p^n v_q^n + 2\sum_p \hat{v}^n_{i,pt} v_p^n + \hat{v}^n_{i,tt})$. After substituting (26) in equation (25) and rearranging terms we get

$$f^{n+1} = f^n + \tau F^n + \tau^2 c_2 a_2 (f_y F)^n + \frac{\tau^2}{2}(v^T f_{yy} v + 2 f_{ty} v + f_{tt})^n + \tau^3 R_f^{k_3}, \quad (27)$$

further using the equalities $\dot{f} = K(y,t) f = f_y v + f_t = F$, $\ddot{f} = \dot{K} f + K^2 f = f_y F + G$, equation (27) can be represented as

$$f^{n+1} = (I + \tau K^n + \tau^2 c_2 a_2 (\dot{K}^n + (K^n)^2)) f^n + \frac{\tau^2}{2}(1 - 2a_2c_2) G^n + \tau^3 R_f^{k_3},$$

But from equation (23): $a_2 c_2 = 1/2$,

$$f^{n+1} = (I + \tau K^n + \frac{\tau^2}{2}(\dot{K} + (K^n)^2)) f^n + \tau^3 R_f^{k_3},$$

Now, taking the hypothesis of the theorem in to account one gets

$$\|f^{n+1}\| \leqslant \left\| I + \tau K + \frac{\tau^2}{2}(\dot{K}^n + (K^n)^2) \right\| \|f^n\| + \tau^3 \left\| R_f^{k_3} \right\| \leqslant q\varepsilon + (1-q)\varepsilon = \varepsilon.$$

3.2.2. When R=3

We can match (21) with (17) and the following set of equations are satisfied:

$$c_1 + c_2 + c_3 = 1,\ a_2c_2 + a_3c_3 = 1/2,\ a_2^2 c_2 + a_3^2 c_3 = 1/3,\ a_2 b_{32} c_3 = 1/6. \quad (28)$$

These are four equations in six unknowns and there exists a two-parameter family of solutions. Combining (21) with (17), and substitute k_1, k_2 and expanding by Taylor we can rewrite Δy^n as

$$\Delta y^n = \tau(c_1 + c_2 + c_3)v^n + \tau^2(a_2c_2 + a_3c_3)F^n + \tau^3 a_2 b_{32} c_3 F^n + \\ + \frac{\tau^3}{2}(a_2^2 c_2 + a_3^2 c_3)G^n + \tau^4 R_v^{k_4} \quad (29)$$

Equation (29) can further be simplified as

$$\Delta y^n = \tau M_1 v^n + \tau^2 M_2 F^n + \tau^3 M_4 F^n + \frac{\tau^3}{2} M_3 G^n + \tau^4 R_v^{k_4}, \quad (30)$$

where $M_1 = c_1 + c_2 + c_3, M_2 = a_2c_2 + a_3c_3, M_3 = c_2a_2^2 + c_3a_3^2, M_4 = a_2c_3b_{32}$. Now, recall the decomposition of vector $f^{n+1} = f(y^{n+1}, t_{n+1})$;

$$f^{n+1} = f^n + f_y^n \Delta y^n + \tau f_t^n + \frac{1}{2}((\Delta y^n)^T f_{y^T y}^n \Delta y^n + 2\tau f_{ty}^n \Delta y^n + \tau^2 f_{tt}^n) + \\ + \frac{1}{3!}(f_{yyy}^n \Delta y^n \Delta y^n \Delta y^n + 3\tau f_{yyt}^n \Delta y^n \Delta y^n + 3\tau^2 f_{ytt}^n \Delta y^n + \tau^3 f_{ttt}^n) + \tau^4 R_f^{k_4}. \quad (31)$$

Theorem 4. *If a solution of* (31) *used a difference scheme of the third order of accuracy* (30) *and for all values of the variables* $y = y^n, t = t_n,\ n = 0, 1, \ldots, N$, *the values* $\tau, \delta > 0$, f^0, *the matrix* $K(y,t)$ *and the remainder in Taylor series expansion* $\tau^4 R^{k_4}$ *satisfy the inequalities:*

$$\left\|f^0\right\| \leqslant \varepsilon,\ \tau^4 \left\|R^{k_4}\right\| \leqslant (1-\delta)\varepsilon,$$

$$\left\| I + \tau K^n + \frac{\tau^2}{2}(\dot{K}^n + (K^n)^2) + \frac{\tau^3}{6}(\ddot{K}^n + 3\dot{K}^n K^n + (K^n)^3) \right\| \leqslant \delta < 1,$$

then $\|f^n\| \leqslant \varepsilon$, *for all* $n = 1, 2, \ldots, N$.

Proof. Substituting (30) in (31) and further simplifying, one gets

$$f^{n+1} = f^n + \tau F^n + \tau^2 M_2 f_y^n F^n + \frac{\tau^2}{2} G^n + \frac{\tau^3}{2} M_3 f_y^n G^n + \\ + \tau^3 M_4 (f_y^n)^2 F^n + \tau^3 M_2 (FI)^n + \frac{\tau^3}{6} H^n + \tau^4 R^{k_4}, \quad (32)$$

where $H^n = (v^n)^3 f_{yyy}^n + 3(v^n)^2 f_{yyt}^n + 3v^n f_{ytt}^n + f_{ttt}^n$. And recall the equalities

$$\dot{f} = K(y,t)f, \quad \ddot{f} = \dot{K}f + K^2 f, \quad \dddot{f} = (\ddot{K} + 3\dot{K}K + K^3)f,$$

$$\dot{f} = f_y v + f_t = F, \quad \ddot{f} = f_y F + G, \quad \dddot{f} = f_y^2 F + f_y G + 3FI + H.$$

$$f^{n+1} = \left(I + \tau K^n + \frac{\tau^2}{2}\left(\dot{K}^n + (K^n)^2\right) + \frac{\tau^3}{6}\left(\ddot{K}^n + 3\dot{K}^n K^n + (K^n)^3\right)\right) f^n + \\ + \tau^4 R^{k_4}. \quad (33)$$

Taking into account the given conditions and assuming that K is constant [2] the conclusion follows

$$\left\|f^{n+1}\right\| \leqslant \left\| I + \tau K^n + \frac{\tau^2}{2}(K^n)^2 + \frac{\tau^3}{6}(K^n)^3 \right\| \|f^n\| + \tau^4 \left\|R^{k_4}\right\| \leqslant \delta\varepsilon + (1-\delta)\varepsilon = \varepsilon.$$

3.2.3. When $R = 4$

Matching equation (21) with (17) we get the following equations:

$$c_1 + c_2 + c_3 + c_4 = 1,$$
$$c_2a_2 + c_3a_3 + c_4a_4 = 1/2, \quad a_2^2c_2 + a_3^2c_3 + a_4^2c_4 = 1/3, \quad c_2a_2^3 + c_3a_3^3 + c_4a_4^3 = 1/4,$$
$$c_3a_2b_{32} + c_4a_2b_{42} + c_4a_3b_{43} = 1/6, \quad c_3a_2b_{32}a_3 + c_4a_2b_{42}a_4 + c_4a_3b_{43}a_4 = 1/8,$$
$$c_3a_2^2b_{32} + c_4a_2^2b_{42} + c_4a_3^2b_{43} = 1/12, \quad c_4a_2b_{32} = 1/24.$$

But $\Delta y^n = \tau h(y^n, t_n, \tau) = \tau(c_1k_1 + c_2k_2 + c_3k_3 + c_4k_4)$. After replacing corresponding expressions for k_1, k_2, k_3, k_4 and simplifying the result we get:

$$\Delta y^n = \tau v^n + \frac{\tau^2}{2}F^n + \frac{\tau^3}{6}(G^n + f_y^n F^n) + \frac{\tau^4}{24}(H^n + (FI)^n + f_y^n G^n + (f_y^n)^2 F^n) + \tau^5 R_v^{k5} \quad (34)$$

and recalling the decomposion of $f^{n+1} = f(y^{n+1}, t_{n+1})$;

$$f^{n+1} = f^n + f_y^n \Delta y^n + \tau f_t^n + \frac{1}{2}f^{(n2)} + \frac{1}{3!}f^{(n3)} + \frac{1}{4!}f^{(n4)} + \tau^5 R_f^{k_5}, \quad (35)$$

where

$$f^{(n2)} = ((\Delta y^n)^T f_{y^T y}^n \Delta y^n + 2\tau f_{ty}^n \Delta y^n + \tau^2 f_{tt}^n),$$
$$f^{(n3)} = (f_{yyy}^n \Delta y^n \Delta y^n \Delta y^n + 3\tau f_{yyt}^n \Delta y^n \Delta y^n + 3\tau^2 f_{ytt}^n \Delta y^n + \tau^3 f_{ttt}^n),$$

$$f^{(n4)} = f_{yyyy}^n \Delta y^n \Delta y^n \Delta y^n \Delta y^n + 4\tau f_{yyyt}^n \Delta y^n \Delta y^n \Delta y^n + 6\tau^2 f_{yytt}^n \Delta y^n \Delta y^n + \\ + 4\tau^3 f_{yttt}^n \Delta y^n + \tau^4 f_{tttt}^n$$

and $R_f^{k_5}$ is obtained from fifth partial derivatives, to be determined for $y = y^n + B\Delta y^n, t = t_n + \theta\tau$. The matrix B and the scalar θ admit the corresponding intermediate values of derivatives.

Theorem 5. *If a solution of* (35) *used a difference scheme of the fourth order of accuracy* (34) *and for all values of the variables* $y = y^n, t = t_n, n = 0, 1, 2, \ldots, N$, *the values* $\tau, \delta > 0$, f^0, *the matrix* $K(y, t)$ *and the remainder in Taylor series expansion* $\tau^5 R^{k_5}$ *satisfy the inequalities:*

$$\left\|f^0\right\| \leqslant \varepsilon, \ \tau^5 \left\|R^{k_5}\right\| \leqslant (1 - \delta)\varepsilon,$$
$$\left\|I + \tau K^n + \frac{\tau^2}{2}(K^n)^2 + \frac{\tau^3}{6}(K^n)^3 + \frac{\tau^4}{24}(K^n)^4\right\| \leqslant \delta < 1,$$

then $\|f^n\| \leqslant \varepsilon$ *for all* $n = 1, 2, \ldots, N$.

Proof. Substitute (34) in (35) and rearranging terms gives:

$$f^{n+1} = f^n + \tau F^n + \frac{\tau^2}{2}(f_y^n F^n + G^n) + \frac{\tau^3}{6}((f_y^n)^2 F^n + f_y^n G^n + 3F^n I^n + H^n) + \\ + \frac{\tau^4}{24}((f_y^n)^3 F^n + (f_y^n)^2 G^n + f_y^n H^n + 4G^n I^n + 6F^n N^n + 7f_y^n F^n I^n + 3f_{yy}^n (F^n)^2 + M^n) + \\ + \tau^5 R^{k_5}, \quad (36)$$

where $N = v^2 f_{yyy} + 2v f_{yyt} + f_{ytt}$ and $M = v^4 f_{yyyy} + 4v^3 f_{yyyt} + 6v^2 f_{yytt} + 4v f_{yttt} + f_{tttt}$.

Let us consider the relation $\dot f = K(y,t)f$ and respective derivatives

$$\dot f = K(y,t)f = f_y v + f_t = F, \quad \ddot f = (\dot K + K^2)f = f_y F + G,$$
$$\dddot f = (\ddot K + 3\dot K K + K^3)f = f_y^2 F + f_y G + 3FI + H,$$

$$\ddddot f = (\dddot K + 4\ddot K K + 3\dot K^2 + 6\dot K K^2 + K^4)f = \\ = f_y^3 F + f_y^2 G + f_y(H + 7FI) + 3f_{yy}F^{+}4GI + 6FN + M.$$

In relation with (36),

$$f^{n+1} = f^n + \tau K^n f^n + \frac{\tau^2}{2}(\dot K^n + (K^n)^2)f^n + \frac{\tau^3}{6}(\ddot K^n + 3\dot K^n K^n + (K^n)^3)f^n + \\ + \frac{\tau^4}{24}(\dddot K^n + 4\ddot K^n K^n + 3(\dot K)n^2 + 6\dot K^n (K^n)^2 + (K^n)^4)f^n + \tau^5 R^{k_5}. \quad (37)$$

Taking into account the given conditions and the fact that K as constant, we conclude

$$\left\|f^{n+1}\right\| \leqslant \left\|I + \tau K^n + \frac{\tau^2}{2}(K^n)^2 + \frac{\tau^3}{6}(K^n)^3 + \frac{\tau^4}{24}(K^n)^4\right\| \|f^n\| + \\ + \tau^5 \left\|R^{k_5}\right\| \leqslant \delta\varepsilon + (1-\delta)\varepsilon = \varepsilon.$$

Example 3. Consider the problem, which we have constructed in example 1 and take $c = 1, r = 1, l = 2$,

$$v_1 = -2\cos\theta_2 + \frac{k\cos\theta_1(\sin\theta_1 - 2\sin\theta_2)}{(\cos\theta_1)^2 + (2\cos\theta_2)^2}, \quad v_2 = -\cos\theta_1 - \frac{2k\cos\theta_2(\sin\theta_1 - 2\sin\theta_2)}{(\cos\theta_1)^2 + (2\cos\theta_2)^2}.$$

We can easily see that the solution using MATLAB is stable for large values of the constant k as shown below in the figure.

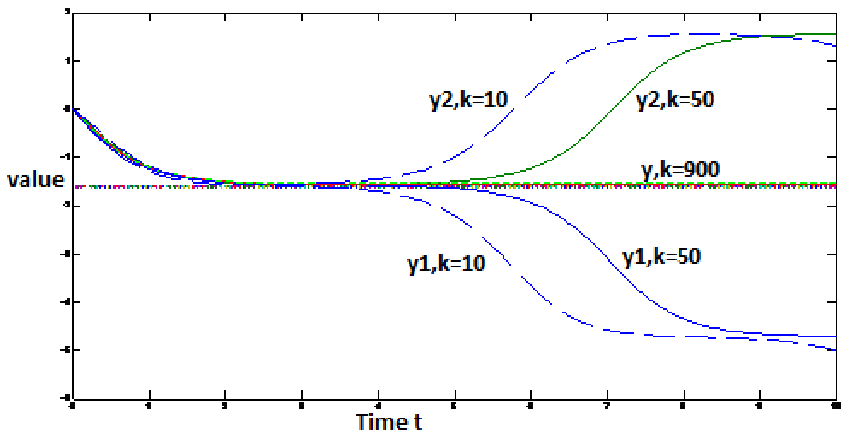

Figure 2. Solution using MATLAB

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Решение дифференциальных уравнений движения для механических систем со связями

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В работе рассматривается задача построения систем дифференциальных уравнений по известным частным интегралам. Приводится метод определения правых частей систем дифференциальных уравнений, основанный на определении общего решения системы линейных алгебраических уравнений с прямоугольной матрицей коэффициентов. Предлагается использовать для численного решения построенной системы дифференциальных уравнений метод Рунге-Кутта. Для рассматриваемой задачи ранее были использованы простейшие разностные схемы первого порядка и метод Рунге Кутта для случая линейных дифференциальных уравнений возмущений связей с постоянными коэффициентами. В статье получены ограничения на коэффициенты уравнений возмущений связей, зависящие от фазовых координат системы, при решении дифференциальных уравнений методом Рунге-Кутта. Подробно рассмотрены случаи разностных уравнений первого порядка, состоящих из нескольких стадий. Получена общая форма условий стабилизации уравнений связей. Метод иллюстрируется на примере решения кинематической задачи кривошипно-шатунного механизма.

Ключевые слова: дифференциальные уравнения, численное интегрирование, кинематические ограничения, стабилизация, ряд Тейлора.